Schwarzschild Solution

Thm (Schwarzschild 1916) The following metric is a spherically symmetric soln of empty space Einstein equations: $G_{ij} = 0 \iff R_{ij} = 0$

\[ ds^2 = -(1 - \frac{2GM}{c^2 r})c^2 (ct)^2 + (1 - \frac{2GM}{r}) dr^2 + r^2 d\Omega^2 \]

Thm (Birkhoff 1923) (S) is the only spherically symmetric soln of $G_{ij} = 0$ in the sense that any other soln of form

\[ ds^2 = A(x^0, r) c^2 dt^2 + D(x^0, r) dx^0 dr + B(x^0, r) dr^2 + C(x^0, r) d\Omega^2 \]

is equivalent under coordinate transformation to (S).

Note: (S) has a singularity at $r = 2GM$ and at $r = 0$. The one at $r = 0$ is essential ($R \to \infty$ as $r \to 0$) but the singularity at $r = 2GM$ is a coordinate singularity.
Note: in limit \( r \to \infty \), (s) approaches
\[
d s^2 = -c^2 \left( d\sigma^2 + d\rho^2 + r^2 d\omega^2 \right)
\]
the flat metric in spherical coordinates.
\[
d s^2 = d\theta^2 + \sin^2 \theta d\phi^2
\]
the metric on unit sphere.

In the Newtonian limit,
\[
\frac{\Phi}{\nu} = -\frac{2GM}{\nu}
\]

\( M \equiv \text{mass of sun} \)

\( G \equiv \text{Newt grav. const that enter the law} \)

\[
F = -\nabla \Phi = -G \frac{M_s M_e}{r^2} \frac{\hat{r}}{r^2}
\]

Check: \[ [\Phi] = \frac{L^2}{T^2} \Rightarrow \left[ \frac{\Phi}{c^2} \right] = 1 \] as required in \( g_{00} \).

Set \( m = \frac{GM}{c^2} \)
Thm.: The invariants $R = R_0$ and $R_{	ext{Schw}}$ are continuous across the Schwarzschild radius. In fact

$$R_{	ext{Schw}}^2 = 4\pi M^2/r$$

(Eq)

Ref. MT-W, pg 822

Thm(b): (Kruskal – Szekeres) Exists a coordinate transformation $(x^0, r) \mapsto (u, v)$ such that in uv-coordinates (5) takes the form

$$ds^2 = f^2(u, v)(dv^2 - du^2) + r^2 d\Omega^2$$

(K)

In fact,

$$f^2(u, v) = \frac{32 M^3}{r} e^{-r/2M}, \quad m = \frac{6M}{c^2},$$

and the coordinate transformation is given
by

\[ u^2 - v^2 = \left( \frac{r}{2m} - 1 \right) e^{r/2m} \]

\[ v = \tanh \frac{x^0}{4m} \]

\[ v^2 - u^2 = \left( 1 - \frac{r}{2m} \right) e^{r/2m} \]

\[ \frac{u}{v} = \tanh \frac{x^0}{4m} \]

\[ r > 2m \quad v > |u| \]

\[ r < 2m \quad v > |u| \]

Since: \( r = 2m \iff u = v \) \( , \) \( u \& v \) are smooth across Schwartz radius \( ) \( & \) \( ) \) smooth everywhere except at \( r = 0 \)\( ) \), which corresponds to \( v^2 - u^2 = +1 \).

Note: singularity \( r = 0 \) is spread out over hyperbola

Note: metric extends smoothly to \( v < |u| \) and \( v < |u| \Rightarrow \) universe of \( (s) \) can be extended to a "mirror universe" not seen in \( (s) \)
We discuss Theorem (3) and Theorem (K):

- **Derivation of (3):**

  Look for solution $G=0 \iff R=0$ of form
  
  $$ds^2 = -A(r) \, d(x^0)^2 + B(r) \, dr^2 + C(r) \, d\Omega^2.$$  

  "The general static spherically symmetric metric."

