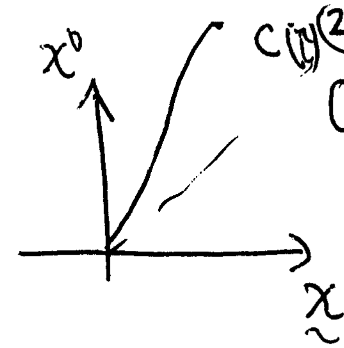
Momentum & Energy $\boxed{\text{SR-III}}$



Consider a mass m moving thru a fixed Lorentz frame $x = (x^0, \dots, x^3)$, $x^0 = ct$.

- classical momentum: $\underset{\sim}{p} = m \dfrac{d\underset{\sim}{x}}{dt}$, $E = \frac{1}{2} m \underline{v}^2$ conserved. These are coord. dependent expressions, not conserved relativistically
- To make momentum relativistically invariant, parameterize path wrt coord indept parameter

$$x = x(\tau) \equiv x \circ c(\tau)$$

$$ds = c\,d\tau\,\sqrt{-\left\{-(dx^0)^2 + \dots + d(x^3)^2\right\}}$$

<u>Defn</u>: The 4-velocity of the particle is

$$u = \frac{dc}{d\tau} \equiv u^i \frac{\partial}{\partial x^i} \in T_{c(\tau)} M^4$$

$$u^i = \frac{dx^i}{d\tau}$$

The 4-momentum of particle $P = mu \in T_{c(\tau)} M^4$

<u>Note</u>: u has dimensions of velocity,

$\langle \frac{u}{c}, \frac{u}{c} \rangle = -1$, u timelike FIP

- Let n ~~mass~~ pt masses $m_1, \dots, m_n$ interact.

Let $p_1 \dots p_n$ be their 4-momenta before interaction, and $q_1, \dots, q_n$ their 4-momenta after interaction.

<u>Defn</u>: We say 4-momentum is conserved in the interaction in L-frame x if

$$\sum_{k=1}^{n} p_k^i = \sum_{k=1}^{n} q_k^i \quad i = 0, \dots, 3.$$

4 quantities separately are conserved.

Theorem: If 4-momentum is conserved in one ~~Lorentz~~ Inertial frame then it is conserved in all I-frames:

Proof: Choose $y^\alpha = A^\alpha_i x^i$ a Lorentz transformation, A^α_i a constant 4x4 matrix. Assume conservation of 4-momentum in x-coordinates:

$$\sum_{k=1}^{n} \left(p^i_k - q^i_k\right) = 0.$$

Then in y-coordinates (p_k, q_k are vectors)

$$p^\alpha_k = A^\alpha_i p^i_k \qquad q^\alpha_k = A^\alpha_i q^i_k.$$

Thus

$$\sum_{k=1}^{n} \left(p^\alpha_k - q^\alpha_k\right) = \sum_{k=1}^{n} A^\alpha_i \left(p^i_k - q^i_k\right) = A^\alpha_i \sum_{k=1}^{n} \left(p^i_k - q^i_k\right)$$

consequence of linearity of transformation alone.

<u>Conclude</u>: "Conservation" is meaningful for the components of a vector, and momentum $p = m\frac{dc}{d\tau}$ is the only "geometric" vector associated with classical momentum.

- We write p in terms of the classical momentum $p_{cl} = m\underset{\sim}{v}$:

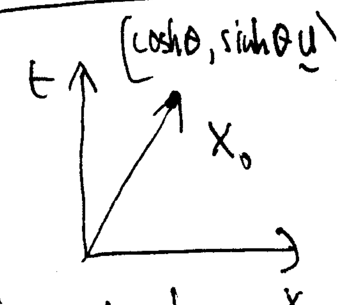
$$p^i = m\frac{dx^i}{d\tau} = m\frac{dx^i}{dt}\frac{dt}{d\tau}$$

But (param. path $c = c(t)$ wrt t:)

$$ds^2 = c^2 d\tau^2 = \left\{-\left(\frac{dx^0}{dt}\right)^2 + \sum_{i=1}^{3}\left(\frac{dx^i}{dt}\right)^2\right\}dt^2$$

$$= \left(-c^2 + |\underset{\sim}{v}|^2\right)dt^2$$

$$\boxed{\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\left|\frac{\underset{\sim}{v}}{c}\right|^2}} \equiv \gamma = \cosh\theta}$$



$$\gamma = 1 + \frac{1}{2}\left|\frac{\underset{\sim}{v}}{c}\right|^2 + O\left(\frac{\underset{\sim}{v}}{c}\right)^4$$

$\Rightarrow$ "$\Delta\tau = 1$ when $\Delta t = \cosh\theta$"

