Kruskal - Szekeres Coordinates:

(S) $$ds^2 = -\left(1-\frac{2M}{r}\right)c^2dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

(K) $$ds^2 = -f^2(u,v)\,(dv^2 - du^2) + r^2 d\Omega^2$$

<u>Theorem</u>: If we let

$$f^2(u,v) = \frac{32M^3}{r}e^{-r/2M},$$

then the mapping $(ct, r) \longmapsto (u,v)$ given by

$$u^2 - v^2 = \left(\frac{r}{2M} - 1\right)e^{r/2M} \quad \leftarrow \text{smooth for } r>0$$

$$u > |v|$$

$$\frac{v}{u} = \tanh\frac{x^0}{4M}$$

maps $2M < r < \infty$, $-\infty < t < +\infty$ to $u > |v|$; and

$$v^2 - u^2 = \left(1 - \frac{r}{2M}\right)e^{r/2M}$$

$$v > |u|$$

$$\frac{u}{v} = \tanh\frac{x^0}{4M}$$

maps $0 < r < 2M$, $-\infty < t < \infty$ to $v > |u|$;

and under each of these coord mappings, the metric (S) is mapped to the metric (K).

Proof: see ABS.

~~Conclude: Since (K) is smooth on r = 2M,~~

Note: (K) is ~~a~~ smooth in uv-coordinates wherever r is a smooth fn of (u, v).

But

$$u^2 - v^2 = \left(\frac{r}{2M} - 1\right) e^{r/2M} = F(r)$$

$$F'(r) = \frac{1}{2M} e^{r/2M} + \left(\frac{r}{2M} - 1\right) \frac{1}{2M} e^{r/2M}$$

$$= \frac{1}{2M} e^{r/2M} \left(1 + \frac{r}{2M} - 1\right)$$

$$F'(r) = \frac{r}{(2M)^2} e^{r/2M} \geq 0, \; = 0 \text{ only at } r = 0$$

$\Rightarrow$ $r = F^{-1}(u^2 - v^2)$ is a smooth fn of

u, v everywhere <u>except</u> at $r=0$.

In uv-space, $\boxed{r=0 \iff u^2-v^2=-1}$

Moreover,

$$\boxed{\begin{aligned} r=2M \iff u^2-v^2&=0 \\ u&=\pm v \end{aligned}}$$

<u>Note</u>: t is <u>not</u> a smooth fn of (u,v) across $r=2M$ as expected:

$$x^0 = 4M \tanh^{-1}\frac{v}{u} \qquad r>2M$$

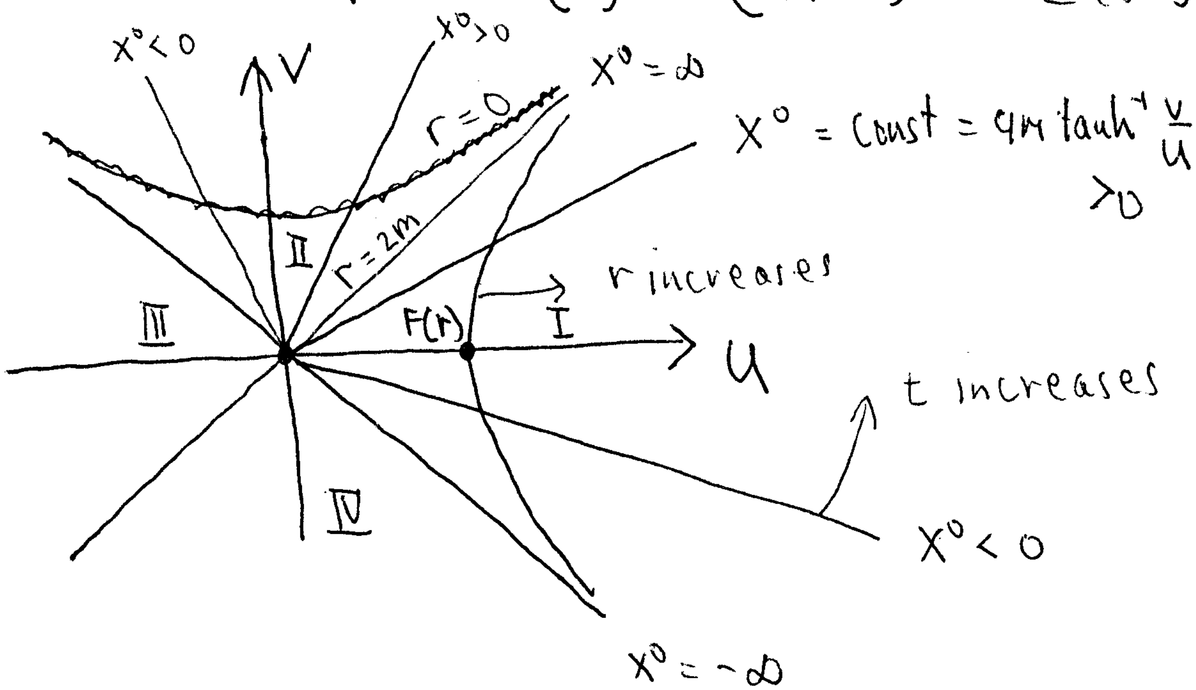
$$x^0 = 4M \tanh^{-1}\frac{u}{v} \qquad r<2M$$

