Schwarzschild Solution

Thm (S) (Schwart 1916) The following metric is a spherically symmetric soln of empty space Einstein equations $G_{ij}=0 \Leftrightarrow R_{ij}=0$

$$(S) \qquad ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)d(ct)^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Thm (Birkoff 1923) (S) is the only spherically symmetric soln of $G_{ij}=0$ in the sense that any other soln of form

$$ds^2 = -A(x^0;r)\,dt^2 + D(x^0;r)\,dx^0 dr + B(x^0;r)\,dr^2 + C(x^0;r)\,d\Omega^2$$

is equivalent under coordinate transformation to (S).

Note: (S) has a singularity at $r=2GM$ and at $r=0$. The one at $r=0$ is essential ($R\to\infty$ as $r\to 0$) but the singularity at $r=2GM$ is a coordinate singularity.

Note: in limit $r \to \infty$, (S) approaches

$$ds^2 = -d(x^0)^2 + dr^2 + r^2 d\Omega^2,$$

the flat metric in spherical coordinates

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

the metric on unit sphere.

In the Newtonian limit,

$$\phi = -\frac{2GM}{r}$$

$\Rightarrow$ $M \equiv$ mass of sun.

$G \equiv$ Newt grav. const that enters the law

$$F = \nabla\phi = -G \frac{M_s M_e}{r^2} \frac{\vec{r}}{|\vec{r}|}$$

Check: $[\varphi] = \frac{L^2}{T^2} \Rightarrow \left[\frac{\phi}{c^2}\right] = 1$ as required in g_{00}.

Set $m = \frac{GM}{c^2}$

<u>Thm</u>: The invariants $R = R^{\sigma}{}_{\sigma}$ and $R_{ijkl}R^{ijkl}$ are continuous across the Schwarzschild radius. In fact

$$R_{ijkl}R^{ijkl} = 48M^2/r^6 \qquad \text{(FIP)}$$

ref M-T-W pg 822

<u>Thm</u>(K): (Kruskal – Szekeres) $\exists$ a coordinate transformation $(x^0, r) \mapsto (u, v)$ such that in uv-coordinates (S) takes the form

$$\text{(K)} \qquad ds^2 = -f^2(u,v)(dv^2 - du^2) + r^2 d\Omega^2$$

In fact,

$$f^2(u,v) = \frac{32M^3}{r} e^{-r/2M} \qquad m = \frac{GM}{c^2}$$

and the coordinate transformation is given by

$$\left.\begin{aligned} u^2 - v^2 &= \left(\frac{r}{2m} - 1\right) e^{r/2m} \\ \frac{v}{u} &= \tanh \frac{x^0}{4m} \end{aligned}\right\} \quad r > 2m \qquad u > |v|$$

$$\left.\begin{aligned} v^2 - u^2 &= \left(1 - \frac{r}{2m}\right) e^{r/2m} \\ \frac{u}{v} &= \tanh \frac{x^0}{4m} \end{aligned}\right\} \quad r < 2m \qquad v > |u|$$

Since: $r = 2m \Leftrightarrow u = v$, u & v are smooth across Schwart radius, & (K) smooth everywhere except at $r = 0$, which corresponds to $v^2 - u^2 = +1$.

Note: singularity $r=0$ is spread out over hyperbola

Note: metric extends smoothly to $u < |v|$ and $v < |u|$ $\Rightarrow$ universe of (S) can be extended to a "mirror universe" not seen in (S)

We discuss Thm (S) & Thm (K):

- Derivation of (S):

Look for soln $G=0 \Leftrightarrow R=0$ of form

$$ds^2 = -A(r)\,d(x^0)^2 + B(r)\,dr^2 + C(r)\,d\Omega^2.$$

"The general static spherically symmetric metric"

- WLOG assume $C(r) = r^2 > 0$ (o.w. not sgn η_{ij})

