☒ Gravitational Collapse - Buchdahl Stability Limit

- Interior Schwarzschild Soln

<u>Wein</u>: $ds^2 = -B(r)\,dt^2 + A(r)\,dr^2 + r^2 d\Omega^2$

W(11.1.8) $$\frac{B'}{B} = -\frac{2p'}{p+\rho} \tag{1}$$

W(11.1.11) $$A = \left(1 - \frac{2GM(r)}{r}\right)^{-1} \tag{2}$$

(11.1.12) $$M(r) = \int_0^r 4\pi \xi^2 \rho(\xi)\,d\xi \tag{3}$$

(11.1.13) $$-r^2 p'(r) = GM\rho\left(1 + \frac{p}{\rho}\right)\left(1 + \frac{4\pi r^3 p(r)}{M}\right)\left(1 - \frac{2GM}{r}\right)^{-1} \tag{4}$$

If we assume an equation of state $p = p(\rho)$, then (3) and (4) give a system of 2 ODE's for (M, ρ).

<u>Stephani</u>: If we let $B = e^{\nu(r)}$,

$$\frac{B'}{B} = \nu' \tag{5}$$

then from (1) we have

$$p' = -\frac{1}{2}\rho\left(1+\frac{p}{\rho}\right)\nu',$$

and substituting this into (4) gives

$$\frac{r^2}{2}\cancel{\rho}\cancel{\left(1+\frac{p}{\rho}\right)}\nu' = GM\cancel{\rho}\cancel{\left(1+\frac{p}{\rho}\right)}\left(1+\frac{4\pi r^3 p}{M}\right)\left(1-\frac{2GM}{r}\right)^{-1}$$

$$\frac{r^2}{2}\nu'\left(1-\frac{2GM}{r}\right) = GM\left(1+\frac{4\pi r^3 p}{M}\right)$$

$$\frac{1}{r}\left(1-\frac{2GM}{r}\right)\nu' - \frac{2GM}{r^3} = 8\pi G p$$

Let $K = 8\pi G$ & obtain

Steph 23.5 $$K p = \frac{1}{r}\left(1-\frac{2GM}{r}\right)\nu' - \frac{2GM}{r^3} \qquad (6)$$

Steph 23.4 $$\nu' = \frac{-2p'}{p+\rho} \qquad (7)$$

where $\rho = \mu c^2$ relates Stephani to Wein

Note: From (4), if $1-\frac{2GM}{r}<0$, then $p'(r)>0 \Rightarrow$ "unstable". This condition is

$$1-\frac{2GM}{r}>0,$$

$$\frac{2GM}{r}<1. \qquad (8)$$

Thus, (8) $\equiv$ $M(r)$ is always less than the mass whose Schwarzschild radius is r, and limits the size of stable spherically symmetric matter distributions*. The estimate (8) can be improved upon as follows.

Theorem: Assume that $p(0)<\infty$, $\rho(0)<\infty$, $\rho'(r)<0$, $B>0$, $A>0$, and $\rho=0$, for $r \geq r_0$. Then

$$\frac{2GM(r_0)}{r_0}<\frac{8}{9} \qquad (9)$$

* It is unclear what $p'>0$ means in $r<2GM$, since $r=\text{const}$ is a timelike surface!

Proof: Let

$$f = B^{1/2} \Rightarrow \frac{f'}{f} = \frac{B'}{2B} = \frac{\nu'}{2}$$

Then we claim that

$$\text{(steph 23.10)} \quad \frac{d}{dr}\left[\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\; f'\right] = \frac{f}{\sqrt{1-\frac{2GM}{r}}}\frac{d}{dr}\frac{GM}{r^3} \qquad (10)$$