$$r=2M \iff u=\pm v \iff x^0=\pm\infty$$

- **Kruskal Diagram** $r > 2M \iff u > |v|$

$t = \text{const} \iff \frac{v}{u} = \tanh \frac{x^0}{4M}$ const

$r = \text{const} \iff u^2 - v^2 = F(r) = \left(\frac{r}{2M} - 1\right) e^{r/2M} \equiv \text{const}$



- $r < 2M \iff v > |u|$

$t = \text{const} \iff \frac{u}{v} = \tanh \frac{x^0}{4M}$ const

$r = \text{const} \iff v^2 - u^2 = \left(1 - \frac{r}{2M}\right) e^{r/2M} = -F(r) > 0$

Note: light rays prop. on null geodesics $\Rightarrow$

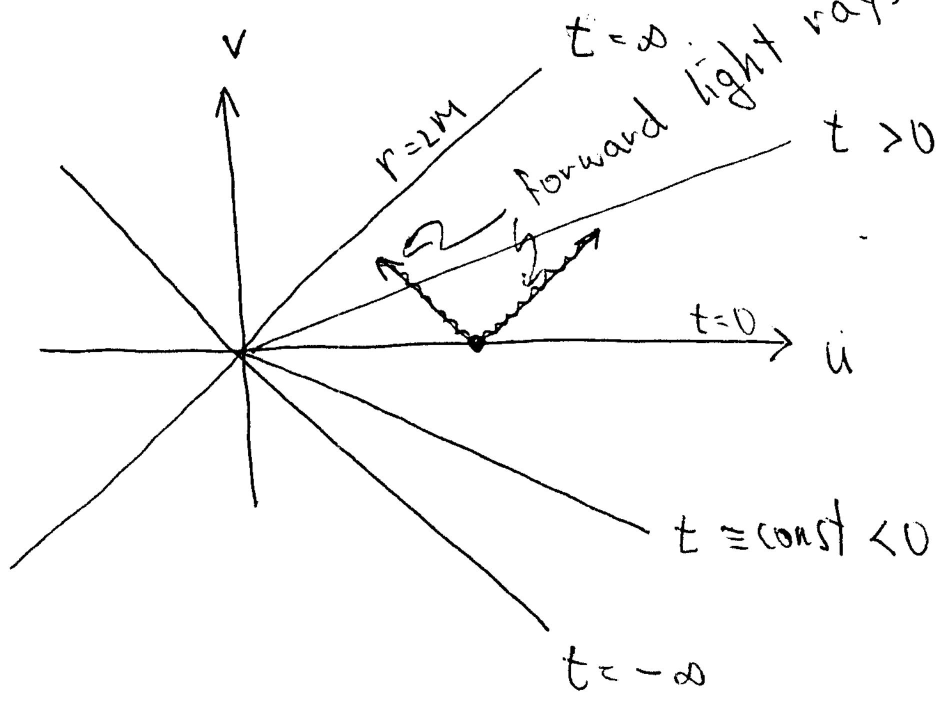
$$ds^2 = -f^2(u,v)(dv^2 - du^2) + r^2 d\Omega^2 = 0$$

