Curvature

1

Curvature: (Intro)

- Einstein Equations: $G = 8\pi T$

$$G = R^{\sigma}_{i\sigma j} - \frac{1}{2}Rg_{ij} \equiv \text{trace of Riemann} - \frac{1}{2}Rg_{ij}$$

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij} = T(\rho, u) \qquad p = p(\rho)$$

$|u| = 1$, 4-unknowns

Equation Count:
(so $\operatorname{div} T = 0$ is relativistic Euler eqn's when $g = \eta$)

$$G[g] = T(\rho, u)$$

10 equations

10 unknown g_{ij}'s
4 unknown ρ, u's

14 unknowns

4 free equations can be imposed to determine the coord system —

$\Rightarrow$ 14 equations 14 unknowns $\Rightarrow$ no more constraints on $G = 8\pi T$ can be imposed

Really $\approx$ only 6 equations ~~of G~~ are evolution any reducing the 10 eqns to six & the 10 metric comps to 6
$\tilde{\Rightarrow}$ 10 eqn's in 10 unknowns

Discovery of equations:

- Guess: $G = \kappa T$ $T \equiv$ "energy-momentum density & fluxes" is the source of curvature

- G should be constructed from R & g in a linear way (simplest)

- G must satisfy div $G = 0$ so evolution satisfies div $T = 0 \Leftrightarrow T^{i\sigma}{}_{;\sigma} = 0$ the covariant div that reduces to $T^{i\sigma}{}_{,\sigma}$ in locally inertial coord frames

We need:

- Derive $R^{\sigma}_{\alpha i j}$ for general connection
(For metric connection it should be a tensor 2nd order in derivatives of g, such that $g \sim \eta$ when $R = 0$)

- Symmetries of R ($\exists$ 20 indept compts)

- Prove Bianchi identities

$$R^{\sigma}_{\alpha [ij;k]} = 0$$

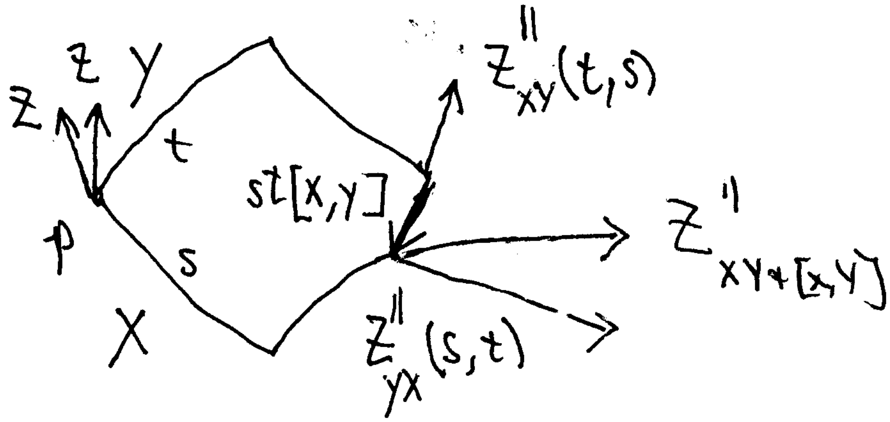
cyclically permute

- <u>Show</u>: $G = R^{\sigma}_{i\sigma j} - \frac{1}{2} R\, g_{ij}$

satisfies $G^{\sigma}_{i;\sigma} \equiv \operatorname{div} G = 0$ as a consequence of Bianchi.

• You can define:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad (*)$$



Can get: "$R(X,Y)Z = \lim_{st \to 0} \dfrac{Z^{\parallel}_{YX} - Z^{\parallel}_{XY+[X,Y]}}{st}$"

(*) operates on vector fields Z, so you must show (*) is a tensor, depends only on Z_p. We find tensor $R^{\sigma}_{\alpha ij}$ fastest way:

Curvature

Curvature:

We have: Y^i $\|$ if

$$dY^i = -\Gamma^i_{jk} Y^j dx^k \qquad (*)$$

<u>Thm</u>: if Γ transforms by

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^i_{jk} \frac{\partial x^j}{\partial y^\beta}\frac{\partial x^k}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i} + \frac{\partial^2 x^\sigma}{\partial y^\alpha y^\beta}\frac{\partial y^\alpha}{\partial x^\sigma}$$

then $Y_\|$ constructed by $(*)$ along any integral curve of X will be a vector at ea. pt:

I.e., X vector field $\Rightarrow$ obtain $Y_\|$ by

solving
$$\frac{dY^i_\|}{d\xi} = -\Gamma^i_{jk} Y^j_\| \frac{dx^k}{d\xi}$$

$$Y^i_\|(0) = Y^i_0$$

(2)

We then defined covariant deriv. by (6)

$$(\nabla_X Y)^i = \lim_{\xi \to 0} \frac{Y^i - Y^i_{\parallel}}{\xi}$$

$$= \frac{dY^i}{d\xi} - \frac{dY^i_{\parallel}}{d\xi} = X(Y^i) + \Gamma^i_{jk} Y^j X^k$$

$$X^\sigma \frac{\partial Y^i}{\partial \sigma}$$

$$\nabla_X f = X(f).$$

$$(\nabla_X \omega)_i = X^\sigma \frac{\partial \omega_i}{\partial x^\sigma} - \Gamma^\sigma_{i\tau} \omega_\sigma X^\tau$$

<u>Notation</u>: $Y^i{}_{;j} = \left(\nabla_{\frac{\partial}{\partial x^j}} Y\right)^i$ etc.

So that $\nabla Y = Y^i{}_{;j}\, dx^j \otimes \frac{\partial}{\partial x^i}$ etc,

$$Y^i{}_{;j} = Y^i{}_{,j} + \Gamma^i_{\sigma j} Y^\sigma$$

• Q: does ∇ commute?

Ex ① Consider $\omega = df = \frac{\partial f}{\partial x^i} dx^i = f_{,i}\, dx^i$
$= f_{;i}\, dx^i$
$= \nabla_i f$

$$\nabla_i \omega^j = \nabla_i \nabla_j f = f_{;i;j} = f_{ij} - \Gamma^{\sigma}_{ij} f_{,\sigma}$$

$$\nabla_j \omega^i = \nabla_j \nabla_i f = f_{;j;i} = f_{ji} - \Gamma^{\sigma}_{ij} f_{,\sigma}$$

$\Rightarrow \nabla_i \nabla_j f - \nabla_j \nabla_i f = 0$ "They commute"

Ex ② Not so for Vector Fields

$$\nabla_i \nabla_j Z - \nabla_j \nabla_i Z \neq 0$$

& we use this to define the curvature.

Thm: $(\nabla_i \nabla_j Z)^{\alpha} - (\nabla_j \nabla_i Z)^{\alpha} = R^{\alpha}_{\beta i j} Z^{\beta}$ where $R^{\alpha}_{\beta i j}$ is Riemann Curvature tensor (*)
Assume $\Gamma^{k}_{ij} = \Gamma^{k}_{ji}$

Note: We know all terms on LHS are a tensors $\forall Z$, but the dependence on derivatives of Z on LHS cancels out $\Rightarrow R^{\alpha}_{\beta i j}$ indept of Z

Cf. Lie Bracket: X, Y vector fields

$$X(Y) - Y(X) = [X, Y]$$

Here $X(Y)$ & $Y(X)$ are <u>not</u> tensors (A difference from (*))
but dependence on non-tensorial 2nd derivatives
<u>cancels</u> out to make tensor.