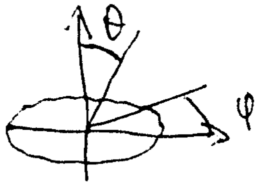
i.e., $r = \psi(\bar{r}) \Rightarrow$

$$ds^2 = -\bar{A}(\bar{r})\,d(x^0)^2 + \bar{B}(\bar{r})\,d\bar{r}^2 + C(\psi(\bar{r}))\,d\Omega^2$$

so choose $\psi(\bar{r})$ st $C \circ \psi = \bar{r}^2$ ✓

This ensures that the 2-sphere at radius r has circumference $2\pi r$

$$2\pi r = \int_0^{2\pi} \sqrt{r^2\,d\Omega^2} = \int_0^{2\pi} r\,d\varphi \quad \leftarrow \text{ at } \theta = \frac{\pi}{2}$$



- Assume $A(r) = e^{\nu(r)}$, $B(r) = e^{\lambda(r)}$ $\Rightarrow$

$$-e^{\nu(r)} d(x^0)^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2$$

- The E-equations are $\Leftrightarrow$ $R_{ij} = 0$:

$$R_{ij} = R^{\sigma}_{i\sigma j} = \Gamma^{\sigma}_{ij,\sigma} - \Gamma^{\sigma}_{i\sigma,j} + \Gamma^{\sigma}_{\tau\sigma}\Gamma^{\tau}_{ij} - \Gamma^{\sigma}_{\tau j}\Gamma^{\tau}_{i\sigma} = 0$$

<u>But</u>: $\Gamma^{\sigma}_{\sigma k} = \{\log\sqrt{-g}\}_{,k}$ from before $\Rightarrow$

$$-g = e^{\nu+\lambda} r^4 \sin^2\theta$$

$$\log\sqrt{-g} = \frac{1}{2}(\nu+\lambda) + 2\log r + \log|\sin\theta|$$

$$\Rightarrow \boxed{\begin{aligned} \Gamma^{\sigma}_{\sigma 0} &= 0 \\ \Gamma^{\sigma}_{\sigma 1} &= \frac{1}{2}(\nu' + \lambda') + \frac{2}{r} \\ \Gamma^{\sigma}_{\sigma 2} &= \frac{\cos\theta}{\sin\theta} = \cot\theta \\ \Gamma^{\sigma}_{\sigma 3} &= 0 \end{aligned}}$$

Thus we can rewrite the field equations: (6B)

$$-R_{ij} = 0 = (\log\sqrt{-g})_{,ij} - \Gamma^{\sigma}_{ij,\sigma} + \Gamma^{\sigma}_{\tau j}\Gamma^{\tau}_{i\sigma} - \Gamma^{\tau}_{ij}(\log\sqrt{-g})_{,\tau}$$

(R_{ij})

- To obtain the Γ^i_{jk} explicitly, we work backwards from the variational formulation:

$$\delta S = \delta \int g_{ij} \dot{x}^i \dot{x}^j \, ds = 0$$

$$\Leftrightarrow \quad \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

But $\delta S = 0 \Leftrightarrow \dfrac{d}{ds}\dfrac{\partial L}{\partial \dot{x}^i} - \dfrac{\partial L}{\partial x^i} = 0$, $L = g_{ij}\dot{x}^i \dot{x}^j$.

In case of (S),

$$L = -e^{\nu}(\dot{x}^0)^2 + e^{\lambda}(\dot{x}^1)^2 + (x^1)^2\{(\dot{x}^2)^2 + \sin^2 x^2 (\dot{x}^3)^2\}$$

$$\Leftrightarrow L = -e^{\nu}\dot{t}^2 + e^{\lambda}\dot{r}^2 + r^2\{\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\}$$

$\boxed{i=0}$ $$\frac{d}{ds}\frac{\partial L}{\partial \dot{x}^0} = \frac{d}{ds}\{-e^{\nu} 2\dot{t}\} = -\cancel{2}\nu' e^{\nu}\dot{r}\dot{t} - \cancel{2}\cancel{e^{\nu}}\ddot{t} = 0 = \frac{\partial L}{\partial x^0}$$

$$\Leftrightarrow \quad \ddot{t} + \nu' \dot{r}\dot{t} = 0$$

$$\Leftrightarrow \boxed{\Gamma^0_{01} = \tfrac{1}{2}\nu' = \Gamma^0_{10}}, \quad \Gamma^0_{ij} = 0 \quad \text{o.w.}$$