If light ray is radial, $d\Omega^2 = 0 \Rightarrow$

$$-dv^2 + du^2 = 0$$

$\Rightarrow$ light rays propagate along $45°$ lines in Kruskal coordinates

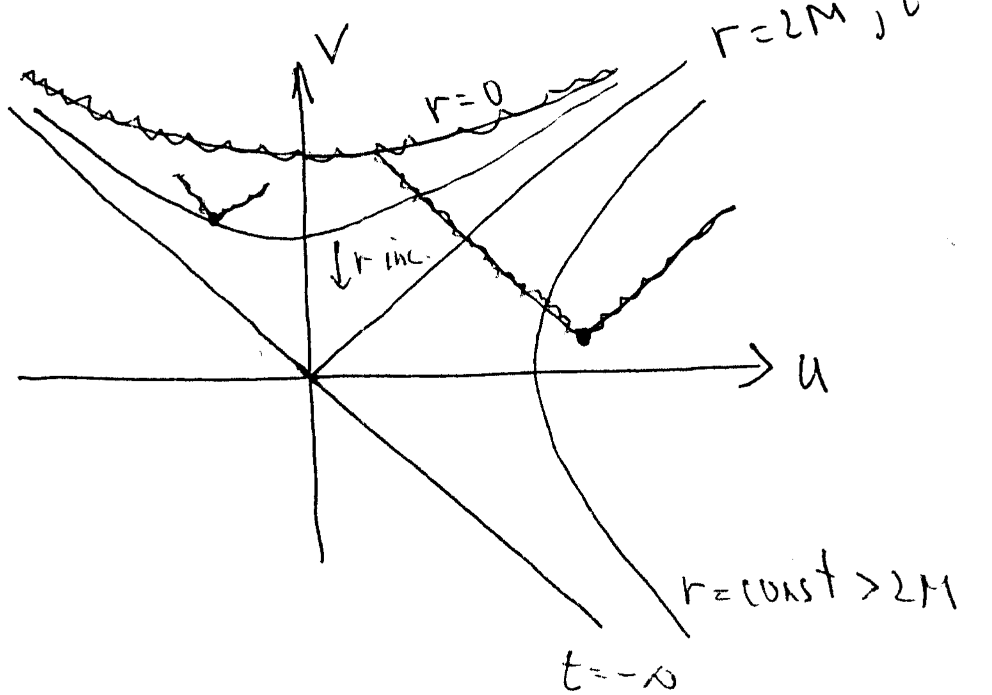
In region I, direction of positive time orients light rays