I.e., $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ in $\underset{\sim}{x}$-coords.

$$[X, Y]_{\underset{\sim}{x}} = \left\{ X^i \frac{\partial}{\partial x^i}(Y^k) - Y^i \frac{\partial}{\partial x^i}(X^k) \right\} \frac{\partial}{\partial x^k}$$

use same dummy var's in
separate terms to reduce # of indices

We now express $[X, Y]_{\underset{\sim}{x}}$ in $\underset{\sim}{y}$-coords &
see that it equals $[X, Y]_{\underset{\sim}{y}}$:

In $\underset{\sim}{y}$-coords: $X^i = \dfrac{\partial x^i}{\partial y^\alpha} X^\alpha, \quad Y^i = \dfrac{\partial x^i}{\partial y^\alpha} Y^\alpha$ (9)

$$[X,Y]_{\underset{\sim}{x}} = \left\{ \underbrace{\frac{\partial x^i}{\partial y^\alpha} X^\alpha \frac{\partial}{\partial x^i}}_{\frac{\partial}{\partial y^\alpha}} \left(\frac{\partial x^k}{\partial y^\beta} Y^\beta \right) - \underbrace{\frac{\partial x^i}{\partial y^\alpha} Y^\alpha \frac{\partial}{\partial x^i}}_{\frac{\partial}{\partial y^\alpha}} \left(\frac{\partial x^k}{\partial y^\beta} X^\beta \right) \right\} \frac{\partial}{\partial x^k}$$

$$= \left\{ X^\alpha \underbrace{\frac{\partial}{\partial y^\alpha} \left(\frac{\partial x^k}{\partial y^\beta}}_{\text{2nd deriv term}} Y^\beta \right) - Y^\alpha \underbrace{\frac{\partial}{\partial y^\alpha} \left(\frac{\partial x^k}{\partial y^\beta}}_{\text{2nd deriv term}} X^\beta \right) \right\} \frac{\partial}{\partial x^k}$$

2nd Deriv: $X^\alpha \dfrac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta} Y^\beta - Y^\alpha \dfrac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta} X^\beta = 0$

$$= \left\{ X^\alpha Y^\beta{}_{,\alpha} \frac{\partial x^k}{\partial y^\beta} - Y^\alpha X^\beta{}_{,\alpha} \frac{\partial x^k}{\partial y^\beta} \right\} \frac{\partial}{\partial x^k}$$

$$= \left\{ X^\alpha \frac{\partial}{\partial y^\alpha} Y^\beta - Y^\alpha \frac{\partial}{\partial y^\alpha} X^\beta \right\} \frac{\partial}{\partial y^\beta} = [X,Y]_{\underset{\sim}{y}}$$ ✓

Pf of (*): $Z^\alpha_{;i;j} - Z^\alpha_{;j;i} = R^\alpha_{\beta ij} Z^\beta$ (10) (6)

$$Z^\alpha_{;i;j} = \left(Z^\alpha_{;i}\right)_{;j} = \left(Z^\alpha_{,i} + \Gamma^\alpha_{i\sigma} Z^\sigma\right)_{;j}$$

$$\begin{aligned} &= Z^\alpha_{,ij} + \Gamma^\alpha_{i\sigma,j} Z^\sigma + \Gamma^\alpha_{i\sigma} Z^\sigma_{,j} \\ &\quad + \boxed{\Gamma^\alpha_{\tau j} Z^\tau_{,i}} + \Gamma^\alpha_{\tau j} \Gamma^\tau_{i\sigma} Z^\sigma \\ &\quad - \Gamma^\tau_{ij} Z^\alpha_{,\tau} - \Gamma^\tau_{ij} \Gamma^\alpha_{\tau\sigma} Z^\sigma \end{aligned} \qquad (A)$$

$$Z^\alpha_{;j;i} = \left(Z^\alpha_{;j}\right)_{;i} = \left(Z^\alpha_{,j} + \Gamma^\alpha_{j\sigma} Z^\sigma\right)_{;i}$$

$$\begin{aligned} &= Z^\alpha_{,ij} + \Gamma^\alpha_{j\sigma,i} Z^\sigma + \Gamma^\alpha_{j\sigma} Z^\sigma_{,i} \\ &\quad + \Gamma^\alpha_{\tau i} Z^\tau_{,j} + \boxed{\Gamma^\alpha_{\tau i} \Gamma^\tau_{j\sigma} Z^\sigma} \\ &\quad - \Gamma^\tau_{ji} Z^\alpha_{,\tau} - \Gamma^\tau_{ji} \Gamma^\alpha_{\tau\sigma} Z^\sigma \end{aligned} \qquad (B)$$

$$(\nabla_i \nabla_j Z)^\alpha - (\nabla_j \nabla_i Z)^\alpha = \left\{\Gamma^\alpha_{\sigma j,i} - \Gamma^\alpha_{\sigma i,j} + \Gamma^\alpha_{\tau i} \Gamma^\tau_{\sigma j} - \Gamma^\alpha_{\tau j} \Gamma^\tau_{\sigma i}\right\} Z^\sigma$$

$$\equiv R^\alpha_{\sigma ij} Z^\sigma$$

- Note: The same argument works in commuting a 1-form:

$$(\nabla_j \nabla_i \omega)_\alpha - (\nabla_i \nabla_j \omega)_\alpha = R_{\alpha ij}{}^{\sigma} \omega_\sigma$$

reverse order
⇝ minus sign

Pf same argument using

$$\omega_{\alpha;i} = \omega_{\alpha,i} - \Gamma_{\alpha i}^{\sigma} \omega_\sigma . \quad \text{(HW)}$$

check

$$W_{\alpha;i;j} = (W_{\alpha;i})_{;j} = \left(W_{\alpha,i} - \Gamma^{\sigma}_{\alpha i} W_\sigma\right)_{;j}$$ (12)

$$= W_{\alpha,ij} - \Gamma^{\sigma}_{\alpha i,j} W_\sigma - \Gamma^{\sigma}_{\alpha i} W_{\sigma,j}$$

$$- \Gamma^{\tau}_{\alpha j} W_{\tau,i} + \Gamma^{\tau}_{\alpha j} \Gamma^{\sigma}_{\tau i} W_\sigma$$

$$- \boxed{\Gamma^{\tau}_{ij} W_{\alpha,\tau}} + \cancel{\Gamma^{\tau}_{ij} \Gamma^{\sigma}_{\alpha\tau}} W_\sigma$$

$$W_{\alpha;j;i} = W_{\alpha,ji} - \Gamma^{\sigma}_{\alpha j,i} W_\sigma - \Gamma^{\sigma}_{\alpha j} W_{\sigma,i}$$

$$- \Gamma^{\tau}_{\alpha i} W_{\tau,j} + \Gamma^{\tau}_{\alpha i} \Gamma^{\sigma}_{\tau j} W_\sigma$$

$$- \boxed{\Gamma^{\tau}_{ji} W_{\alpha,\tau}} + \cancel{\Gamma^{\tau}_{ji} \Gamma^{\sigma}_{\alpha\tau}} W_\sigma$$

$$W_{\alpha;i;j} - W_{\alpha;j;i} = \left\{\Gamma^{\sigma}_{\alpha j,i} - \Gamma^{\sigma}_{\alpha i,j} + \Gamma^{\tau}_{\alpha j}\Gamma^{\sigma}_{\tau i} - \Gamma^{\tau}_{\alpha i}\Gamma^{\sigma}_{\tau j}\right\} W_\sigma$$

different order ≈ minus sign

$$= R^{\sigma}_{\alpha i j} W_\sigma$$

• <u>Contraction Lemma</u>: If $R^{\alpha}_{\sigma i j} Z^{\sigma}$ is a $\binom{1}{2}$-tensor for every vector field $Z^{\sigma}\frac{\partial}{\partial x^{\sigma}}$, then $R^{\alpha}_{\sigma i j}$ transforms as a $\binom{1}{3}$-tensor.

Pf. (HW)

This holds in full generality:

$$T^{i_1 \cdots i_\ell \, i_{\ell+1} \cdots i_n}_{j_1 \cdots j_k \, j_{k+1} \cdots j_m} \; A^{j_{k+1} \cdots j_m}_{i_{\ell+1} \cdots i_n}$$

a $\binom{\ell}{k}$-tensor $\forall\, A \in \mathcal{T}_{n-\ell}$, the T is a $\binom{n}{m}$-tensor.

<u>Cor</u>: $R^{\alpha}_{\beta i j}$ transforms like a $\binom{1}{3}$-tensor

Riemann Curvature Tensor.