$\boxed{i=1}$

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{x}^1} = \frac{d}{ds}\frac{\partial L}{\partial \dot{r}} = \frac{d}{ds}\left\{2e^{\lambda}\dot{r}\right\}$$

$$= 2\lambda' e^{\lambda}\dot{r}^2 + 2e^{\lambda}\ddot{r}$$

$$\frac{\partial L}{\partial x^1} = \frac{\partial L}{\partial r} = -\nu' e^{\nu}\dot{t}^2 + \lambda' e^{\lambda}\dot{r}^2 + 2r\left\{\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right\}$$

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{x}^1} - \frac{\partial L}{\partial x^1} = \ddot{r} + \frac{1}{2}\lambda'\dot{r}^2 + \frac{1}{2}\nu' e^{\nu-\lambda}\dot{t}^2 - e^{-\lambda} r\dot{\theta}^2$$

$$- r\sin^2\theta\,\dot{\phi}^2 e^{-\lambda} = 0$$

$$\boxed{\begin{aligned} \Gamma^1_{00} &= \frac{1}{2}\nu' e^{\nu-\lambda} \\ \Gamma^1_{11} &= \frac{1}{2}\lambda' \\ \Gamma^1_{22} &= -e^{-\lambda} r \\ \Gamma^1_{33} &= -e^{-\lambda} r\sin^2\theta \end{aligned}}$$

$\Gamma^1_{ij} = 0$ o.w.

$\boxed{i=2}$ $$\frac{d}{ds}\frac{\partial L}{\partial \dot{x}^2} = \frac{d}{ds}\left\{2r^2\dot{\theta}\right\} = 4r\dot{r}\dot{\theta} + 2r^2\ddot{\theta}$$

$$\frac{\partial L}{\partial x^2} = \frac{\partial L}{\partial \theta} = 2r^2 \sin\theta \cos\theta \, \dot{\varphi}^2$$

$$0 = \frac{d}{ds}\frac{\partial L}{\partial \dot{x}^2} - \frac{\partial L}{\partial x^2} = \cancel{2}r^2\ddot{\theta} + \overset{2}{\cancel{4}}r\dot{r}\dot{\theta} - \cancel{2}r^2 \sin\theta\cos\theta\,\dot{\varphi}^2$$

$$\Leftrightarrow \quad \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = 0$$

$$\boxed{\begin{array}{l} \Gamma^2_{12} = \frac{1}{r} = \Gamma^2_{21} \\ \Gamma^2_{33} = -\sin\theta\cos\theta \end{array}} \qquad \Gamma^2_{ij} = 0 \quad \text{o.w.}$$

$\boxed{i=3}$ $$\frac{d}{ds}\frac{\partial L}{\partial \dot{\varphi}^2} = \frac{d}{ds}\left\{2r^2\sin^2\theta\,\dot{\varphi}\right\} = 4r\sin^2\theta\,\dot{r}\dot{\varphi} + 4r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\varphi} + 2r^2\sin^2\theta\,\ddot{\varphi}$$

$$\frac{\partial L}{\partial \varphi} = 0$$

$$0 = \frac{d}{ds}\frac{\partial L}{\partial \dot{\varphi}^2} - \frac{\partial L}{\partial \varphi} \Leftrightarrow \quad \ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi} = 0$$

$$\boxed{\begin{array}{l} \Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31} \\ \Gamma^3_{23} = \cot\theta = \Gamma^3_{32} \end{array}} \qquad \Gamma^3_{ij} = 0 \quad \text{o.w.}$$

- Thus from (R_{ij}):

$$-R_{00} = 0 = \cancel{\{\log\sqrt{-g}\}_{,00}} - \Gamma^{\sigma}_{00,\sigma} + \Gamma^{\sigma}_{\tau 0}\Gamma^{\tau}_{0\sigma} - \Gamma^{\tau}_{00}(\log\sqrt{-g})_{,\tau}$$