- WLOG assume $C(r) = r^2 > 0$ (or not $\text{sgn} I_{ij}$)

  i.e., $r = \Psi(\tilde{r}) \Rightarrow$

  $$ds^2 = -A(\tilde{r}) \, d(x^0)^2 + B(\tilde{r}) \, d\tilde{r}^2 + C(\Psi(\tilde{r})) \, d\Omega^2$$

  so choose $\Psi(\tilde{r})$ s.t. $C \circ \Psi = \tilde{r}^2$.

  This ensures that the 2-sphere at radius $r$ has circumference: $2\pi r$

  $$2\pi r = \int_0^{2\pi} \sqrt{r^2 + \omega^2} \, d\theta = \int_0^{2\pi} r \, d\theta \iff \theta = \frac{\pi}{2}$$
• Assume \( A(r) = e^\nu(r) \), \( B(r) = e^\lambda(r) \) \Rightarrow

\[-e^{\nu(r)} \, d(x_0)^2 + e^{\lambda(r)} \, dr^2 + r^2 \, d\Omega^2\]

• The E-equations are \( \Rightarrow \) \( R_{ij} = 0 \):

\[R_{ij} = R_{i0j} = \Gamma^\sigma_{i0} - \Gamma^\sigma_{0i0} + \Gamma^\sigma_{0i} \Gamma^r_{\sigma r} - \Gamma^r_{\sigma i} \Gamma^r_{i0} = 0\]

But: \( \Gamma^\sigma_{0k} = \{ \log \sqrt{-g} \} \), \( \kappa \) from before \( \Rightarrow \)

\[-g = e^{\nu + \lambda} \, r^4 \sin^2 \Theta\]

\[\log \sqrt{-g} = \frac{1}{2} (\nu + \lambda) + 2 \log r + \log |\sin \Theta|\]

\[\begin{array}{c}
\Gamma^b_{00} = 0 \\
\Gamma^b_{01} = \frac{1}{2} (\nu' + \lambda') + \frac{2}{r} \\
\Gamma^b_{02} = \frac{\cos \Theta}{\sin \Theta} = \cot \Theta \\
\Gamma^b_{03} = 0 \\
\Gamma^b_{11} = \frac{1}{2} (\nu' + \lambda') + \frac{2}{r} \\
\Gamma^b_{12} = \frac{\cos \Theta}{\sin \Theta} = \cot \Theta \\
\Gamma^b_{13} = 0 \\
\Gamma^b_{22} = \frac{1}{2} (\nu' + \lambda') + \frac{2}{r} \\
\Gamma^b_{23} = 0 \\
\Gamma^b_{33} = \frac{1}{2} (\nu' + \lambda') + \frac{2}{r} \end{array}\]
Thus we can rewrite the field equations:

\[-R_{ij} = 0 = (\log \sqrt{g})_{ij} - \Gamma^\sigma_{ij,\sigma} + \Gamma^\tau_{ij} \Gamma^\tau_{i,\nu} - \Gamma^\tau_{ij} (\log \sqrt{g})_\tau\]

\((R_{ii})\)
To obtain the $\Gamma^i_{jk}$ explicitly, we work backwards from the variational formulation:

$$\delta S = \delta \int g_{ij} \dot{x}^i \dot{x}^j \, ds = 0$$

$$\Rightarrow \dot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$$

But $\delta S = 0 \Rightarrow \frac{d}{ds} \frac{\delta L}{\delta \dot{x}^i} - \frac{\delta L}{\delta x^i} = 0$, $L = g_{ij} \dot{x}^i \dot{x}^j$.