To see this, eliminate p from (6) & (7): first diff (6),

$$\frac{Kp'}{2} = \frac{d}{dr}\left\{\frac{1}{r}\left(1-\frac{2GM}{r}\right)\frac{f'}{f}\right\} - \frac{d}{dr}\frac{GM}{r^3} \qquad (11)$$

and rewrite (7) as

$$-p' = \frac{f'}{f}(p+\rho)$$

which is

$$\frac{Kp'}{2} = -\frac{K}{2}\frac{f'}{f}(p+\rho) \qquad (12)$$

But the RHS of (11) equals:

$$\frac{d}{dr}\left\{\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\; f'\,\frac{\sqrt{1-\frac{2GM}{r}}}{f}\right\} - \frac{d}{dr}\frac{GM}{r^3}$$

$$= \frac{d}{dr}\left\{\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\; f'\right\}\frac{\sqrt{1-\frac{2M}{r}}}{f} - \frac{d}{dr}\frac{GM}{r^3}$$

$$+ \frac{1}{r}\sqrt{1-\frac{2GM}{r}}\; f'\,\frac{d}{dr}\left(\frac{\sqrt{1-\frac{2M}{r}}}{f}\right)$$

$$= \frac{\sqrt{1-\frac{2GM}{r}}}{f}\left[\frac{d}{dr}\left(\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\; f'\right) - \frac{f}{\sqrt{1-\frac{2GM}{r}}}\frac{d}{dr}\frac{GM}{r^3}\right.$$

$$\left. + \frac{ff'}{r}\frac{d}{dr}\left(\frac{\sqrt{1-\frac{2GM}{r}}}{f}\right)\right]$$

But

$$\frac{d}{dr}\frac{\sqrt{1-\frac{2GM}{r}}}{f} = -\frac{f'}{f^2}\sqrt{1-\frac{2GM}{r}} + \frac{1}{2}\frac{1}{f}\frac{1}{\sqrt{1-\frac{2GM}{r}}}\left(-\frac{2GM'}{r} + \frac{2GM}{r^2}\right)$$

$$= -\frac{f'}{f^2}\sqrt{1-\frac{2GM}{r}} + \frac{1}{2}\frac{1}{f}\frac{1}{\sqrt{1-\frac{2GM}{r}}}\left(\frac{-K\rho r^2}{r} + \frac{2GM}{r^2}\right)$$

$$= -\frac{f'}{f^2}\sqrt{1-\frac{2GM}{r}} + \frac{1}{f}\frac{r}{2}\frac{1}{\sqrt{1-\frac{2GM}{r}}}\left(-K\rho + \frac{2GM}{r^3}\right)$$

Thus setting RHS of (11) = RHS (12) gives

$$\frac{f}{\sqrt{1-\frac{2GM}{r}}}\left(-\frac{K}{2}\right)\frac{f'}{f}(p+\cancel{\rho}) = \left\{\frac{d}{dr}\left(\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\,f'\right) - \frac{f}{\sqrt{1-\frac{2GM}{r}}}\frac{d}{dr}\frac{GM}{r^3}\right\}$$

$$+\frac{ff'}{r}\left\{-\frac{f'}{f^2}\sqrt{1-\frac{2GM}{r}} + \frac{r}{2}\frac{1}{f}\frac{1}{\sqrt{1-\frac{2GM}{r}}}\left(-\cancel{K\rho}+\frac{2GM}{r^3}\right)\right\}$$

$\Leftrightarrow$ (Let $A = \left(1-\frac{2GM}{r}\right)^{-1}$)

$$-\frac{K}{2}A^{1/2}f'p = \{\quad\} - \frac{(f')^2}{rf}A^{-1/2} + f'A^{1/2}\frac{GM}{r^3} \qquad (13)$$

But from (6)

$$-\frac{K}{2}A^{1/2}f'p = -\frac{1}{2}A^{1/2}f'\left(\frac{1}{r}A^{-1}\frac{2f'}{f} - \frac{2GM}{r^3}\right)$$

$$= -\frac{1}{r}A^{-1/2}\frac{(f')^2}{f} + f'A^{1/2}\frac{GM}{r^3} \equiv \text{cf RHS (13)} \qquad (14)$$

Thus from (14) and (13),

$$0 = \{\} = \frac{d}{dr}\left(\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\,f'\right) - \frac{f}{\sqrt{1-\frac{2GM}{r}}}\frac{d}{dr}\frac{GM}{r^3}$$ which proves claim $\square$ (10)