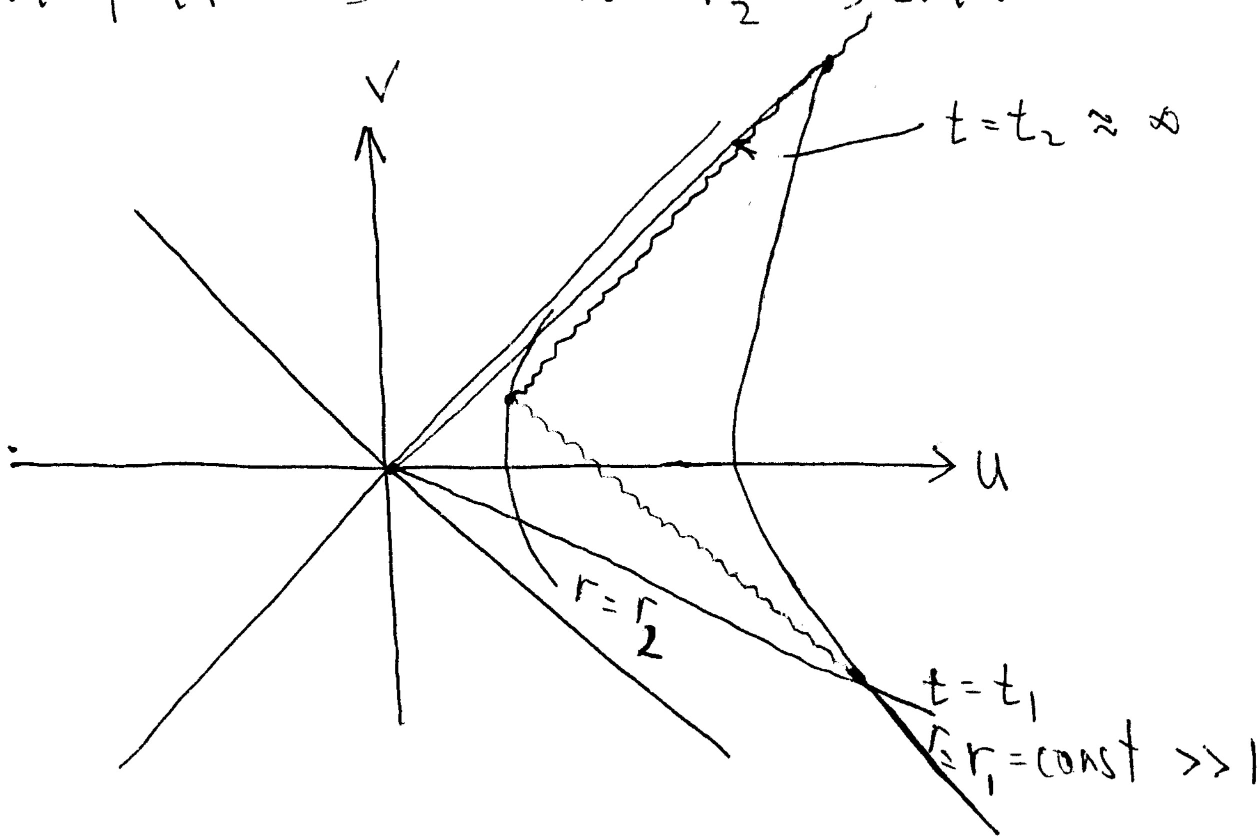
- In side Schw-radius, $r<2M$, r is the time marker

Q: is $\frac{\partial}{\partial r}$ forward pointing or $-\frac{\partial}{\partial r}$?

Ans: the light ray crosses $r=2M$ in uv-coordinates & continues smoothly to decreasing values of $r<2M$ $\Rightarrow$ $-\frac{\partial}{\partial r}$ is positive-time directed by continuity (only way to determine this since $r<2M$ has no cont. connection to $r>2M$ in tr-coords:



• From picture: consider a light ray sent in from $r_1 \gg 1$ to $r_2 \gtrsim 2M$, $r_2 > 2M$, and then reflected back to r_1. Then the change in time Δt betw. sending the signal and receiving it $\to \infty$ as $r_2 \to 2M$:



• <u>From the picture</u>: At $r=2M$, the forward light cone is tangent to the sphere of symmetry $r=2M$, and inside $r<2M$, the forward light cone points <u>entirely</u> toward <u>decreasing</u> values of r. $\Rightarrow$ $r=2M$ is a <u>black hole</u>

~~• From picture: expect inward pointing light rays to hit the singularity $r=0$.~~

~~Check:~~

<u>Moreover</u>: all timelike geodesics starting inside $r=2M$, are directed <u>within</u> the light cone $\Rightarrow$ will hit singularity in <u>finite</u> Δt. From geodesic equation, (G),

$$\dot{r} = -\sqrt{L^2c^2 + \left(1-\frac{h}{r^2}\right)\left(1-\frac{2M}{r^2}\right)},$$

as $r\to\infty$, $\dot{r}\to-\infty$ $\Rightarrow$ r hits in <u>finite</u> proper time.