In case of (8),

$$L = -e^\nu (\dot{x}^0)^2 + e^\lambda (\dot{x}^1)^2 + (\dot{x}^1)^2 \{ (\dot{x}^2)^2 + \sin^2 x^2 (\dot{x}^3)^2 \}$$

$$\Rightarrow L = -e^\nu \dot{t}^2 + e^\lambda \dot{v}^2 + r^2 \{ \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \}$$

$$\dot{\nu} = 0 \Rightarrow \frac{d}{ds} \frac{\delta L}{\delta \dot{x}^0} = \frac{d}{ds} \{-e^\nu 2 \dot{t} \} = -2e^\nu \dot{e} \dot{v} + e^\nu \dot{t} - 2e^\nu \dot{t} = 0$$

$$\Rightarrow \dot{t} + \nu \dot{t} = 0$$

$$\Rightarrow \Gamma^0_{01} = \frac{1}{2} \nu \dot{t} = \Gamma^0_{10}, \quad \Gamma^0_{ij} = 0 \quad \text{otherwise.}$$
\[ \dot{\gamma} \quad i = 1 \]

\[ \frac{d}{ds} \frac{\partial L}{\partial \dot{\alpha}^{i}} - \frac{\partial L}{\partial \alpha^{i}} = \frac{d}{ds} \{ 2 e^{\lambda} \dot{r} \} \]

\[ = 2 \lambda' e^{\lambda} \dot{r}^2 + 2 e^{\lambda} \ddot{r} \]

\[ \frac{\partial L}{\partial x^1} = \frac{\partial L}{\partial r} = -v' \dot{v} \dot{t}^2 + \lambda' e^{\lambda} \ddot{r}^2 + 2 r \{ \dot{\phi}^2 + \sin^2 \theta \phi \ddot{\phi} \} \]

\[ \frac{d}{ds} \frac{\partial L}{\partial \dot{\alpha}^{1}} - \frac{\partial L}{\partial \alpha^{1}} = \ddot{r} + \frac{1}{2} \lambda' \ddot{r}^2 + \frac{1}{2} v' e^{\lambda - \lambda} \ddot{t}^2 - e^{\lambda} r \dot{\phi}^2 \]

\[ -r \sin^2 \theta \dot{\phi}^2 e^{\lambda} = 0 \]

\[ \Gamma_{00} = \frac{1}{2} v' e^{v - \lambda} \]

\[ \Gamma_{11} = \frac{1}{2} \lambda' \]

\[ \Gamma_{22} = -\dot{e}^\lambda \dot{r} \]

\[ \Gamma_{33} = -\dot{e}^\lambda r \sin^2 \theta \]

\[ \Gamma_{ij} = 0 \quad \forall \omega. \]
\[ i=2 \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{x}^i} = \frac{d}{ds} \left\{ 2 r^2 \dot{\theta}^2 \right\} = 4 r \ddot{\theta} + 2 r^2 \dddot{\theta} \]

\[ \frac{\partial L}{\partial x} = \frac{\partial L}{\partial \dot{x}} = 2 r^2 \sin \theta \cos \theta \dot{\phi}^2 \]

\[ 0 = \frac{d}{ds} \frac{\partial L}{\partial \ddot{x}^i} - \frac{\partial L}{\partial x} \]

\[ \implies \quad \ddot{\theta} + \frac{2}{r} \dddot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0 \]

\[ \Gamma^2_{12} = \frac{1}{r} = \Gamma^2_{21} \quad \Gamma^2_{i,j} = 0 \quad \text{ow.} \]

\[ \Gamma^2_{33} = -\sin \theta \cos \theta \]

\[ i=3 \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{x}^i} = \frac{d}{ds} \left\{ 2 r^2 \sin^2 \theta \dot{\phi}^2 \right\} = 4 r \sin^2 \theta \dot{\phi} \]

\[ + 4 r^2 \sin^2 \theta \cos \theta \sin \theta \dot{\phi} + 2 r^2 \sin^2 \theta \ddot{\phi} \]

\[ \frac{\partial L}{\partial \dot{\phi}} = 0 \]

\[ 0 = \frac{d}{ds} \frac{\partial L}{\partial \ddot{x}^i} - \frac{\partial L}{\partial x} \]

\[ \implies \quad \ddot{\phi} + \frac{2}{r} \dddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0 \]

\[ \Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31} \quad \Gamma^3_{i,j} = 0 \quad \text{ow.} \]

\[ \Gamma^3_{23} = \cot \theta = \Gamma^3_{32} \]
Thus from \((R_{ij})\):