Cor: $(\nabla_X \nabla_Y Z)^\alpha - (\nabla_Y \nabla_X Z)^\alpha = R^\alpha_{\sigma i j} Z^\sigma X^i Y^j + \nabla_{[X,Y]} Z^\alpha$

Pf: Mimicking (A) we get

$$\nabla_Y \nabla_X Z^\alpha = \nabla_Y (\nabla_X Z)^\alpha = \nabla_Y \left(X^i Z^\alpha_{,i} + X^i \Gamma^\alpha_{i\sigma} Z^\sigma \right)$$

$$= Y^j \frac{\partial}{\partial x^j}(X^i)\left(Z^\alpha_{,i} + \Gamma^\alpha_{i\sigma} Z^\sigma\right) + \text{terms linear in } X^i, Y^j$$

Thus

$$(\nabla_X \nabla_Y Z)^\alpha - (\nabla_Y \nabla_X Z)^\alpha = R^\alpha_{\sigma i j} Z^\sigma X^i Y^j$$

$$+ \Big\{ X^j \frac{\partial}{\partial x^j}(Y^i)\left(Z^\alpha_{,i} + \Gamma^\alpha_{i\sigma} Z^\sigma\right)$$

$$- Y^j \frac{\partial}{\partial x^j}(X^i)\left(Z^\alpha_{,i} + \Gamma^\alpha_{i\sigma} Z^\sigma\right)$$

$$= [X,Y]^i \left(Z^\alpha_{,i} + \Gamma^\alpha_{i\sigma} Z^\sigma\right) = \left(\nabla_{[X,Y]} Z\right)^\alpha \Big\}$$

$$= R^\alpha_{\sigma i j} Z^\sigma X^i Y^j + \nabla_{[X,Y]} Z^\alpha \checkmark$$

Curvature as a "Curl" plus a "Commutator"

- Think of $\Gamma^\alpha_{\beta i}$ as $\underset{n\times n}{\left(\Gamma^\alpha_\beta\right)_i} = \left(\underset{\sim}{\Gamma}\right)_i$ matrix @ each i

$$R^\alpha_{\beta ij} = \Gamma^\alpha_{\beta j,i} - \Gamma^\alpha_{\beta i,j} + \Gamma^\alpha_{\tau i}\Gamma^\tau_{\beta j} - \Gamma^\alpha_{\tau j}\Gamma^\tau_{\beta i}$$

$$\underbrace{\underset{\sim}{\Gamma}_{j,i} - \underset{\sim}{\Gamma}_{i,j}}_{\text{Curl}} + \underbrace{\underset{\sim}{\Gamma}_i\underset{\sim}{\Gamma}_j - \underset{\sim}{\Gamma}_j\underset{\sim}{\Gamma}_i}_{\text{Commutator}}$$

$$R^\alpha_{\beta ij} = -R^\alpha_{\beta ji}$$

$\approx \Gamma^\alpha_{\beta i}dx^i \approx$ matrix valued 1-form

$R^\alpha_{\beta ij}\,dx^i\wedge dx^j \approx$ matrix valued 2-form

- Geometric Interpretation of Curvature Tensor —

$$R^{\alpha}_{\beta i j} X^i Y^j = R(X,Y)^{\alpha}_{\beta}$$

so

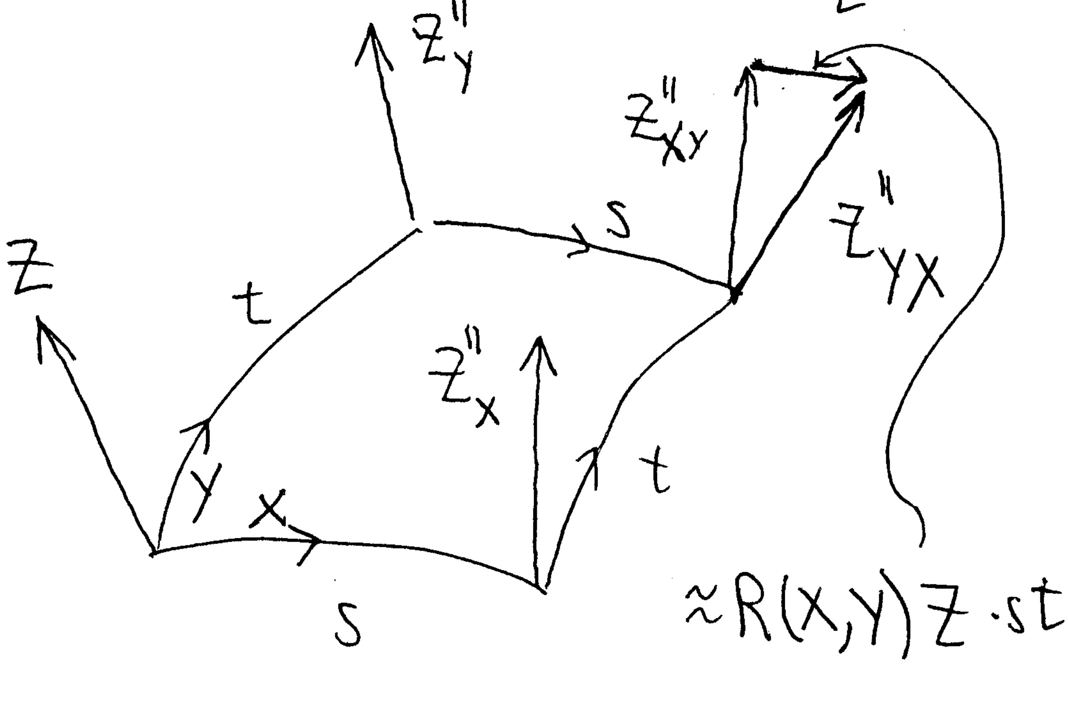
$$R(X,Y)Z = R(X,Y)^{\alpha}_{\beta} Z^{\beta} \frac{\partial}{\partial x^{\alpha}} \quad \text{(a vector field)}$$

- Assume $[X,Y]=0 \Leftrightarrow X, Y$ are coord. vector fields

say $X = x^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^k}$, $Y = y^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^\ell}$ some coord system $\underset{\sim}{y}$

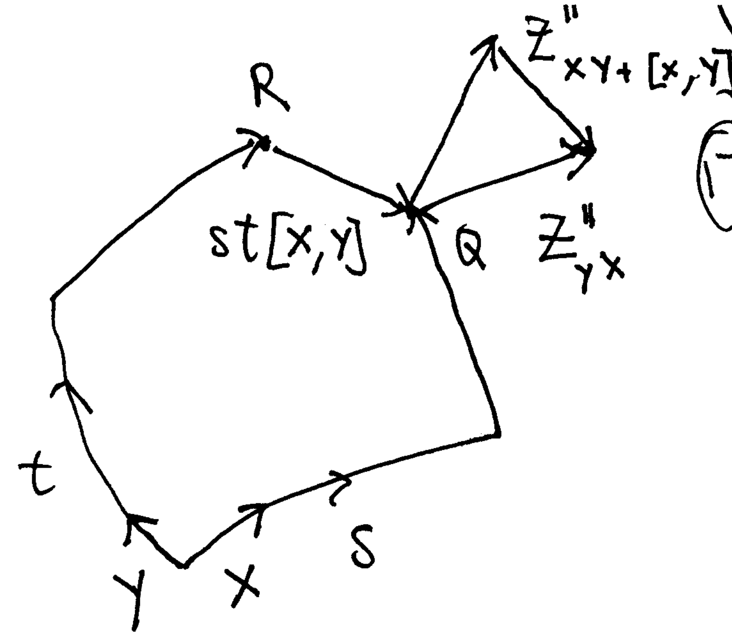
<u>Then</u>: $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \lim_{st \to 0} \frac{Z''_{YX} - Z''_{XY}}{st}$

<u>Picture</u>:



In General:

When $[X,Y] \neq 0$



(17)

$$R(X,Y)Z = \lim_{s,t \to 0} \frac{Z''_{YX} - Z''_{XY+[X,Y]}}{st}$$

$$= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

[Correct argument using

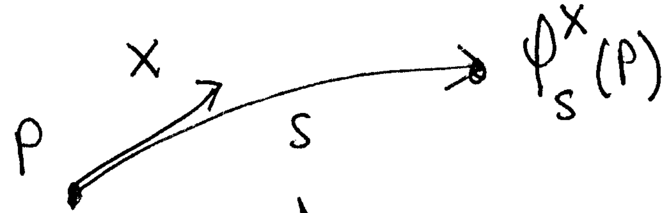
$(st)\nabla_{[X,Y]} Z_R = Z_Q - Z''_{[X,Y]}$ + hot's]

Geometric Interpretation of $[X,Y]$

X, Y vector fields viewed in $\underset{\sim}{x}$-coordinates

$\varphi^X_s(p) \equiv$ "pt s units along int curve of X from p"

$$\frac{d}{ds}\varphi^X_s(p) = X_p$$



Restrict to values in $\underset{\sim}{x}$-coords & write

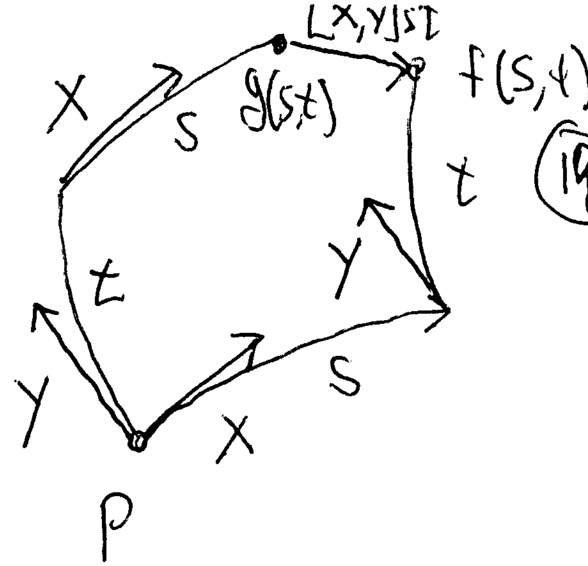
$$f(s,t) = \varphi^Y_t \circ \varphi^X_s(p) \qquad (s,t) \longmapsto \underset{\sim}{x} \in \mathbb{R}^4$$

$$g(s,t) = \varphi^X_s \circ \varphi^Y_t(p) \qquad (s,t) \longmapsto \underset{\sim}{x} \in \mathbb{R}^4$$

<u>Thm</u>: $[X,Y]_p = \lim\limits_{s,t\to 0} \dfrac{f(s,t)-g(s,t)}{st}$; or more precisely

$$f(s,t) - g(s,t) = [X,Y]_p st + O(st^2 + ts^2)$$

P.f. Picture:



Follows by Taylor's Thm:

$$f(s,t) = P + \frac{\partial f}{\partial s}(0,0)s + \frac{\partial f}{\partial t}(0,0)t$$

$$+ \frac{1}{2}(s,t)\begin{bmatrix} f_{ss} & f_{st} \\ f_{ts} & f_{tt} \end{bmatrix}_{(0,0)}(s,t) + O(s^2t + ts^2)$$

$$g(s,t) = P + g_s(0,0)s + g_t(0,0)t$$

$$+ \frac{1}{2}(s,t)\begin{bmatrix} g_{ss} & g_{st} \\ g_{ts} & g_{tt} \end{bmatrix}_{(0,0)}\begin{bmatrix} s \\ t \end{bmatrix} + O(s^2t + ts^2)$$

But easy to see: $f_s(0,0) = g_s(0,0)$ $f_t(0,0) = g_t(0,0)$

$f_{ss}(0,0) = g_{ss}(0,0)$ $f_{tt}(0,0) = g_{tt}(0,0)$

(These involve derivatives along same curve.

$\therefore$ $$f(s,t) - g(s,t) = (f_{st} - g_{st})_{(0,0)}st + O(s^2t + st^2)$$

15

20

But:

$$f_{st}(0,0) = \frac{\partial^2}{\partial s\,\partial t} f(s,t)\Big|_{(0,0)} = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\varphi^Y_t \circ \varphi^X_s(p)\Big|_{(0,0)}$$

$$= \frac{\partial}{\partial s} Y^i\left(\varphi^X_s(p)\right)\Big|_{s=0} = X^a \frac{\partial}{\partial x^a}\left(Y^i\right)\Big|_p$$

take $\underset{\sim}{x}$-coords of everything

$$= X(Y)^i$$

Conclude:

$$\left[f(s,t) - g(s,t)\right]^i = \left(X(Y)^i - Y(X)^i\right) st + O(s^2t + t^2s)$$

expressed in $\underset{\sim}{x}$-coords

$\underset{\sim}{x}$-coords

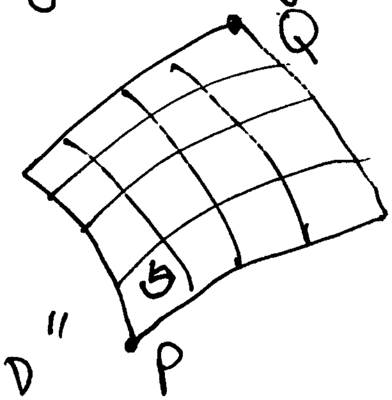
$$= [X,Y]^i st + O(s^2t + t^2s)$$

$$\boxed{[X,Y] = \lim_{s,t\to 0} \frac{f(s,t) - g(s,t)}{st}}$$

Cor. $[X,Y] \equiv 0$ iff X, Y are coord vector fields, i.e., $\exists\, \underset{\sim}{y}$ st $X = \frac{\partial}{\partial y^i}$, $Y = \frac{\partial}{\partial y^j}$

<u>Cor</u>: $[X,Y] \equiv 0$ in $\mathcal{U}$ iff X, Y are coord vector fields in $\mathcal{U}$, ie, $\exists\ \underset{\sim}{y}$ st $X = \frac{\partial}{\partial y^i}$ $Y = \frac{\partial}{\partial y^j}$

Pf. $[X,Y]=0$ iff $f(s,t)=g(s,t)$



"Error in closing @ $Q = \sum \frac{st}{N^2}$ errors of order $[X,Y]st + O\left(\frac{s^2t}{N^3}+\frac{st^2}{N^3}\right) \to 0$"

(s,t) are coords on surface thru P

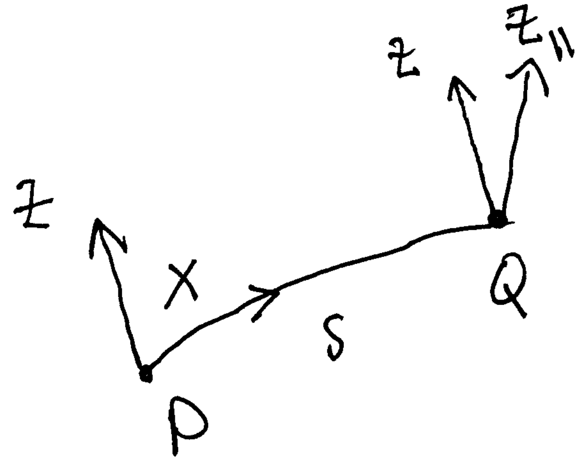
Easy to extend to a foliation of surfaces ✓

<u>Cor</u>: <u>Frobenius</u> <u>Integrability</u> <u>Thm</u>:

$X_1, \dots, X_n$ are coord vector fields iff $[X_i, X_j] \equiv 0$

Geometric Interp of $R(X,Y)Z$:

