Shock Waves in GR. / Oppenheimer-Snyder Model

- Three relavent metrics

(S) $d\bar{s}^2 = -\left(1-\frac{2GM}{\bar{r}}\right)d\bar{t}^2 + \left(1-\frac{2GM}{\bar{r}}\right)^{-1}d\bar{r}^2 + \bar{r}^2 d\Omega^2$

(Schwarzschild)

(IS) $d\bar{s}^2 = -B(\bar{r})d\bar{t}^2 + A(\bar{r})d\bar{r}^2 + \bar{r}^2 d\Omega^2$

(Interior Schwarzschild)

(R-W) $ds^2 = -dt^2 + R^2(t)\left\{\frac{1}{1-kr^2}dr^2 + r^2 d\Omega^2\right\}$

(Robertson-Walker)

- (S) is the "only" empty space spherically symmetric soln of $G=0$.
- (IS) is a spherically symm. soln of $G=\kappa T$ with stationary source $T^{ij}=(\rho+p)u^i u^j + p g^{ij}$

- (R-W) is the ~~metric~~ that ~~is~~ maximally symmetric at each fixed t. "isotropic" ~~about each point~~ and homogeneous about each point.

- (R-W) is the maximally symmetric homogeneous isotropic space of constant positive curvature at $t=t_0$. In fact, $r \to \alpha r$ changes K by a factor α^2 and R by a factor α. When $K=1$,

$$\text{3-d Ricci Scalar Curvature} = \left[n(n-1) R^2\right]^{-1}$$

Ref Wein Ch 13

I-S A, B, p, ρ all fn's of r alone: $G = \kappa T \Leftrightarrow$

Comoving $\Leftrightarrow u^i = (\sqrt{B}, 0, 0, 0)^i$ is assumed

(1) $$\frac{B'}{B} = -\frac{2p'}{p+\rho}$$

(2) $$A = \left(1 - \frac{2GM(r)}{r}\right)^{-1}$$

(3) $$M(r) = \int_0^r 4\pi \xi^2 \rho(\xi)\, d\xi \Leftrightarrow M' = 4\pi r^2 \rho$$

(4) $$-r^2 p'(r) = GM\rho \left(1 - \frac{p}{\rho}\right)\left(1 + \frac{4\pi r^3 p(r)}{M}\right)\left(1 - \frac{2GM}{r}\right)^{-1}$$

(3) + (4) + $p = p(\rho)$ yield 2 equn's in 2 unknowns:

$$\begin{pmatrix} \rho \\ M \end{pmatrix}' = F\begin{pmatrix} \rho \\ M \end{pmatrix}$$

☒ (R-W) We assume fluid is co-moving,

$\Leftrightarrow u^i = (1,0,0,0)^i$. Then $G = \kappa T \Rightarrow$

$$3\ddot{R} = -4\pi G(\rho + 3p)R \qquad (0,0)$$

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi G(\rho - p)R^2 . \qquad (i,i)$$

Eliminating $\ddot{R}$ yields

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 . \qquad (5)$$

$\operatorname{div} T = 0$ (a consequence of $G = \kappa T$, & hence of $(0,0)$ and (i,i)) yields

$$p = \frac{-\frac{d}{dt}(\rho R^3)}{3R^2\dot{R}} \qquad (6)$$

Given $p = p(\rho)$ (eqn of state), (5) + (6) give 2 eqns in 2 unknowns (R, ρ) of form

$$\begin{pmatrix} \dot{R} \\ \dot{\rho} \end{pmatrix} = F\begin{pmatrix} R \\ \rho \end{pmatrix} \qquad R \equiv R(t) \quad \rho \equiv \rho(t)$$

Matching R-W and I-S solutions:

- Let $(\bar{t}, \bar{r})$ be coords for I-S Soln with equn of state $\bar{p} = \bar{p}(\bar{\rho})$.

  Let (t, r) be coords for R-W Soln with equn of state $p = p(\rho)$

Q: Can we create a dynamical solution of $G = \kappa T$ by <u>matching</u> the I-S soln to the R-W soln across a "shock" interface?

I.e. can we make the metrics g_{IS} and g_{RW} continuous across a shock interface?