Now from (6),

$$K P = \frac{1}{r}\left(1 - \frac{2GM}{r}\right)\frac{f'}{2f} - \frac{2GM}{r^3},$$

if $\rho' < 0$ and $\rho(0)$ and $p(0)$ are finite, then

$$\frac{1}{r^3} M = \frac{1}{r^3}\int_0^r 4\pi \rho \xi^2 \, d\xi \leq \frac{1}{r^3}\frac{4\pi}{3}\rho(0) r^3 < \infty$$

which ~~implies~~ together with $1 - \frac{2GM}{r} > 0$ implies

$$\frac{f'}{2rf} < \infty . \tag{15}$$

Now assuming $\rho' < 0$,

$$\frac{d}{dr}\frac{M}{r^3} = \frac{M'}{r^3} - 3\frac{M}{r^4} = \frac{4\pi\rho r^2}{r^3} - 3\frac{M}{r^4}$$

But $M(r) = \frac{4\pi}{3}\bar{\rho}(r) r^3$ where $\bar{\rho}(r)$, the ave. density inside r satisfies $\bar{\rho}(r) > \rho(r)$ since $\rho' < 0$.

Thus $$\frac{d}{dr}\frac{M}{r^3} = \frac{4\pi}{r}(\rho - \bar{\rho}) < 0. \tag{16}$$

Moreover, by (10), and the fact that

$$\frac{d}{dr}\frac{GM}{r^3} < \infty,$$

we conclude

$$\frac{d}{dr}\left\{\frac{f'}{r}\left(1-\frac{2GM}{r}\right)^{1/2}\right\} < \infty,$$

and so (assuming $A>0$)

$$\frac{f'}{r} < \infty. \tag{16A}$$

Thus the assumptions $p<\infty$, $\rho<\infty$, $\rho'<0$ and $A\neq 0$ lead to (15) and (16A), which imply

$$f(r) < \infty \tag{16B}$$

Using 16, ~~Therefore~~, (10) implies

$$\frac{d}{dr}\left[\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\, f'(r)\right] \leq 0 \qquad (17)$$

Assume now that $r = r_0$, the radius of star, the metric is Schwarz metric $\Rightarrow$

$$A(r_0) = \frac{1}{1-\frac{2GM_0}{r_0}}, \quad B(r_0) = 1 - \frac{2GM_0}{r_0}. \qquad (18)$$

Then

$$f^2(r_0) = 1 - \frac{2GM_0}{r_0} \quad f'(r_0) = \frac{GM_0}{r_0^2}\frac{1}{\sqrt{1-\frac{2GM_0}{r_0}}} \qquad (19)$$

$\Rightarrow$ by (17)

$$\int_r^{r_0} \frac{d}{dr}\frac{1}{r}\sqrt{1-\frac{2GM}{r}}\, f'(r)\,dr = \frac{1}{r_0}\sqrt{1-\frac{2GM_0}{r_0}}\, f'(r_0) - \frac{1}{r}\sqrt{1-\frac{2GM}{r}}\, f'(r) \leq 0$$

$$\Leftrightarrow \frac{1}{r_0}\frac{GM_0}{r_0^2} \leq \frac{1}{r}\sqrt{1-\frac{2GM}{r}}\, f'(r)$$

$$\Leftrightarrow f'(r) \geq \frac{r}{r_0^3}\frac{GM_0}{\sqrt{1-\frac{2GM}{r}}} \qquad (20)$$

Integrating (20) between ~~[illegible]~~ 0 & r_0 leads to

$$f(r_0)-f(0)=\int_0^{r_0} f'(r)\,dr \geq \int_0^{r_0} \frac{r}{r_0^3}\frac{GM_0}{\sqrt{1-\frac{2GM}{r}}}\,dr$$

$$f(0) \leq \left(1-\frac{2GM_0}{r_0}\right)^{1/2} - \frac{GM_0}{r_0^3}\int_0^{r_0}\frac{r\,dr}{\sqrt{1-\frac{2GM}{r}}}. \qquad (21)$$