- Assume $[X,Y]=0 \Leftrightarrow X,Y$ coord vector fields in sense

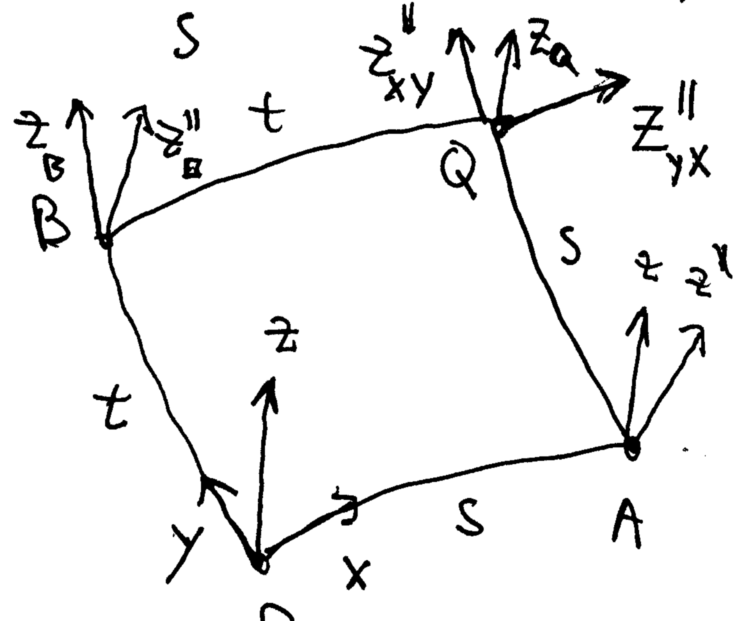


- Define

$$\Phi^X_s Z_p = Z_{\varphi^X_s(p)} - Z^{\parallel}_{\varphi^X_s(p)}$$

So: $$\frac{d}{ds}\Phi^X_s Z_p = \lim_{s\to 0} \frac{Z_{\varphi^X_s(p)} - Z^{\parallel}_{\varphi^X_s(p)}}{s} = \nabla_X Z_p$$

$$F(s,t) = \underbrace{\Phi^Y_t \circ \underbrace{\Phi^X_s Z_p}_{Z_A - Z^{\parallel}_A}}_{Z_Q - Z^{\parallel}_{YX}}$$



$$G(s,t) = \underbrace{\Phi^X_s \circ \underbrace{\Phi^Y_t Z_p}_{Z_B - Z^{\parallel}_B}}_{Z_Q - Z^{\parallel}_{XY}} \Rightarrow \boxed{F(s,t) - G(s,t) = Z^{\parallel}_{XY} - Z^{\parallel}_{YX}}$$

- Similar argument as [,] case:

$$\left.\frac{\partial F}{\partial s}\right|_p = \left.\frac{\partial G}{\partial s}\right|_p, \quad \frac{\partial^2 F}{\partial s^2} = \frac{\partial^2 G}{\partial s^2}, \quad \frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 G}{\partial t^2}$$

because these involve ∥-translation along <u>same</u> curve. ∴ Taylor

$$F(s,t) \underset{\substack{\uparrow \\ \text{in } \underset{\sim}{x}}}{=} \underset{\sim}{x}(p) + \left.\frac{\partial F}{\partial s}\right|_{(0,0)} s + \left.\frac{\partial F}{\partial t}\right|_{(0,0)} t$$

$$+ \frac{1}{2}(s,t)\begin{bmatrix} F_{ss} & F_{st} \\ F_{ts} & F_{tt} \end{bmatrix}_{(0,0)} \begin{pmatrix} s \\ t \end{pmatrix} + O(s^2 t + s t^2)$$

$$G(s,t) = \cdots$$

$$F(s,t) - G(s,t) = \left(F_{st} - G_{st}\right)_{(0,0)} st + O(s^2 t + s t^2)$$

$$F_{st}\Big|_{(0,0)} = \frac{\partial}{\partial s}\underbrace{\frac{\partial}{\partial t}\Phi^{Y}_{t}}\circ\Phi^{X}_{s} Z_p$$

$$= \frac{\partial}{\partial s}\nabla_Y\left(\Phi^X_s Z_p\right)$$

∇_Y does not depend on how Y changes, only on Y_p $\rightsquigarrow$

$$= \nabla_Y\left\{\frac{\partial}{\partial s}\Phi^X_s Z_p\right\} = \nabla_Y\nabla_X Z_p$$

$$G_{st}\Big|_{(0,0)} = \nabla_X\nabla_Y Z_p$$

$$\therefore\quad Z''_{XY} - Z''_{YX} = F(s,t) - G(s,t) = \left(\nabla_Y\nabla_X Z_p - \nabla_X\nabla_Y Z_p\right)s\,t + O(st^2+s^2t)$$

$$\boxed{\nabla_Y\nabla_X Z_p - \nabla_X\nabla_Y Z_p = \lim_{s,t\to 0}\frac{Z''_{YX} - Z''_{XY}}{st}}$$

$$R(X,Y)Z_p = \nabla_X\nabla_Y Z_p - \nabla_Y\nabla_X Z_p = \lim_{s,t\to 0}\frac{Z''_{YX} - Z''_{XY}}{st}$$

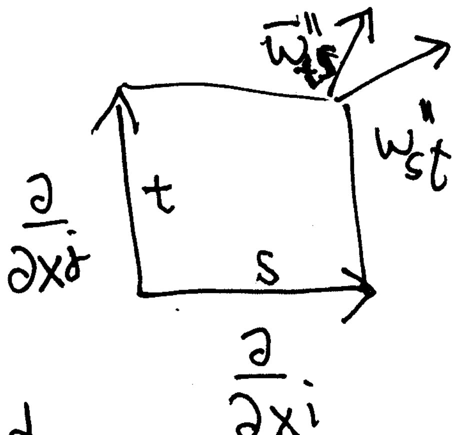
$[X,Y]=0$

✓

Similar argument gives —

Thm. $$\underbrace{\left(\nabla_i \nabla_j \omega - \nabla_j \nabla_i \omega\right)_k}_{R^{\sigma}_{kij}\,\omega_\sigma} st = \left(\omega'' - \bar{\omega}''\right)_k + O(st^2 + s^2 t)$$

Cor: (The main point) If $R^i_{jkl} \equiv 0$, then $\parallel$-translation of vectors & 1-forms is indept of path.



"Proof" $$\left(\bar{\omega}''_{ts} - \omega''_{st}\right)_k = R^{\sigma}_{kij}\,\omega_\sigma\, st + O(s^2 t + s t^2)$$

If $R^{\sigma}_{kij} = 0$, then as in the argument for $[X,Y]$, $\bar{\omega}'' - \omega'' \equiv 0$ ✓

{ I.e., "decompose $\bar{\omega}''_{ts} - \omega''_{st}$ into a sum of N^2 terms of order $\frac{1}{N^3}$ by discretizing the rectangle $s\frac{\partial}{\partial x^i} \times t\frac{\partial}{\partial x^j}$" }

Cor 1 If $R^i_{jkl} \equiv 0$, then $\|$-translation of ω is indept of path (FIP) 24

(For a metric connection)

<u>Theorem</u> If $R^i_{jkl} \equiv 0$, then spacetime is <u>flat</u>: I.e., $\exists$ a coord system in which

$$g_{ij} \equiv \eta_{ij}$$

<u>Proof</u>: Choose $\omega(0)$ in x-coords. $\|$-translate it to construct a covector field in a nbhd. By <u>Cor 1</u>, $\omega(x)$ is defined <u>indept</u> of path.

<u>Claim</u>: $\omega_i = df_i$ for some function f.

