Red Shift : Ref A-B-S pg 136 don't motivate 4.150, which follows from const sp of light

Starting point: Assume we are in the regime where classical Newtonian gravitation applies. In this regime,

$$ds^2 = -g_{00}\,d(x^0)^2 + d\underset{\sim}{x}^2 \qquad \underset{\sim}{x} = (x^1, x^2, x^3),$$

and

$$g_{00} = 1 + \frac{2\varphi}{c^2}, \qquad x^0 = ct,\ s = c\tau$$

where φ corresponds to the Newtonian gravitational potential, $\Delta\varphi = 4\pi\rho$. In the earth-sun system, $\varphi \sim -\frac{1}{r}$ in $\varphi \ll 0$ near sun, and $\to 0$ at infinity. We show that in a static gravitational field, when $\varphi = \varphi(r)$, there is a red-shift that satisfies

(1) $$\frac{\lambda_2}{\lambda_1} = \frac{\alpha_2}{\alpha_1},$$

(2) $$\nu_2 \alpha_2 = \nu_1 \alpha_1$$

(3) $$\Delta\nu = (\nu_2 - \nu_1) = -\nu_1 \frac{\Delta\varphi}{c^2}$$ $\Delta\nu = \nu_2 - \nu_1$, $\Delta\varphi = \varphi_2 - \varphi_1$

where $\alpha \equiv \alpha(r) = \sqrt{1 + \frac{2\varphi(r)}{c^2}}$

$$\nu = \frac{n}{\Delta\tau} = \left.\frac{\#\text{ of waves}}{\text{proper time change}}\right|_{\text{at fixed } r}$$

λ = wavelength, (in this case Δr is proper distance)

and (1) - (3) apply to radial waves at any two radial distances r_1 and r_2, being emitted at r_1 and received at r_2. Moreover, these are valid asymptotically in the limit of high frequency, indept of separation distance $|r_2 - r_1|$, under the assumption that the wave crests move at the speed of light in O-N frames.

<u>Proof</u>: First, in a frame fixed with r,

$$d\tau = \alpha\, dt,$$

so an O-N frame is obtained by taking

$$\bar{\underset{\sim}{x}} = \underset{\sim}{x}, \quad \bar{t} = \alpha_0 t, \qquad \text{so that}$$

$$ds = -d(c\bar{t})^2 + d\bar{\underset{\sim}{x}}^2$$

at $\begin{cases} \underset{\sim}{x} = \underset{\sim}{x}_0 \\ t = t_0 \end{cases}$. Thus, if $\bar{r}(\bar{t}) = |\underset{\sim}{x}(\bar{t})|$ is the wave crest position, then at $t = t_0$,

$$c = \frac{d\bar{r}}{d\bar{t}} = \frac{dr}{\alpha\, dt} \Rightarrow \boxed{\frac{dr}{dt} = \alpha c}$$

gives the coord speed of wave crests under assumption of "constant speed of light."

- Now assume wave crests leave surface of sun ($r = r_1$), and are received at earth ($r = r_2$)

Say: the wave crests leave sun at $t = t_1$, and arrive at $t = t_2$. Then assuming a wave crest moves with speed $\frac{dr}{dt} = \alpha c$, we obtain $\frac{dr}{\alpha(r)} = c\,dt$ for wave crests, $\Rightarrow$

$$\underbrace{c(t_2 - t_1) = \int_{r_1}^{r_2} \frac{dr}{\alpha(r)}}_{\text{crest 1}} \quad , \quad \underbrace{c(t_2 - t_1) = \int_{r_1+\lambda_1}^{r_2+\lambda_2} \frac{dr}{\alpha(r)}}_{\text{crest 2}} \qquad (A)$$

and subtracting we obtain

$$0 = \int_{r_1+\lambda_1}^{r_2+\lambda_2} \frac{dr}{\alpha(r)} - \int_{r_1}^{r_2} \frac{dr}{\alpha(r)}$$

By changing to t instead of τ we have global coord fn that compares waves at diff places

$$= \int_{r_2}^{r_2+\lambda_2} \frac{dr}{\alpha(r)} - \int_{r_1}^{r_1+\lambda_1} \frac{dr}{\alpha(r)} \qquad (B)$$

which in the limit $\lambda_i \ll \alpha'$ gives asymptotically,

$$0 = \frac{\lambda_2}{\alpha_2} - \frac{\lambda_1}{\alpha_1} \Rightarrow (1). \checkmark \qquad (C)$$

We also know

$$\underset{\substack{\uparrow \\ \text{freq. of crests}}}{\nu} = \frac{n}{\Delta\tau} \Rightarrow \nu \cdot \lambda = \frac{n\lambda}{\Delta\tau} = \overset{\text{speed of crests}}{\frac{dr}{d\tau}} = \frac{dr}{dt}\frac{dt}{d\tau}$$

$$= \alpha c \cdot \frac{1}{\alpha} = c \quad \text{as expected}$$

(could take as assumption)

Thus

$$c = \nu_2 \lambda_2 = \nu_1 \lambda_1 \quad \Rightarrow \quad \nu_2 \alpha_2 = \nu_1 \alpha_1$$

which is 2.