0

$$= -\Gamma^{1}_{00,1} + 2\Gamma^{1}_{00}\Gamma^{0}_{01} - \Gamma^{1}_{00}\log(\sqrt{-g})_{,1}$$

$$= \left\{-\tfrac{1}{2}\nu' e^{\nu-\lambda}\right\}' + \tfrac{1}{2}\nu'\nu' e^{\nu-\lambda} - \left(\tfrac{1}{2}\nu' e^{\nu-\lambda}\right)\cdot$$

$$\left(\tfrac{1}{2}(\nu'+\lambda') + \tfrac{2}{r}\right)$$

$$\Rightarrow \quad \boxed{\nu'' + \tfrac{1}{2}(\nu')^2 - \tfrac{1}{2}\lambda'\nu' + \frac{2\nu'}{r} = 0} \qquad (0,0)$$

<u>Similarly</u>:

$$-R_{11} = \tfrac{1}{2}\left(\nu'' + \tfrac{1}{2}(\nu')^2 - \tfrac{1}{2}\lambda'\nu' - \frac{2\lambda'}{r}\right)$$

$$\boxed{\nu'' + \tfrac{1}{2}(\nu')^2 - \tfrac{1}{2}\lambda'\nu' - \frac{2\lambda'}{r} = 0} \qquad (1,1)$$

$$\boxed{-R_{22} = \frac{\partial^2}{\partial\theta^2}(\log|\sin\theta|) + (e^{-\lambda} r)' + 2(-e^{-\lambda}) + \cot^2\theta + e^{-\lambda} r\left(\frac{\lambda'+\nu'}{2} + \frac{2}{r}\right) = 0} \quad (22)$$

- we show that the 3-equations (0,0) (1,1) and (2,2) are consistent & determine the soln to within a constant M. One can then check (FIP) that all other equations $R_{ij}=0$ are identically satisfied on this solution:

(In fact, $R_{ij} \equiv 0$ $i \neq j$, $R_{33} = \sin^2\theta\, R_{22}$)

<u>Note</u>: $\frac{\partial^2}{\partial\theta^2}(\log|\sin\theta|) = \frac{\partial}{\partial\theta}\cot\theta = -\csc^2\theta$

$$-\csc^2\theta + \cot^2\theta = -1$$

$\Rightarrow$ (2,2) is equiv to

$$\boxed{0 = -1 + (e^{-\lambda} r)' - 2e^{-\lambda} + e^{-\lambda} r\left(\frac{\lambda'+\nu'}{2} + \frac{2}{r}\right)} \quad (2,2)$$

Subtract (1,1) from (0,0):

$$\frac{2\nu'}{r} + \frac{2\lambda'}{r} = 0$$

$$\nu' + \lambda' = 0$$

$$(\nu + \lambda) = \text{const.} = C. \qquad (1)$$

Note: wlog, we can take $C = 0$ because a change of coords $\bar{t} = \alpha t$ takes metric (S) to

$$ds^2 = -\underbrace{\alpha^2 e^{\nu}}\, d\bar{t}^2 + e^{\lambda} dr^2 + r^2 d\Omega^2$$

$$= -e^{\nu + \beta} d\bar{t}^2 + e^{\lambda} dr^2 + r^2 d\Omega^2$$

$\Rightarrow$ if $\beta = c$, in $\bar{t}$-coords $\nu + \lambda = 0$.

Thus: If $\exists$ a soln satis (1) with C, then it is $\equiv$ to a soln satis (1) with $C = 0$ ✓

Thus

$$\nu = -\lambda . \qquad (2)$$

Substituting into (1,1) gives

$$\boxed{\cancel{\nu'' + \tfrac{1}{2}(\nu')^2 - \tfrac{1}{2}\lambda'\nu' + \tfrac{2\nu'}{r}}}$$

$$\lambda'' - \tfrac{1}{2}(\lambda')^2 - \tfrac{1}{2}(\lambda')^2 + \frac{2\lambda'}{r} = 0$$

$$\lambda'' - (\lambda')^2 + \frac{2\lambda'}{r} = 0$$

$$\Leftrightarrow \qquad (re^{-\lambda})'' = 0$$

$$\Rightarrow \qquad (re^{-\lambda})' = C \equiv \text{constant}. \qquad (3)$$

Thus: (0,0) and (1,1), two <u>second order</u> equations in $\nu(r)$ & $\lambda(r)$ $\Rightarrow$ determines ν & λ to within 4 arb. consts <u>in general</u>. Then $c = 0$ in (2) reduces these to <u>two</u> in (3).