I.e., ω parallel $\Rightarrow$

$$0 = \omega_{i;j} = \omega_{i,j} - \Gamma^\sigma_{ij}\,\omega_\sigma$$

$$\Rightarrow \quad \omega_{i,j} = \Gamma^\sigma_{ij}\,\omega_\sigma .$$

$\therefore \quad d\omega = d(\omega_i dx^i) = \omega_{i,j}\, dx^j \wedge dx^i$

$$= \underbrace{\Gamma^{\sigma}_{ij}}_{\text{symm}} \omega_\sigma \underbrace{dx^j \wedge dx^i}_{\text{anti-symm}} = 0$$

$\therefore$ $\omega = df$ for some fn (scalar) f. ✓

• Similarly construct $n=4$ parallel indept vector fields $\omega^1, \dots, \omega^n$ in a nbhd of zero, & get n coordinate functions $y^1, \dots, y^n$ satisfying

$$dy^\alpha = \omega^\alpha \qquad \omega^\alpha = \omega^\alpha_i dx^i$$

$$y(P_0) = 0.$$

<u>Claim</u>: in y-coordinates, $g \equiv$ constant

This is suffient to conclude the theorem because then a constant matrix transformation Λ takes g to η : $z^\mu = \Lambda^\mu_\alpha y^\alpha$, $g_{\mu\nu} = \Lambda_\mu g_{\alpha\beta} \Lambda_\nu \equiv \eta_{\mu\nu}$

• To prove the claim, note

$$\frac{\partial y^\alpha}{\partial x^j} = y^\alpha_{,j} \equiv w^\alpha_j$$

$$y^\alpha_{,jk} = w^\alpha_{j,k} = \Gamma^\sigma_{ij} w^\alpha_\sigma = \Gamma^\sigma_{ij} y^\alpha_{,\sigma}$$

Therefore, the components $g_{\alpha\beta}$ of the metric in y-coordinates satisfy

$$g_{ij} = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} = g_{\alpha\beta}\, y^\alpha_{,i}\, y^\beta_{,j}$$

so

$$g_{ij,k} = g_{\alpha\beta,\gamma} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$$

$$+ g_{\alpha\beta}\, y^\alpha_{,ik}\, \underbrace{y^\beta_{,j}}_{w^\alpha_j} + g_{\alpha\beta}\, \underbrace{y^\alpha_{,i}}_{w^\beta_i}\, y^\beta_{,jk}$$

$$= g_{\alpha\beta,\gamma} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} + g_{\alpha\beta} \Gamma^\sigma_{ik} w^\alpha_\sigma w^\beta_j$$

$$+ g_{\alpha\beta} \Gamma^\sigma_{jk} w^\beta_\sigma w^\alpha_i$$

g_{ij}

or

$$g_{ij,k} - \underbrace{g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\sigma}\frac{\partial y^\beta}{\partial x^j}}_{g_{\sigma j}}\Gamma^\sigma_{ik} - \underbrace{g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^\sigma}}_{g_{i\sigma}}\Gamma^\sigma_{jk}$$

$$= g_{\alpha\beta,\gamma}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}$$

or

$$0 = \nabla_k g_{ij} = g_{\alpha\beta,\gamma}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} = 0$$

$$\forall\, i,j,k$$

$$0 = g_{\alpha\beta,\gamma} \quad \forall\, \alpha,\beta,\gamma \Rightarrow g_{\alpha\beta} \equiv \text{const}$$

<u>Conclude</u>: $R^\sigma_{ijk} \equiv 0 \Rightarrow$ spacetime is flat !

<u>Conclude</u>: $R^\sigma_{ijk} \equiv 0 \Rightarrow$ spacetime is flat is essentially a generalized Frobenius Thm for covectors !

Restrict to metric connection:

We have:

$$R^{\alpha}_{\beta ij} = \left\{\Gamma^{\alpha}_{\beta j,i} - \Gamma^{\alpha}_{\beta i,j}\right\}_{I} + \left\{\Gamma^{\alpha}_{\tau i}\,\Gamma^{\tau}_{\beta j} - \Gamma^{\alpha}_{\tau j}\,\Gamma^{\tau}_{\beta i}\right\}_{II}$$

In Locally Lorentz coords at p (Riemann Normal coords)

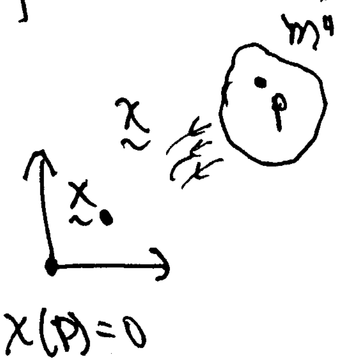
$x(p) = 0$

$g_{ij}(0) = \eta_{ij}$

$g_{ij,k}(0) = 0$

$$\Rightarrow \Gamma^{i}_{jk} = \tfrac{1}{2} g^{i\sigma}\left\{-g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j}\right\} = O(|x|)$$

$$\Rightarrow \{\ \}_{II} = O(|x|^2)$$



$$\Rightarrow R^{\alpha}_{\beta ij}(0) = \Gamma^{\alpha}_{\beta j,i}(0) - \Gamma^{\alpha}_{\beta i,j}(0)$$

$$R^{\alpha}_{\beta ij}(x) = \Gamma^{\alpha}_{\beta j,i}(x) - \Gamma^{\alpha}_{\beta i,j}(x) + O(|x|^2)$$

ASIDE:

Write: $\omega^{\alpha}_{\beta} = \Gamma^{\alpha}_{\beta i}\, dx^{i}$ "Lie Alg Valued 1-form"

Valid if $\Gamma^{\alpha}_{\beta i}$ const $\forall\, i \Rightarrow$ valid to $O(x^2)$

$$d\omega^{\alpha}_{\beta} = \Gamma^{\alpha}_{\beta i,j}\, dx^{j} \wedge dx^{i} = \sum_{j<i}\left(\Gamma^{\alpha}_{\beta i,j} - \Gamma^{\alpha}_{\beta j,i}\right) dx^{j} \wedge d$$

$$= \left(\Gamma^{\alpha}_{\beta i,j} - \Gamma^{\alpha}_{\beta j,i}\right) dx^{j} \otimes dx^{i}$$

<u>Conclude</u>: $\{\ \}_{I}$ is a generalized curl.

$$\{\ \}_{II} = \Gamma_i \Gamma_j - \Gamma_j \Gamma_i$$

$$\Gamma_i \equiv 4 \times 4 \text{ matrix } \left(\Gamma^{\alpha}_{\tau}\right)_i$$

row

colm

$\Rightarrow$ $\{\}_{II}$ is a commutator., and corrects for the fact that dw is not a tensor because Γ is not a tensor.

Symmetries of R (Assume Γ symmetric)

① $R^{\alpha}_{\beta \underbrace{j i}} = - R^{\alpha}_{\beta \underbrace{i j}}$

P.f. $R^{\alpha}_{\beta i j} Z^{\beta} = Z^{\beta}_{;i;j} - Z^{\beta}_{;j;i}$ ✓

② $R^{\alpha}_{[\beta i j]} = R^{\alpha}_{\beta i j} + R^{\alpha}_{j \beta i} + R^{\alpha}_{i j \beta}$

cyclidly permute

P.f $R^{\alpha}_{\beta i j} = \underbrace{\Gamma^{\alpha}_{\beta j, i}}_{\text{symmetric}} - \Gamma^{\alpha}_{\beta i, j} + \Gamma^{\alpha}_{\tau i} \underbrace{\Gamma^{\tau}_{\beta j}}_{\text{symmetric}} - \Gamma^{\alpha}_{\tau j} \Gamma^{\tau}_{\beta i}$

Claim: Any set of #'s $B_{\beta j i} = A_{\beta j i} - A_{\beta i j}$ (*)

symm

with $A_{\beta j i} = A_{j \beta i}$

satisfies $B_{[\beta j i]} = 0$ i.e. $B_{[\beta j i]} = \cancel{A_{\beta j i}} - \cancel{A_{\beta i j}} + \cancel{A_{i \beta j}} - \cancel{A_{i j \beta}} + \cancel{A_{j i \beta}} - \cancel{A_{j \beta i}}$

Since $A_{\beta j i} = \Gamma^{\alpha}_{\beta j, i}$ & $\bar{A}^{\alpha}_{\beta j i} = \Gamma^{\alpha}_{\tau i} \Gamma^{\tau}_{\beta j}$

are symm in βj, & $R^{\alpha}_{\beta i j} = B^{\alpha}_{\beta j i} + \bar{B}^{\alpha}_{\beta j i}$,

② follows ✓

Let $R_{\alpha\beta ij} = g_{\alpha\sigma} R^{\sigma}{}_{\beta ij}$.