\[- \mathbf{R}_{00} = 0 = \left\{ \log \sqrt{-g} \right\}_{,00} - \Gamma_{00}^{\sigma} \Gamma_{00}^{\circ} - \Gamma_{00}^{\circ} \Gamma_{00}^{\circ} - \Gamma_{00}^{\circ} \left( \log \sqrt{-g} \right)_{,00} \]

\[= -\Gamma_{00}^{11} + 2 \Gamma_{00}^{1} \Gamma_{00}^{0} - \Gamma_{00}^{1} \log (\sqrt{g})_{,1} \]

\[\left\{ -\frac{1}{2} V' e^{V'} \lambda^2 \right\}' + \frac{1}{2} V' V' e^{V'} - \left( \frac{1}{2} V' e^{V'} \lambda \right)^{1} \]

\[\left( \frac{1}{2} (V' + \lambda') + \frac{2}{r} \right) \]

\[\Rightarrow \]

\[V'' + \frac{1}{2} (V')^2 - \frac{1}{2} \lambda' V' + \frac{2 V'}{r} = 0 \quad (0,0) \]

Similarly:

\[- \mathbf{R}_{11} = \frac{1}{2} \left( V'' + \frac{1}{2} (V')^2 - \frac{1}{2} \lambda' V' + \frac{2 \lambda'}{r} \right) \]

\[V'' + \frac{1}{2} (V')^2 - \frac{1}{2} \lambda' V' - \frac{2 \lambda'}{r} = 0 \quad (1,1) \]
\[-R_{22} = \frac{\partial^2}{\partial \theta^2} (\log |\sin \theta|) + (e^{-\lambda} r)' + 2(-e^{-\lambda}) + \cot^2 \theta + e^{-\lambda} r \left( \frac{\chi' + \nu'}{2} + \frac{2}{r} \right) = 0 \]

- We show that the 3-equations (0,0) (1,1) and (2,2) are consistent and determine the solution to within a constant \( m \). One can then check (F10) that all other equations \( R_{ij} = 0 \) are identically satisfied on this solution:

\[
\text{Note: } \frac{\partial^2}{\partial \theta^2} (\log |\sin \theta|) = \frac{2}{\cot \theta} = -\csc^2 \theta
\]

\[-m^2 e^\theta + \cot^2 \theta = -1\]

\[
\Rightarrow (2,2) \text{ is equivalent to } 0 = -1 + (e^{-\lambda} r)' - 2 e^{-\lambda} + e^{\lambda} r \left( \frac{\chi' + \nu'}{2} + \frac{2}{r} \right)
\]
Subtract (1,1) from (0,0):

\[ \frac{2\nu}{r} + \frac{2\chi}{r} = 0 \]

\[ \nu + \chi = 0 \]

\[ (\nu + \chi) = \text{const.} = C. \] \hspace{1cm} (i)

\textbf{Note:} w.l.o.g., we can take $C = 0$ because a change of coords $\tilde{T} = \alpha T$ takes metric (5) to

\[ ds^2 = -\alpha^2 e^\nu d\tilde{T}^2 + e^\lambda dr^2 + r^2 d\Sigma^2 \]

\[ = -e^{\nu+\beta} d\tilde{T}^2 + e^\lambda dr^2 + r^2 d\Sigma^2 \]

\implies \text{if } \beta = C, \text{ in } \tilde{T}-\text{coords } \nu + \lambda = 0. \]

Thus: If $\exists$ a soln satis \( \omega \) with $C$, then it is $\equiv$ to a soln satis \( \omega \) with $C = 0$. \( \checkmark \)
Thus

\[ V = -\lambda. \]  

(2)

Substituting into (131) gives

\[ \chi'' + \frac{1}{r}(\chi')^2 - \frac{1}{2} (\chi')^2 + \frac{2 \chi'}{r} + \frac{2 \chi''}{r} = 0 \]

\[ \chi'' - \frac{1}{2} (\chi')^2 + \frac{2 \chi'}{r} = 0 \]

\[ (re^{-\chi'})'' = 0 \]

\[ (re^{-\chi'})' = C = \text{constant}. \]  

(3)

Thus, \((0, 0)\) and \((1, 1)\), two second order equations in \(V(r)\) & \(\lambda(r)\) determine \(V\) & \(\lambda\) to within 4 arb. constants in general. Then \(C = 0\) in (2) reduces these to two in (3).