<u>Idea</u>: define a smooth mapping

$$\Psi : (r, t) \to (\bar{r}, \bar{t})$$

so that the metric g_{RW} ~~map~~ agrees with g_{IS} on a surface $r = r(t)$ <u>when</u> <u>mapped</u> by Ψ to $\bar{r}\bar{t}$-coordinates

- $$d\bar{s}^2 = -B(r)\,d\bar{t}^2 + A(r)\,d\bar{r}^2 + \bar{r}^2 d\Omega^2 \qquad (\Phi\text{-S})$$

$$ds^2 = -dt^2 + R(t)^2\left\{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right\}$$

In order that 2-spherés of symmetry agree under Φ, ask that

$$\bar{r} = r\,R(t) = \Phi_2(t,r) \qquad \left(\begin{matrix}\text{determining}\\ \Phi_2\end{matrix}\right) \qquad (8)$$

This implies that

$$d\bar{r} = R\,dr + \dot{R}\,r\,dt$$

which yields

$$dr^2 = \frac{1}{R^2}d\bar{r}^2 + \frac{\dot{R}^2}{R^2}r^2 dt^2 - 2\frac{\dot{R}}{R^2}\bar{r}\,dt\,d\bar{r}$$

$\Rightarrow$ in t $\bar{r}$ coords, the R-W metric is given by

$$ds^2 = -\left\{1 - \frac{\dot{R}^2\bar{r}^2}{R^2 - k\bar{r}^2}\right\}dt^2 + \frac{R^2}{R^2-k\bar{r}^2}d\bar{r}^2 + \frac{2R\dot{R}\bar{r}}{R^2-k\bar{r}^2}dt\,d\bar{r} + \bar{r}^2 d\Omega^2 \qquad (9)$$

This is non-diagonal.

Idea: define a mapping $\bar{t} = \bar{t}(t, r)$ that diagonalizes (9), and equate this diagonal form of (R-W) to (I-S):

- Lemma: if

$$d\tilde{s}^2 = -C(t,\bar{r})\,dt^2 + D(t,\bar{r})\,d\bar{r}^2 + 2E(t,\bar{r})\,dt\,d\bar{r} \tag{10}$$

and $\psi = \psi(t, \bar{r})$ satisfies

$$\frac{\partial}{\partial \bar{r}}(\psi C) = -\frac{\partial}{\partial t}(\psi E)$$

then

$$d\bar{t} = \psi(t,\bar{r})\{C\,dt - E\,d\bar{r}\}$$

is exact, can be integrated to $\bar{t} = \bar{t}(t, \bar{r})$, and in $(\bar{t}, \bar{r})$ coords (10) becomes

$$d\tilde{s}^2 = -(\psi^{-2}C^{-1})\,d\bar{t}^2 + \left(D + \frac{E^2}{C}\right)d\bar{r}^2 \tag{11}$$

For (9),

$$C = 1 - \frac{\dot{R}^2 \bar{r}^2}{R^2 - k\bar{r}^2} \quad D = \frac{R^2}{R^2 - k\bar{r}^2} \quad E = \frac{R\dot{R}\bar{r}}{R^2 - k\bar{r}^2}.$$

If we now ask that the coeff of $d\bar{r}^2$ in (11) = coeff of $d\bar{r}^2$ in I-S, we obtain (after simplification)

$$M(\bar{r}) = \frac{4\pi}{3} \rho(t) \bar{r}^3 \qquad (12)$$

This defines "shock surface" $\bar{r} = \bar{r}(t)$.

To match the $d\bar{t}^2$ coeff's as well, we need

$$\psi^{-2} C^{-1} = B$$

$$\Rightarrow \qquad \psi^2 = \frac{1}{CB}. \qquad (13)$$

But ψ solves

$$\frac{\partial}{\partial \bar{r}} \psi C = -\frac{\partial}{\partial t} \psi E \tag{14}$$

in $(t, \bar{r})$-coords. So we can satisfy

$$\psi^2 = \frac{1}{CB} \text{ on initial surface } \bar{r} = \bar{r}(t) \tag{15}$$

(which determines values of C and B on surface) and then solve PDE (14) with i-values on surface.

Miracle: the shock surface equation

$$M = \frac{4\pi}{3} \rho \bar{r}^3 \tag{*}$$

uncouples from the difficult ψ equation $\Rightarrow$ we can pose the ivp for ψ!