(3) $R_{\alpha\beta ij} = -R_{\beta\alpha ij}$

$= -R_{\alpha\beta ji}$ (from above)

i.e. antisymm in 1st 2 & last 2 indices

Proof: in Loc Lorentz coords,

$\Gamma_{\alpha\beta j} = \frac{1}{2}\{-g_{\beta j,\alpha} + g_{\alpha\beta,j} + g_{j\alpha,\beta}\}$

$$R_{\alpha\beta ij} = \Gamma_{\alpha\beta j,i} - \Gamma_{\alpha\beta i,j}$$

$$= \frac{1}{2}\{-g_{\beta j,\alpha i} + \cancel{g_{\alpha\beta,ji}} + g_{j\alpha,\beta i}\}$$

$$- \{-g_{\beta i,\alpha j} + \cancel{g_{\alpha\beta,ij}} + g_{i\alpha,\beta j}\}$$

$$R_{\alpha\beta ij} = \frac{1}{2}\{-g_{\beta j,\alpha i} + g_{\beta i,\alpha j} + g_{\alpha j,\beta i} - g_{\alpha i,\beta j}\} \quad (*)$$

from which (3) follows at once.

(Note: antisymm. is a coord indept prop of a tensor)

(FIP)

(4) $R_{\alpha\beta ij} = R_{ij\alpha\beta}$ "Symm. under pr. exch."

Pf. This follows from (*). in Loc. Lor. fr. But Symm under pair exch is a coord. indept property of a tensor. ✓ (FIP)

Thm: Symmetries (1) - (4) $\Rightarrow \exists$ 20 indept entries in the Curvature tensor. (FIP)

Project: Show $\exists$ a metric with exactly 20 indept entries: [See David Meldgin]

- $R_{\alpha\beta ij}$ $\quad 4\times 4\times 4\times 4 = 256$ entries
- $\left.\begin{array}{l} R_{\alpha\beta ij} = -R_{\beta\alpha ij} \\ R_{\alpha\beta ij} = -R_{\alpha\beta ji} \end{array}\right\}$ $\quad \underbrace{4\times 4}_{6} \times \underbrace{4\times 4}_{6}$ $\quad \leq 36$ indept entries
- $R_{\alpha\beta ij} = R_{ij\alpha\beta}$ $\quad \Rightarrow \leq 21$ indept entries (FIP)
- $R_{\alpha[\beta ij]} = 0$ $\quad \Rightarrow$ 20 indept entries

Ricci Tensor

$$R_{ij} = R^{\sigma}{}_{i\sigma j}$$

<u>Thm</u>: R_{ij} is the only non-trivial 2 tensor which can be obtained from the 4-tensor $R^{i}{}_{jkl}$ by contraction (modulo raising/lowering & trivial interchanges)

<u>Pf</u>: If $A_{ij} = -A_{ji}$, then

$$A^{i}{}_{j} = g^{\sigma i} A_{\sigma j} = -g^{\sigma i} A_{j\sigma} = -A_{j}{}^{i} \qquad (A)$$

$$A^{i}{}_{i} = \underbrace{g^{\sigma i}}_{\text{symm}} \underbrace{A_{\sigma i}}_{\text{antisym}} = 0 \qquad (B)$$

<u>Thus</u>: ① $R^{\sigma}{}_{\sigma ij} = R_{ij}{}^{\sigma}{}_{\sigma} = 0$ by (B)

② $R^{\sigma}{}_{i\sigma j} = -R_{i}{}^{\sigma}{}_{\sigma j} = R_{i}{}^{\sigma}{}_{j\sigma} = -R^{\sigma}{}_{ij\sigma}$

③ Since $R_{\alpha\beta ij} = R_{ij\alpha\beta}$, ② is equiv. to raising the latter indices & contr. ✓

<u>Note</u>: $R_{ij} = R_{ji} \Rightarrow R^{i}{}_{j} = R_{j}{}^{i} \equiv R^{i}_{j}$ (Not $R^{i}{}_{j} = R^{j}{}_{i}$!)

I.e., $R_{ij} = R^{\sigma}{}_{i\sigma j} = g^{\tau\sigma} R_{\tau i \sigma j} = g^{\tau\sigma} R_{\sigma j \tau i} = R_{ji}$

<u>Scalar Curvature</u>: $R = R^{\sigma}{}_{\sigma}$

<u>Note</u>: Since there's only "one way" to contract $R^{i}{}_{jkl}$ once, $\exists$ essentially only one way to contract it twice $\Rightarrow$ R is quite natural.

Bianchi Identities :

Theorem : $R^i{}_{j[k\ell;m]} = 0 = R^i{}_{jk\ell;m} + R^i{}_{jmk;\ell} + R^i{}_{j\ell m;k}$

$\uparrow$ cyclically permute

Proof: in loc Lorentz frame,

$$\text{(B)} \quad \frac{1}{2} R^i{}_{jk\ell}(x)\, dx^\ell \wedge dx^k = d\left(\Gamma^i{}_{jk}(x)\, dx^k\right) + O(|x|^2)$$

This is true for each i and j ; i.e., treating $\left(\Gamma^i{}_j\right)_k dx^k$ as a covector. Since d is a 1st order operator, and $d^2 = 0$, we take d of both sides of (B) and obtain

$$d\left(R^i{}_{jk\ell}\, dx^\ell \wedge dx^k\right) = 0 + O(|x|)$$

$$\Leftrightarrow R^i{}_{jk\ell,m}\, dx^m \wedge dx^\ell \wedge dx^k = O(|x|)$$

Fact: $A_{ijk}\, dx^i \wedge dx^j \wedge dx^k = A_{[ijk]}\, dx^{|i} \wedge dx^j \wedge dx^{k|}$

cyclic perm

$|ijk|$

increasing order

$A_{ijk} = -A_{jik}$

(FIP)

$\Rightarrow \quad R^i{}_{j[kl,m]} = O(|x|)$.

Conclude: in loc Lorentz coord system

$$R^i{}_{j[kl,m]}(0) = 0.$$

~~Thus, in general,~~

$$\cancel{R^i{}_{j[kl;m]} = 0} \quad \cancel{(*)}$$

~~since this is a~~

<u>But</u> $A_{[ijk]} = 0$ is a coord indept property of a tensor: i.e.

$$A_{[\alpha\beta\gamma]} = A_{[ijk]} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^k}{\partial y^\gamma}$$

$$\Rightarrow A_{[\alpha\beta\gamma]} = 0 \text{ iff } A_{[ijk]} = 0$$

$\therefore$ $R^i{}_{jkl;m}$ is a tensor that agrees with $R^i{}_{jkl,m}$ in Loc Lorentz fr; and $R^i{}_{j[kl;m]} = 0$ in this frame $\Rightarrow$

$$\boxed{R^i{}_{j[kl;m]} = 0}$$

in every frame.

$\boxed{\text{Fill In :}}$

$\boxed{\mathbb{R}^4}$

$\Lambda^0 \equiv$ zero forms are functions

$$f \in \Lambda^0 : df = f_{,i}\, dx^i \in T^*M$$

$\Lambda^1 \equiv T^*M$

$$\omega \in \Lambda^1 : w = w_i\, dx^i$$

$$d\omega = \omega_{i,j}\, dx^j \wedge dx^i = -w_{i,j}\, dx^i \wedge dx^j$$

$$= -w_{i,j}(dx^i \otimes dx^j - dx^j \otimes dx^i)$$

$$= (w_{j,i} - w_{i,j})\, dx^i \otimes dx^j \in \Lambda^2 \subseteq \mathcal{J}^0_2(M)$$