We show that (2,2) reduces these two to <u>one</u> !!

- (2,2) reads:

$$0 = -1 + \left(e^{-\lambda} r\right)' - 2e^{-\lambda} + r e^{-\lambda}\left(\frac{\underbrace{\cancel{\lambda' + \nu'}}_{0}}{2} + \frac{2}{r}\right)$$

$\Longleftrightarrow$

$$1 = \left(e^{-\lambda} r\right)' - \cancel{2e^{-\lambda}} + \cancel{2e^{-\lambda}}$$

$\therefore$ $c = 1$ in (3) !

<u>Thus</u>:

$$e^{-\lambda} r = r - 2m$$

$$e^{-\lambda} = 1 - \frac{2m}{r}$$

$$\boxed{\begin{aligned} e^{\lambda} &= \left(1 - \frac{2m}{r}\right)^{-1} \\ e^{\nu} = e^{-\lambda} &= \left(1 - \frac{2m}{r}\right) \end{aligned}} \qquad (4)$$

$$\Rightarrow \quad ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 .$$

Claim all other equations in $R_{ij}=0$ are identically satisfied when e^{λ} e^{ν} are given by (4) (a miracle) (FIP)

Actually: $R_{ij}\equiv 0$ $i\neq j$ and $R_{33}=R_{22}\sin^2\theta$

- Ex: we see what happened to the 4 free constants of integration in the ~~rel~~ two 2nd order equations (0.0) & (1,1) for (ν,λ):

① $(0,0)+(1,1) \Rightarrow \lambda'+\nu'=C \Rightarrow (0,0)$ & $(1,1)$ are equivalent to a degenerate system:

$$\lambda'+\nu'=C \qquad \text{1st order}$$

$$(1,1) \qquad \text{2nd order}$$

$\rightsquigarrow$ expect 3 constants

② guage freedom in t allows us to set $C=0$

$\rightsquigarrow$ 2 remaining constants

~~(2,2) is consistent~~

3 (2,2) is consistent with (0,0) & (1,1) (why ??) but restricts the two free constants to a single constant m.

<u>Problem</u>: What general principles are working behind this to make all the equations consistent, ~~and~~ with the interesting reduction of free parameters?

(I have seen no satisfying answer to this)

The Schwarzschild Singularity

(S) $$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

- singularity at $r = 2M$
- $R \equiv 0 \Rightarrow R$ is cont through the singularity
- Calculation: $R_{ijkl} R^{ijkl} = \frac{48 M^2}{r^6}$

continuous thru $r = 2M$.

- note: $r > 2M \Rightarrow t$ timelike ~~positively~~

  $r < 2M \Rightarrow r$ timelike

But: assuming $\frac{\partial}{\partial t}$ is positively oriented at $r = \infty$, by cont. we know $\frac{\partial}{\partial t}$ pos oriented for $r > \frac{2M}{r}$.

Q: how do we know whether $\frac{\partial}{\partial r}$ or $-\frac{\partial}{\partial r}$ is pos oriented for $r < 2M$ when metric is not cont. thru $r = 2M$?