Conclude: if $\bar{r} = \bar{r}(t)$ is non-char. for (14) (which we show it is) then (*) defines the "shock surface across which the metrics match.

• Note: (*) gives a global cons. of mass principle:

$$M(\bar{r}) = \frac{4\pi}{3}\rho(t)\,\bar{r}^3 \qquad *$$

total mass inside $\bar{r}$ if I-S soln is continued on in to origin

total mass inside $\bar{r}$ if $\rho \equiv \rho(t) \equiv$ const. ie "at fixed t"

• set $\bar{r} = \bar{r}(t)$ & diff (*):

$$M(\bar{r}(t)) = \frac{4\pi}{3}\rho(t)\,\bar{r}(t)^3$$

[$M' = 4\pi\bar{\rho}\bar{r}^2$

$$\cancel{4\pi}\,\bar{\rho}\,\cancel{\bar{r}^2}\,\dot{\bar{r}} = \frac{\cancel{4\pi}}{3}\dot{\rho}\,\bar{r}^{\cancel{3}} + \cancel{4\pi}\,\rho\,\cancel{\bar{r}^2}\,\dot{\bar{r}}$$

$$(\bar{\rho} - \rho)\,\dot{\bar{r}} = \frac{\dot{\rho}\,\bar{r}}{3}$$

$$\boxed{\frac{d\bar{r}}{dt} = \frac{\dot{\rho}\,\bar{r}}{(\bar{\rho}-\rho)\,3}}$$

gives shock speed in $(t, \bar{r})$-words.

• <u>Note</u>: if $\bar{\rho} = \bar{p} = 0$ (IS → S) we have Oppenheimer - Snyder Case ≈ matched surface is ∂ of star. Then ~~[illegible]~~ $M(\bar{r}) = M_0$ &

$*$ ⟹

$$M_0 = \frac{4\pi}{3} \rho(t)\, \bar{r}(t)^3 = \frac{4\pi}{3} \rho(t)\, R(t)^3 r(t)^3$$

~~Problem: Is energy a~~

$$\Rightarrow \quad r(t) = \left(\frac{3}{4\pi} \frac{M_0}{\rho(t) R(t)^3} \right)^{1/3}$$

From

$$p = - \frac{\frac{d}{dt}(\rho R^3)}{3R^2 \dot{R}} \qquad (6)$$

we have that when $p = 0$ (O-S case)

$$\rho R^3 = \text{const} = \rho(0) \quad \text{if } R(0) = 1 \checkmark$$

$$\Rightarrow \quad r(t) = \text{Const.} = \left(\frac{3}{4\pi} \frac{M_0}{\rho(0)} \right)^{1/3}$$

3 Conservation

We have: given any (I-S) ~~soln~~ metric with equn of state $\bar{p} = \bar{p}(\bar{\rho})$ and (R-W) metric with $p = p(\rho)$, $\exists$ a coord mapping $(t, r) \mapsto (\bar{t}, \bar{r})$,

$$\bar{r} = rR$$
$$\bar{t} = \bar{t}(t, r)$$

such that the metrics match in a Lip cont fashion along shock surface

$$M(\bar{r}) = \frac{4\pi}{3} \rho \bar{r}^3$$

Q: does cons. of energy & momentum hold across the surface.

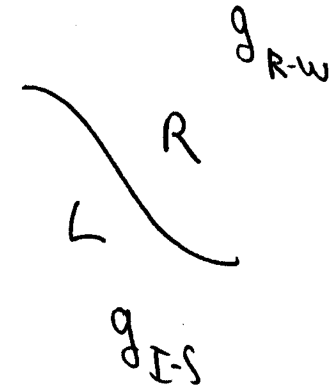
Ist Guess: $G = \kappa T$, $\operatorname{div} G \equiv 0$ by construction

$\Rightarrow$ soln's of $G = \kappa T$ satisfy $\operatorname{div} T = 0 \Leftrightarrow$ local cons of energy-mom in smooth soln's

Might expect: since $\operatorname{div} G_{RW} = 0$ on one side of shock & $\operatorname{div} G_{IS} = 0$ on other, and g Lipschitz con't across the shock $\overset{??}{\Longrightarrow}$ G satisfies the weak form of cons. at each pt on the shock

$$[G^{ij}]\, n_i = 0$$

$$[G] = G_R - G_L$$



"Rankine–Hugoniot Jump Relations"

Not so: but following is true:

Theorem (ISRAEL) Assume two metrics match Lip cont across a shock Σ. Let K denote the second fundamental form on the shock surface:

$$K : T_\Sigma \to T_\Sigma \quad X \longmapsto \nabla_n X$$

so that K is determines separately by the covariant derivative ∇ on each side of Σ.