For 3, note that

$$\cancel{\nu_2 \alpha_2 = \nu_2 \frac{\alpha_2}{\alpha_1} \alpha_1 = \nu_1 \alpha_1}$$

$$\boxed{\nu_2 \alpha_2 = \nu_1 \alpha_1}$$

$$\Rightarrow \quad \nu_2 = \frac{\alpha_1}{\alpha_2} \nu_1 = \frac{\sqrt{1 + \frac{2\varphi_1}{c^2}}}{\sqrt{1 + \frac{2\varphi_2}{c^2}}} \nu_1$$

(D)

$$\frac{\sqrt{1+\frac{2\varphi_1}{c^2}}}{\sqrt{1+\frac{2\varphi_2}{c^2}}} \sim \frac{1 + \frac{\varphi_1}{c^2} + O(\,)^2}{1 + \frac{\varphi_2}{c^2} + O(\,)^2} \sim \frac{1 + \frac{\varphi_1}{c^2} + O(\,)^2}{\left(1 + \frac{\varphi_2}{c^2}\right)\left(1 + O(\,)^2\right)} = \left(1 + \frac{\varphi_1}{c^2}\right)\left(1 - \frac{\varphi_2}{c^2}\right) + \text{HOT}$$

$$= 1 - \frac{\varphi_2 - \varphi_1}{c^2} + \text{HOT} = 1 - \frac{\Delta\varphi}{c^2} + \text{HOT}$$

$\therefore$ in limit of weak fields,

$$\nu_2 = \nu_1 - \nu_1 \frac{\Delta\varphi}{c^2}$$

$$\nu_2 - \nu_1 = -\frac{\nu_1}{c^2}\Delta\varphi$$

$$\Delta\nu = -\frac{\nu_1}{c^2}\Delta\varphi.$$

$\varphi \sim -\frac{GM}{r}$

earth sun

Since when $2 \equiv$ earth, $1 \equiv$ sun, $\Delta\varphi = \varphi_2 - \varphi_1 > 0$, we get that $\Delta\nu < 0$ when light moves out of grav. pot. well, $\Rightarrow \nu_2 - \nu_1 < 0$

$\Rightarrow \nu_2 < \nu_1 \Rightarrow$ shift to the red ✓

- **Note**: (1), (2), (3) apply when the metric is static

Def'n: g_{ij} is static if $g_{i0} = 0$ $i \neq 0$ and g_{ij} is indep't of x^0. (no assumption $|g_{ij}| \ll 1$!)

To see this: let $x^i(\xi)$ denote the path of a light ray in x-coordinates. Then

$$-g_{00}(\dot{x}^0)^2 + g_{ij}\dot{x}^i\dot{x}^j = 0 \qquad i,j = 1,..3.$$

Let $dr = \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, d\xi$. If we reparameterize $x^i(\cdot)$ wrt arclength dr, then $\xi = r$ &

$$-g_{00}(\dot{x}^0)^2 + 1 = 0$$

$$c\frac{dt}{dr} = \frac{dx^0}{dr} = \frac{1}{\sqrt{g_{00}}} = \frac{1}{\alpha}$$

$$\Rightarrow \frac{dr}{dt} = \alpha c \quad \Rightarrow \text{(A), (B), (C) follow} \checkmark$$

Note (2), involving Newtonian limit, only applies when (D) is valid.

Lemma: if g is static,

then geodesics for which $\frac{d\underset{\sim}{x}}{dt} \ll 1$ solve

(to leading order in $\frac{d\underset{\sim}{x}}{dt}$)

$$\frac{d^2 x^i}{d(x^0)^2} = \left(g_{00}^{1/2}\right)_{,i}$$

(EIP)

Application: Schwarzschild metric:

$$\text{(S)} \quad ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Here: $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$ area element on unit sphere $\theta \equiv x^2$ $\varphi \equiv x^3$ $x^0 = ct$. $x^1 = r$.

Conclude: (S) is static, with singularity at $r = 2GM$. ($r \sim$ km for sun, cm for earth)

Note: as $r \to \infty$, $ds^2 \sim -d(x^0)^2 + dr^2 + r^2 d\Omega^2$ which is the metric $ds^2 = \eta_{ij} dx^i dx^j$ in spherical coordinates $\Rightarrow$ "asymptotically flat".

Note: $\varphi = -\frac{GM}{r} \equiv$ Newtonian grav. potential

$\Rightarrow$ $M \sim$ mass of sun $G \sim$ Newtons grav. const.

Thm: (S) solves $G_{ij}[g] = 0$ $\Rightarrow$ "empty space soln of E-eqn's that models solar system pt mass at $r=0$"

Claim: the red shift at ~~∞~~ $r=\infty$ for light emitted from a source near Schwarz radius $\to \infty$ as source tends to $r=2GM$.

Proof: (S) ~~stationary~~ static $\Rightarrow$

$$\boxed{\cancel{\Delta\nu = \nu(\infty) - \nu(r)} = -\cancel{\nu(r)} \quad \frac{\quad}{c^2}}$$

$$\nu(\infty) = \nu(r)\frac{\sqrt{g_{00}(r)}}{\sqrt{g_{00}(\infty)}} = \nu(r)\sqrt{1-\frac{2GM}{r}}$$

$$\nu_2\sqrt{(g_{00})_2} = \nu_1\sqrt{(g_{00})_1} \qquad \to 0 \text{ as } r \to \frac{2GM}{r}$$

$\Rightarrow$ ∞-red-shift ✓