Extension of Schwarzschild Universe by Kruskal Coordinates:

we have:

(I) $\dfrac{u}{v} = \tanh \dfrac{x^0}{4M}$ — defines a 1-1 onto mapping from

$u^2 - v^2 = \left(\frac{r}{2M} - 1\right) e^{r/2M}$

$$\begin{bmatrix} -\infty < x^0 < \infty \\ 2M < r < \infty \end{bmatrix} \longrightarrow \begin{bmatrix} u > |v| \end{bmatrix}$$

---

same function

(II) $\dfrac{u}{v} = \tanh \dfrac{x^0}{4M}$ — defines a 1-1 onto mapping from

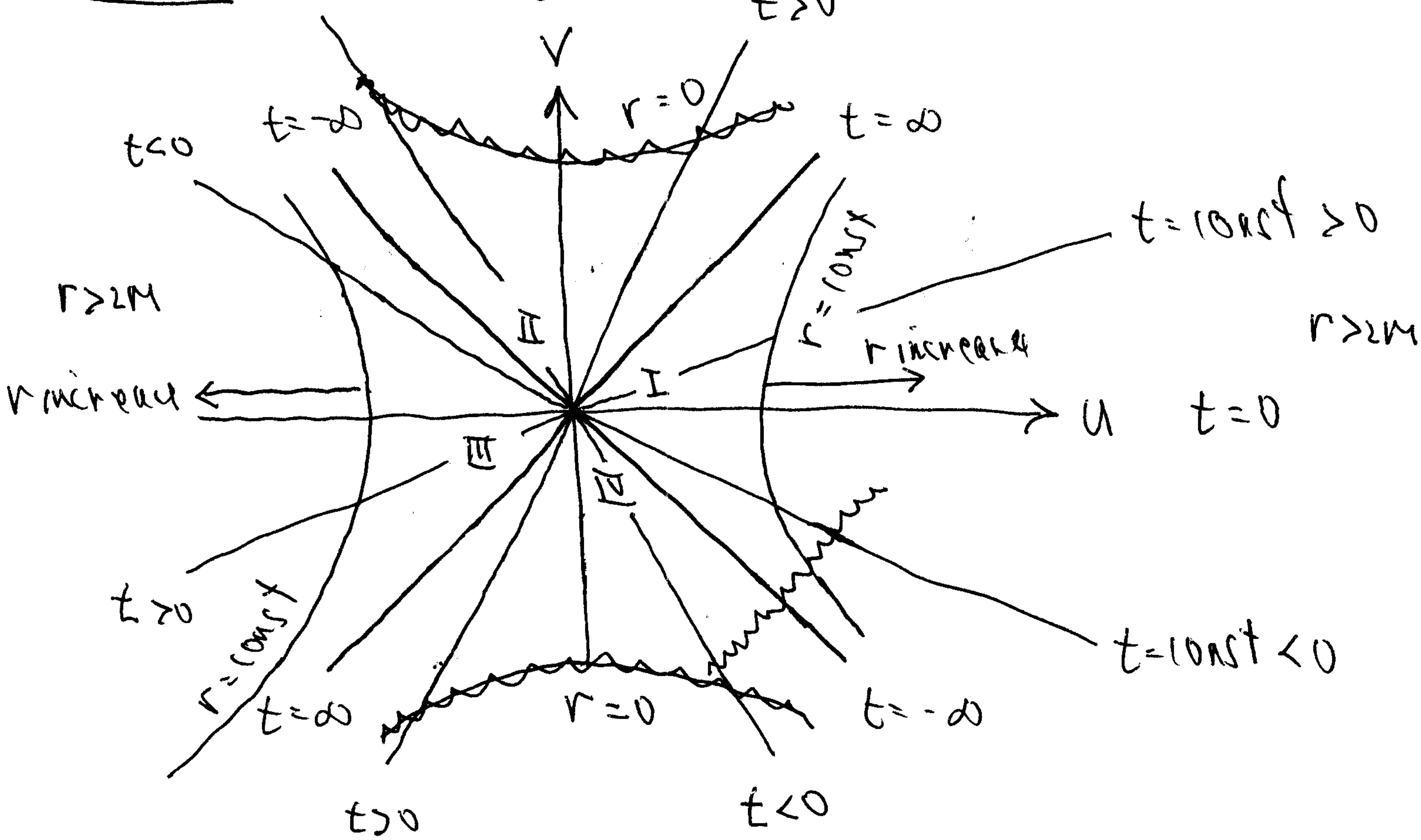
$v^2 - u^2 = \left(1 - \frac{r}{2M}\right) e^{r/2M}$

$$\begin{bmatrix} -\infty < x^0 < \infty \\ 0 < r < 2M \end{bmatrix} \longrightarrow \begin{bmatrix} v > |u| \\ v^2 - u^2 < 1 \end{bmatrix}$$

Together these define 1-1 onto mapping from $r \geq 0$ to $u + v > 0$.