Assume that $[K]=0$ at each pt on Σ.

Then

① $\exists$ a $C^{1,1}$ coord transformation in a nbhd of each pt on Σ st the metric is C^1 across Σ in the new coords.

$\approx>$ "G = a second order operator on g contains jumps, but no δ-fn singularities on Σ."

② $[G^{ij}]n_i = 0$ at each pt on Σ.

◈ We show that for the spherically symmetric case, $[K]=0$ across Σ iff

$$(*) \qquad [T^{ij}]n_i n_j = 0$$

<u>Conclude</u>: Conservation $[T^{ij}]n_i = 0$

$\Leftrightarrow$ <u>two</u> nontrivial conditions <u>reduces</u> to the single condition (*) when the metric is Lipschitz cont across the shock.

We use this derive the following condition for conservation:

$$[T^{ij}]n_i n_j = (\bar{p}+\rho)\dot{r}^2 - \frac{\bar{\rho}+\bar{p}}{R^2\psi^2C^2}\dot{\bar{r}}^2 + (p-\bar{p})\frac{1-kr^2}{R^2} = 0 \qquad \text{(Cons)}$$

<u>Note</u>: in case of Oppenheimer-Snyder:

$\bar{p} = \bar{\rho} = 0 \Rightarrow$

$$[T^{ij}]n_i n_j = \rho \dot{r}^2 + p\,\frac{1-kr^2}{R^2} = 0$$

$\Rightarrow \dot{r} = 0$ and $p = 0$ $\quad (1-kr^2 > 0)$

$\Rightarrow$ "can only match to empty space Schwarzschild, conservatively when $p=0$, in which case shock surface is $\dot{r}=0$."

- When $p \neq 0$, we assume O-S soln is fixed: then R-W satisfies

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 \qquad (A)$$

$$p = \frac{-\frac{d}{dt}(\rho R^3)}{3R^2\dot{R}} . \qquad (B)$$

To accomodate (Cons), we let $p = p(t)$ be determined by (B), in which case the

ODE is

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 \qquad (C)$$

$$(\bar{p}+\bar{\rho})\dot{r}^2 - \frac{\bar{\rho}+\bar{p}}{R^2\psi^2 C^2}\dot{\bar{r}}^2 + (p-\bar{p})\frac{1-kr^2}{R^2} = 0 \qquad (D)$$

holds on shock surface.

<u>But</u>: on shock surface, $\bar{r}(t) = R(t)r(t)$,

$\Rightarrow$ $\bar{p}(\bar{r})$, $\bar{\rho}(\bar{r})$, $\bar{r}$ can be replaced with fn's of (r, R). <u>Moreover</u>, on shock surface,

$$\psi^2 C^2 = \frac{A(\bar{r})}{B(\bar{r})(1-kr^2)}$$

$\Rightarrow$ also a fn of (R, r). <u>Moreover</u>,

$$M = \frac{4\pi}{3}\rho\,\bar{r}^3$$

$$\Rightarrow \rho(t) = \frac{M(Rr)}{(Rr)^3} \equiv \frac{M(R(t)r(t))}{(R(t)r(t))^3}$$

Conclude: (C) & (D) give two autonomous ODE's in $R(t)$ and the shock position $r(t)$

$$\begin{pmatrix} \dot{r} \\ \dot{R} \end{pmatrix} = F\begin{pmatrix} r \\ R \end{pmatrix}.$$

These determine R-W metric $(R(t))$ and pressure $p(t)$ thru

$$p = \frac{-\frac{d}{dt}(\rho R^3)}{3R^2\dot{R}}$$

Problems: ① local soln near $\dot{r}=0$ $\dot{R}=0$ $R=1$ (lack of Lip. cont)

② Global behavior of shock