$\Lambda^2 \subseteq \mathcal{J}^0_2(M)$

$$\omega \in \Lambda^2 : \omega = \omega_{ij}\, dx^i \wedge dx^j$$

Eg: $\omega \in \Lambda^2 \Leftrightarrow w = \bar{w}_{ij}\, dx^i \otimes dx^j$

st $\bar{w}_{ij} = -\bar{w}_{ji}$

Basis for Λ^2 is $\{dx^i \wedge dx^j : i < j\}$

$\dim \Lambda^2 = \binom{n}{2}$

So $$\omega = \underbrace{\tfrac{1}{2}\bar{w}_{ij}}_{w_{ij}} \underbrace{(dx^i \otimes dx^j - dx^j \otimes dx^i)}_{dx^i \wedge dx^j}$$

$$dw = w_{ij,k}\, dx^k \wedge dx^i \wedge dx^j$$

$$= w_{ij,k}\, dx^i \wedge dx^j \wedge dx^k$$ (pick up minus sign) under interchange)

$$= w_{ij,k} \{ dx^i \otimes dx^j \otimes dx^k - dx^i \otimes dx^k \otimes dx^j + dx^k \otimes dx^i \otimes dx^j - dx^k \otimes dx^j \otimes dx^i + dx^j \otimes dx^k \otimes dx^i - dx^j \otimes dx^i \otimes dx^k \}$$

$$= \{ w_{ij,k} - \underbrace{w_{ik,j} + w_{ki,j}}_{2w_{ki,j}} - w_{kj,i} + w_{jk,i} - w_{ji,k} \} \cdot dx^i \otimes dx^j \otimes dx^k$$

Since $w_{\alpha\beta} = -w_{\beta\alpha}$ (w started in Λ^2)

$$dw = 2w_{[ij,k]}\, dx^i \otimes dx^j \otimes dx^k \in \Lambda^3$$

<u>Thm</u>: $d^2 = 0$

<u>Cor</u>: If $w \in \Lambda^3$ & $w = d^2\alpha \in \Lambda^1$ then

$$\boxed{w_{[ij,k]} = 0}$$ ✓

Differential Forms (general)

$\Lambda^k \subseteq \mathcal{T}^0_k$ antisymmetric:

$$\omega_{\pi(i_1)\cdots\pi(i_k)} = (-1)^\pi \omega_{i_1\cdots i_k}$$

$$\omega = \omega_{i_1\cdots i_k} \underbrace{dx^{i_1}\otimes\cdots\otimes dx^{i_k}} \qquad i_1,\dots,i_k \in \{1,\dots,n\}$$

collect all permutations of the same k indices together to get basis

$$\underbrace{dx^{i_1}\wedge\cdots\wedge dx^{i_k}} = \sum (-1)^\pi dx^{\pi(i_1)}\otimes\cdots\otimes dx^{\pi(i_k)}$$

(= 0 if $i_j = i_k$ defn)

we only need the one with increasing $i_1 \cdots i_k$

$dx^{|i_1}\wedge\cdots\wedge dx^{i_k|}$ = one with $i_1 < i_2 < \cdots < i_k$

Then $\omega = \omega_{i_1\cdots i_k} dx^{|i_1}\wedge\cdots\wedge dx^{i_k|}$

- $\{dx^{i_1}\wedge\cdots\wedge dx^{i_k} : i_1 < \cdots < i_k \le n\}$ is a basis for Λ^k
- $dx^{i_1}\wedge\cdots\wedge dx^{i_k}[X_1 \cdots X_k] = \det_{i_1\cdots i_k}\left[\begin{matrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{matrix}\right]$

Also:

$$\omega = \frac{1}{k!}\, \omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

sum over all
indices, any order

Defn: $\alpha = \alpha_{|\lambda|}\, dx^{\lambda} \in \Lambda^{k}$, $\beta = \beta_{|\mu|}\, dx^{\mu} \in \Lambda^{\ell}$

(1) $\alpha \wedge \beta = \alpha_{|\lambda|}\beta_{|\mu|}\, dx^{\lambda} \wedge dx^{\mu}$

(2) $dx^{\lambda} \wedge dx^{\mu} \equiv dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_k} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_\ell}$

Thm: $\alpha \wedge \beta \in \Lambda^{k+\ell}$ is defined indept of coords

Defn: $d\alpha = \left(\alpha_{|\lambda|,k}\, dx^{k}\right) \wedge dx^{\lambda} \in \Lambda^{k+1}$

Thm: $d^2 = 0$ & $d\alpha = 0$ in star shaped domain
$\Rightarrow \alpha = d\omega$ some $\omega \in \Lambda^{k-1}$

We look for a 2-tensor G_{ij} constructable from R_{ij} and g_{ij} such that $(\operatorname{div} G)_j = G^{\sigma}_{j;\sigma} = 0$. We have

<u>Bianchi:</u> $$R^i_{j[kl;m]} = 0$$

$$\left\{ R^i_{jkl;m} + R^i_{jmk;l} + R^i_{jlm;k} \right\} = 0$$

contract i & m:

$$g^{jk} \left\{ R^{\sigma}_{jkl;\sigma} + R^{\sigma}_{j\sigma k;l} + R^{\sigma}_{jl\sigma;k} \right\} = 0$$

contract in $j\,k$:

The covariant deriv. of g is zero $\Rightarrow$

$$\text{B)} \quad \left(g^{jk} R^{\sigma}_{jkl}\right)_{;\sigma} + \left(g^{jk} R^{\sigma}_{j\sigma k}\right)_{;l} + \left(g^{jk} R^{\sigma}_{jl\sigma}\right)_{;k} = 0$$

$$g^{jk} R^{\sigma}_{jkl} = g^{jk} g^{\sigma\tau} R_{\tau jkl} = -g^{\sigma\tau} g^{jk} R_{j\tau kl}$$

$$= -g^{\sigma\tau} R_{\tau l} = -R^{\sigma}_{l}$$

$$g^{jk} R^{\sigma}{}_{j\sigma k} = g^{jk} R_{jk} = R$$ 15 43

$$g^{jk} R^{\sigma}{}_{jl\sigma} = - g^{jk} R^{\sigma}{}_{j\sigma l} = - g^{jk} R_{jl} = - R^{k}{}_{l}$$

so (B) becomes:

$$- R^{\sigma}{}_{l;\sigma} + R_{;l} - R^{k}{}_{l;k} = 0$$

$$\Leftrightarrow \quad R^{\sigma}{}_{l;\sigma} - \frac{1}{2} R_{;l} = 0$$

$$\Leftrightarrow \quad R^{l\sigma}{}_{;\sigma} - \frac{1}{2}\left(g^{l\sigma} R\right)_{;l} = 0$$

$$\Leftrightarrow \quad \left(R^{l\sigma} - \frac{1}{2} g^{l\sigma} R\right)_{;l} = 0$$

$G^{ij} = R^{ij} - \frac{1}{2} g^{ij} R$ satisfies $\operatorname{div} G = 0$

$$\boxed{G = R - \frac{1}{2} g \underset{\sim}{R}}$$

Theorem: $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$ is the only nontrivial 2-tensor constructable from R^{σ}_{ijk}, using raising & lowerings & contractions such that $\operatorname{div} G = 0$ as a consequence of the Bianci Identities.

III Einstein Equation $G = \kappa T$ (E)

$T \equiv$ stress energy tensor $\begin{bmatrix} e & e\text{-flux} \\ \underset{\sim}{p} & \underset{\sim}{p}\text{-flux} \end{bmatrix}_{ij}$

$G \equiv$ Einstein Curvature tensor

$\operatorname{div} G = 0 \Rightarrow$ soln's of (E) satisfy

$\operatorname{div} T = 0 \approx\!> $ conservation (local) of energy and momentum.

17

45

- In empty space: $T=0 \Rightarrow G=0$

<u>Claim</u>: $G_{ij}=0$ iff $R_{ij}=0$

<u>pf</u>:

$$G_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij}$$

$$G^{\sigma}{}_{\sigma} = R^{\sigma}{}_{\sigma} - \tfrac{1}{2} R \underbrace{g^{\sigma}{}_{\sigma}}_{id^{\sigma}{}_{\sigma}=4} = - R$$

$\therefore$ $G_{ij}=0 \Rightarrow G^{\sigma}{}_{\sigma}=0 \Rightarrow R^{\sigma}{}_{\sigma}=0=R \Rightarrow G_{ij}=R_{ij}=0$

$R_{ij}=0 \Rightarrow G_{ij}=0$ (easy) ✓ (Homework)

<u>Conclude</u>: $R_{ij}=0$ empty space field eqn's

Replaces: $\Delta \Phi = 0$ Φ the gravitational potential

$\nabla \Phi = - \frac{\text{force}}{\text{mass}}$ in Newton Theory.