We show that (2, 2) reduces these to one!!
$(2,2)$ reads:

$$0 = -1 + (e^{-\lambda} r)' - 2 e^{-\lambda} + r e^{-\lambda} \left( \frac{\lambda'}{e^{\lambda}} + \frac{2}{r} \right)$$

$$
\Rightarrow
$$

$$1 = (e^{-\lambda} r)' - 2 e^{-\lambda} + 2 e^{-\lambda}$$

$\therefore$ c = 1 in (3)!

Thus:

$$e^{-\lambda} r = r - 2m$$

$$e^{-\lambda} = 1 - \frac{2m}{r}$$

$$e^{\lambda} = \left(1 - \frac{2m}{r} \right)^{-1}$$

$$e^v = e^{-\lambda} = \left(1 - \frac{2m}{r} \right)$$

$$ds^2 = -\left(1 - \frac{2m}{r} \right) dt^2 + \left(1 - \frac{2m}{r} \right)^{-1} d\rho^2 + r^2 d\Omega^2.$$
Claim: all other equations in $R_{ij} = 0$ are identically satisfied when $e^x e^y$ are given by (4) (a miracle). (F10)

Actually: $R_{ij} = 0$ if $i \neq j$ and $R_{33} = R_{22} \sin^2 \Theta$

Ex: we see what happened to the 4 free constants of integration in the set two 2nd order equations (6.1) & (6.4) for $(\nu, \lambda)$:

1. $(0,0) + (1,1) \Rightarrow x' + \nu' = c \Rightarrow (0,0) & (1,1)$

are equivalent to a degenerate system:

$$x' + \nu' = c$$

1st order

$$(1,1)$$

2nd order

\[\Rightarrow\] expect 3 constants

\[\Rightarrow\] gauge freedom in $\xi$ allows us to set $c = 0$

\[\Rightarrow\] 2 remaining constants

\[\Xi(\nu, \lambda)\] is consistent
(3) \((2,2)\) is consistent with \((0,0)\) & \((1,1)\) (why??) but restricts the two free constants to a single constant \(M\).

**Problem:** What general principles are working behind this to make all the equations consistent, and with the interesting reduction of free parameters?

(I have seen no satisfying answer to this)
The Schwarzschild Singularity

\[ ds^2 = -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2 \]

- Singularity at \( r = 2M \)
- \( r = 0 \Rightarrow R \) is cont through the singularity
- Calculation: \( R_{\text{ijkl}} = \frac{48 M^2}{r^6} \)
  - continuous thru \( r = 2M \).
- note: \( r > 2M \Rightarrow t \) timelike
  - \( r < 2M \Rightarrow r \) timelike

But: assuming \( \frac{\partial}{\partial t} \) is positively oriented
at \( r = \infty \), by cont. we know \( \frac{\partial}{\partial t} \) pos. oriented
for \( r > \frac{2M}{r} \).

Q: how do we know whether \( \frac{\partial}{\partial r} \) or \( \frac{\partial}{\partial r} \) is
pos. oriented for \( r < 2M \) when metric is not
cont. thru \( r = 2M \)?
Redshift: \( \frac{\lambda_2}{\lambda_1} = \frac{\sqrt{1 - \frac{2m}{r_2}}}{\sqrt{1 - \frac{2m}{r_1}}} \)

\( r_2 = \infty \implies \lambda_2 = \left(1 - \frac{2m}{r_1}\right)^{-\frac{1}{2}} \lambda_1 \)

\( \implies \lambda_2 \to \infty \text{ as } r_1 \to 2m \)
\( \implies \infty \text{-redshift} \)

Geodesic: \( ds^2 = -(1 - \frac{2m}{r}) c^2 dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\Omega^2 \)