- Red Shift: $\dfrac{\lambda_2}{\lambda_1} = \dfrac{\sqrt{1-\frac{2M}{r_2}}}{\sqrt{1-\frac{2M}{r_1}}}$

$r_2 = \infty \Rightarrow \lambda_2 = \left(1-\frac{2M}{r_1}\right)^{-1/2} \lambda_1$

$\Rightarrow \lambda_2 \to \infty$ as r_1 (emitter) $\to 2M$

$\Rightarrow \infty$-red-shift

- Geodesics: $ds^2 = -\left(1-\frac{2M}{r}\right)c^2 dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$

Consider geodesic $(t(s), r(s), \theta(s), \phi(s))$:

Divide (s) by $ds^2 \Rightarrow$

$$1 = -\left(1-\frac{2M}{r}\right)c^2 \dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right) \quad (A)$$

Obtain the 3 other equations for geodesic from E-L equation directly:

$$0 = \delta \int \left\{ -\left(1-\frac{2M}{r}\right) c^2 \dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2\left[\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2\right] \right. ds$$

$\Rightarrow$ 3 E-L equations $\dfrac{d}{ds}\dfrac{\partial L}{\partial \dot{x}^i} - \dfrac{\partial L}{\partial x^i} = 0 \quad i = 0, 2, 3$

$$\frac{d}{ds}\left(r^2 \dot{\theta}\right) = r^2 \sin\theta \cos\theta\, \dot{\varphi}^2 \tag{B}$$

$$\frac{d}{ds}\left(r^2 \sin^2\theta\, \dot{\varphi}\right) = 0 \tag{C}$$

$$\frac{d}{ds}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right] = 0 \tag{D}$$

<u>Note</u>: (A) – (D) give 4 2nd order equn's for (t, r, θ, φ) along geodesic.

If initially $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, then (B) is identically satisfied by $\theta = \frac{\pi}{2}$, & (A) – (D) reduces to 3 equn's in (t, r, φ):

$$1 = -\left(1-\frac{2m}{r}\right)c^2\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 \qquad (A')$$

$$\frac{d}{ds}\left(r^2\dot{\varphi}\right) = 0 \qquad (C')$$

$$\frac{d}{ds}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right] = 0 \qquad (D')$$

Then (C') gives

$$2\frac{dA}{dt} = r^2\dot{\varphi} = h \equiv \text{const} \qquad (E)$$

$\Leftrightarrow$ "equal area in equal time"

Also, (D') becomes

$$\dot{t} = \left(1-\frac{2M}{r}\right)^{-1} L \qquad L \equiv \text{const.} \qquad (F)$$

<u>Note</u>: substituting (E), (F) into (A') gives a 1st order ODE in r:

$$1 = -L^2c^2\left(1-\frac{2M}{r}\right)^{-1} + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + \frac{h}{r^2}$$

$\Leftrightarrow$

$$\dot{r}^2 = \left(1-\frac{2M}{r}+L^2c^2\right) - \frac{h}{r^2}\left(1-\frac{2M}{r}\right)$$

Thus the equations for r & t become:

$$\dot{r} = \pm\sqrt{\left(L^2c^2+1-\frac{2M}{r}\right) - \frac{h}{r^2}\left(1-\frac{2M}{r}\right)} \qquad (G)$$

$$\dot{t} = \left(1-\frac{2M}{r}\right)^{-1} L \qquad (H)$$

<u>Note</u>: choosing the minus sign in (G), it is clear that as $r \to 2M+$, (G) $\sim \dot{r} = -L^2c^2$ and (H) $\sim \dot{t} = \infty \Rightarrow$ the r equation is regular, but in the t-equn the, ~~value but~~ $t \to \infty$ as $r \to 2M$.

<u>Note</u>: letting $r = \frac{1}{u}$, and obtaining the equation for $r = r(\varphi)$ through the replacement of s by φ as indep't variable using (E)

$$\frac{d\varphi}{ds} = \frac{h}{r^2},$$

leads to the <u>exact</u> <u>equation</u>:

$$u'' + u = \frac{m}{h^2} + 3mu^2 \qquad m = \frac{G_0 M}{c^2}$$

for the geodesic. The corresponding $u = u(\varphi)$ in Newton's theory is (Binet Equn)

$$u'' + u = \frac{G_0 M}{H^2} \approx \frac{m}{h^2} \quad \text{in classical limit}$$

$\Rightarrow$ <u>Same</u> except for nonlinear quadratic term $3mu^2$. <u>Ref</u>: Te/Tracy Monthly

A-B-S