Estimate the integral as follows: Let $\bar{\rho}$ denote the <u>average</u> density inside radius r, so that

$$M = \frac{4\pi}{3}\bar{\rho}\, r^3.$$

Then $\bar{\rho}(r)$ is <u>decreasing</u> because $\rho' < 0$, and so

$$M(r) \leq M(r_0) = \frac{4\pi}{3}\bar{\rho}(r_0)\, r_0^3 \qquad \bar{\rho}(r) \geq \bar{\rho}(r_0)$$

$$\bar{\rho}(r_0) = \frac{3}{4\pi}\frac{M_0}{r_0^3}$$

$$M(r) = \frac{4\pi}{3}\bar{\rho}(r)\, r^3 \geq \frac{4\pi}{3}\bar{\rho}_0\, r^3 \geq \frac{M_0 r^3}{r_0^3}$$

Substituting this into (21) gives

$$f(0) \leq \left(1 - \frac{2GM_0}{r_0}\right)^{1/2} - \frac{GM_0}{r_0^3} \int_0^{r_0} \frac{r\,dr}{\sqrt{1 - \frac{2G}{r}\frac{M_0 r^3}{r_0^3}}}$$

$$\int_0^{r_0} \frac{r\,dr}{\sqrt{1 - \frac{2GM_0 r^2}{r_0^3}}} = -\frac{r_0^3}{4GM_0} \int_0^{r_0} \frac{du}{\sqrt{u}} = -\frac{r_0^3}{\cancel{4}_{2} GM_0} \cancel{2} u^{1/2} \Big]_{r=0}^{r=r_0}$$

$$u = 1 - \frac{2GM_0 r^2}{r_0^3}$$

$$du = -\frac{4GM_0}{r_0^3} r\,dr$$

$$= -\frac{r_0^3}{2GM_0} \left(1 - \frac{2GM_0 r^2}{r_0^3}\right)^{1/2} \Bigg]_{r=0}^{r=r_0} = -\frac{r_0^3}{2GM_0}\left(\left(1 - \frac{2GM_0}{r_0}\right)^{1/2} - 1\right)$$

$$\therefore\ 0 \leq f(0) \leq \left(1 - \frac{2GM_0}{r_0}\right)^{1/2} + \frac{\cancel{GM_0}}{\cancel{r_0^3}} \frac{r_0^3}{2GM_0} \left\{\left(1 - \frac{2GM_0}{r_0}\right)^{1/2} - 1\right\}$$

$$\leq \left(1 - \frac{2GM_0}{r_0}\right)^{1/2} + \frac{1}{2}\left\{\left(1 - \frac{2GM_0}{r_0}\right)^{1/2} - 1\right\}$$

$$\Rightarrow \quad 0 \leq \frac{3}{2}\left(1 - \frac{2GM_0}{r_0}\right)^{1/2} - \frac{1}{2}$$

Thus

$$\frac{1}{3} \leq \left(1 - \frac{2GM_0}{r_0}\right)^{1/2}$$

$$\frac{1}{9} \leq 1 - \frac{2GM_0}{r_0}$$

$$\boxed{\frac{2GM_0}{r_0} \leq \frac{8}{9}}$$

as claimed.

<u>Application</u>: a stable spherical star can provide a maximum red shift factor of 3

P.f. at surface, $\sqrt{B} = \sqrt{1 - \frac{2GM_0}{r_0}} \geq \sqrt{1 - \frac{8}{9}} = \frac{1}{3}$

$$\Rightarrow \quad \frac{\lambda_2}{\lambda_1} = \frac{\sqrt{B_2}}{\sqrt{B_1}} \leq \left(\sqrt{B_1}\right)^{-1} \leq \left(\frac{1}{3}\right)^{-1} = 3$$

$B_2 = 1$ @ ∞

$$\Rightarrow \quad \boxed{\lambda_2 \leq 3\lambda_1} \quad !!$$

<u>Note</u>: $\frac{\lambda_2}{\lambda_1}$ dimensionless