Consider geodesic \((t(s), r(s), \theta(s), \phi(s))\):
Divide (5) by \( ds^2 = \)

\[ 1 = -(1 - \frac{2m}{r}) c^2 \dot{t}^2 + (1 - \frac{2m}{r})^{-1} \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \]

Obtain the 3 other equations for geodesics from E-L equation directly.
\[ 0 = \delta \int \left\{ -\left(1 - \frac{2m}{r}\right)c^2 \dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + r^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right) \right\} ds \]

\[ \Rightarrow 3 \text{ E-L equations} \quad \frac{d}{ds} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \quad i = 0, 2, 3 \]

\[ \frac{d}{ds} \left( r^2 \dot{\theta} \right) = r^2 \sin \theta \cos \theta \dot{\phi}^2 \quad (B) \]

\[ \frac{d}{ds} \left( r^2 \sin^2 \theta \dot{\phi} \right) = 0 \quad (C) \]

\[ \frac{d}{ds} \left[ (1 - \frac{2m}{r}) \dot{t} \right] = 0 \quad (D) \]

**Note:** (A) - (D) give 4 2nd order eqn's for (t, r, \theta, \phi) along geodesic.

If initially \( \theta = \frac{\pi}{2} \), \( \dot{\theta} = 0 \), then (B) is identically satisfied by \( \theta = \frac{\pi}{2} \), & (A) - (D) reduces to 3 eqn's in (t, r, \phi):
\[ 1 = -(1 - \frac{2M}{r}) c^2 \dot{t}^2 + (1 - \frac{2M}{r})^{-1} \dot{r}^2 + r^2 \phi^2 \]  

(E')

\[ \frac{d}{ds} (r^2 \phi) = 0 \]  

(c')

\[ \frac{d}{ds} \left[ (1 - \frac{2M}{r}) \dot{t} \right] = 0 \]  

(c')

Then, (c') gives

\[ 2 \frac{dM}{dt} = r^2 \phi = h = \text{const}^\dagger \]

(E)

\[ \text{"equal area in equal time"} \]

Also, (D') becomes

\[ \dot{t} = (1 - \frac{2M}{r})^{-1} L \]

\[ L = \text{const}^\dagger \]  

(F)
Note: substituting (E), (F) into (A) gives a 1st order ode in \( r \):

\[
1 = -L^2 C^2 \left(1 - \frac{2M}{r}\right)^{-1} + \left(1 - \frac{2M}{r}\right)^{-1} \frac{h}{r^2}
\]

\[\Rightarrow\]

\[
r^2 = \left(1 - \frac{2M}{r} + L^2 C^2\right) - \frac{h}{r^2} \left(1 - \frac{2M}{r}\right)
\]

Thus the equations for \( r \) & \( t \) become:

\[
r' = \pm \sqrt{\left(L^2 C^2 + 1 - \frac{2M}{r}\right) - \frac{h}{r^2} \left(1 - \frac{2M}{r}\right)} \quad (G)
\]

\[
t' = \left(1 - \frac{2M}{r}\right)^{-1} L \quad (H)
\]

Note: choosing the minus sign in (G), it is clear that as \( r \to 2M + \), (G) \( \sim \) \( r' = -L^2 C^2 \) and (H) \( \sim \) \( t = \infty \Rightarrow \) the \( r \) equation is regular; but in the t-equn the t \( \to \infty \) as \( r \to 2M \).
Note: Letting $r = \frac{1}{u}$, and obtaining the equation for $r = r(\phi)$ through the replacement of $s$ by $\phi$ as independent variable using (E)

$$\frac{d\phi}{ds} = \frac{h}{r^2}$$

leads to the exact equation:

$$u'' + u = \frac{m}{\hbar^2} + 3mu^2, \quad m = \frac{G_0 M}{c^2}$$

for the geodesic. The corresponding $u = u(\phi)$ in Newton's theory is (Binet's Equ)

$$u'' + u = \frac{G_0 M}{H^2} \approx \frac{M}{\hbar^2} \text{ in classical limit}$$

Except for nonlinear quadratic term $3mu^2$. Ref: Te/Tracy Monthly A-BS