<u>Note</u>: given (x^0, r), if (u, v) solves I or II then so does $(-u, -v)$, so I, II <u>also</u> define a 1-1 onto mapping from $r \geq 0$ to $u + v < 0$

Picture :



Conclude : III, IV are connected smoothly to I, II in uv-coords

Note: if $r=\infty$ in I sets $\frac{\partial}{\partial t}$ as direction of forward time, then $\frac{\partial}{\partial t} \sim \frac{\partial}{\partial V} \Rightarrow \frac{\partial}{\partial v}$ sets forward time, & this continues smoothly thru whole soln.

Conclude : Direction of forward time

I	$\frac{\partial}{\partial t}$	III	$-\frac{\partial}{\partial t}$
II	$-\frac{\partial}{\partial r}$	IV	$\frac{\partial}{\partial r}$

<u>Note</u>: light rays emitted from $r=0$ in IV cross into I at time $t=-\infty$

Q: what is the significance of this in presence of ∞ red shift?

<u>Note</u>: time-like geodesics cannot move from region I into regions III or IV in forward time. ~~and similarly~~

<u>Note</u>: time-like geodesics from I cannot enter III in forward or backward time & visa-versa

$\Rightarrow$ universe III is completely separate from universe I.

Possibility of "Wormhole" connecting I & III?
(Wheeler)

M-T-W

34

From the Kruskal-Szekeres diagram and the 45-degree nature of its radial light rays, one sees that any particle that ever finds itself in region IV of spacetime must have been "created" in the earlier singularity; and any particle that ever falls into region II is doomed to be crushed in the later singularity. Only particles that stay forever in one of the asymptotically flat universes I or III, outside the gravitational radius ($r > 2M$), are forever safe from the singularities.

Creation and destruction in the singularities

Some investigators, disturbed by the singularities at $r = 0$ or by the "double-universe" nature of the Schwarzschild geometry, have proposed modifications of its topology. One proposal is that the earlier and later singularities be identified with each other, so that a particle which falls into the singularity of region II, instead of being destroyed, will suddenly reemerge, being ejected, from the singularity of region IV. One cannot overstate the objections to this viewpoint: the region $r = 0$ is a physical singularity of infinite tidal gravitation forces and infinite Riemann curvature. Any particle that falls into that singularity must be destroyed by those

Nonviable proposals for modifying the topology of Schwarzschild spacetime

forces. Any attempt to extrapolate its fate through the singularity using Einstein's field equations must fail; the equations lose their predictive power in the face of infinite curvature. Consequently, to postulate that the particle reemerges from the earlier singularity is to make up an *ad hoc* mathematical rule, one unrelated to physics. It is conceivable, but few believe it true, that any object of finite mass will modify the geometry of the singularity as it approaches $r = 0$ to such an extent that it can pass through and reemerge. However, whether such a speculation is correct must be answered not by *ad hoc* rules, but by concrete, difficult computations within the framework of general relativity theory (see Chapter 34).

A second proposal for modifying the topology of the Schwarzschild geometry is this: one should avoid the existence of two different asymptotically flat universes by identifying each point (v, u, θ, ϕ) with its opposite point $(-v, -u, \theta, \phi)$ in the Kruskal-Szekeres coordinate system. Two objections to this proposal are: (1) it produces a sort of "conical" singularity (absence of local Lorentz frames) at $(v, u) = (0, 0)$, i.e., at the neck of the bridge at its moment of maximum expansion; and (2) it leads to causality violations in which a man can meet himself going backward in time.

One good way for the reader to become conversant with the basic features of the Schwarzschild geometry is to reread §§31.1–31.4 carefully, reinterpreting everything said there in terms of the Kruskal-Szekeres diagram.