Thus:

$$p^0 = m \frac{dx^0}{dt}\frac{dt}{d\tau} = mc\gamma$$

$$p^i = m \frac{dx^i}{dt}\frac{dt}{d\tau} = m v^i \gamma$$

$$\Rightarrow \quad p^0 = mc\left(1 + \frac{1}{2}\left|\frac{\underset{\sim}{v}}{c}\right|^2 + \text{hot}\right)$$

$$p^i = m v^i \left(1 + \frac{1}{2}\left|\frac{\underset{\sim}{v}}{c}\right|^2 + \text{hot}\right) \qquad \text{hot} \sim \left|\frac{\underset{\sim}{v}}{c}\right|^4$$

$$\Leftrightarrow \quad c p^0 = mc^2 + \frac{1}{2} m |\underset{\sim}{v}|^2 + O\left(\frac{\underset{\sim}{v}}{c}\right)^4$$

$$p^i = m v^i + O\left(\frac{\underset{\sim}{v}}{c}\right)^2$$

Conclude: $\left|\frac{\underset{\sim}{v}}{c}\right| \ll 1$, cp^0 agrees with classical momentum, and p^i agrees with classical K.E. to within the constant mc^2. $[cP_0]$ = energy

$[P_i]$ = momentum

CONCEPTUAL LEAP:

ENERGY $= c\, p^0$

MOMENTUM $= \underset{\sim}{p}$

are the true conserved physical quantities.

- Since $E = mc^2 + \underbrace{\frac{1}{2} m |\underset{\sim}{v}|^2}_{\text{Kinetic energy}} + O\left(\frac{|\underset{\sim}{v}|^4}{c^2}\right)$,

if one imagines that masses can change during interactions, conservation of E implies that mc^2 is the amt of "rest" energy that can be converted into kinetic energy during interactions. Since $mc^2 \gg mv^2$, small changes in rest mass $\Rightarrow$ large releases of K.E with obvious implications.

- EINSTEIN viewed the above as the greatest prediction of his theory.

<u>Conservation Laws</u>: $\rho(x) = \frac{\text{mass}}{\text{3-volume}}$ in frame fixed with fluid particle. Here, let $u = \frac{dx}{ds}$, (dim'less 4-vel). $\rho u \equiv$ 4-mass-flux vector. Assume <u>dust</u> $p=0$

- in L-frame fixed with fluid, $u = (1,0,0,0) \Rightarrow$

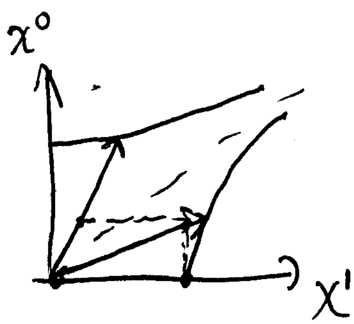
$$\rho u^0 = \rho = \frac{\text{mass}}{dx^1 dx^2 dx^3}$$

- under L-transformation, ($u \equiv X_1$ unit timelike vector)

$$\begin{bmatrix} 1 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & & \\ 0 & & id & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 1 \\ \bar{x} \\ 1 \end{bmatrix}$$

Assume fluid <u>fixed</u> rel to $\bar{x}$-coord system

$$u^\alpha = (\cosh\theta, \sinh\theta, 0, 0) = \left(\gamma, \gamma \frac{v^1}{c}, 0, 0\right)$$



$$L_{\bar{x}} = \gamma L_x \qquad d\bar{x}^1 = \gamma\, dx^1, \quad d\bar{x}^2 = dx^2, \quad d\bar{x}^3 = dx^3$$

$$\Rightarrow \rho u = \left(\rho\gamma, \rho\gamma \frac{v^1}{c}, 0, 0\right)$$

$$\rho\gamma = \frac{d\,\text{mass}}{d\bar{x}^1 d\bar{x}^2 d\bar{x}^3} \cdot \gamma = \frac{d\,\text{mass}}{\cancel{\gamma} dx^1 dx^2 dx^3} \cancel{\gamma} \equiv$$

mass density as measured in moving frame. x

Here – ρ is $\frac{\text{mass}}{\text{3-vol}}$ in the frame $\bar{x}$ moving <u>with</u> the fluid

$$\rho u^1 = \rho\gamma \frac{v_1}{c} \equiv \text{"mass flux"} \equiv \frac{d\,\text{mass}}{dx^2 dx^3\, c\,dt} = \text{"mass per time thru area with normal } \frac{\partial}{\partial x^1}\text{"}$$

$\Rightarrow$ conclude: conservation of mass is expressed in any L-frame by

$$\int_{\partial \boxed{4}} \rho u \cdot n \, dA = \text{mass(top)} - \text{mass(bottom)} + \text{amt passing out thru } \partial$$

$$= 0$$

$$\underset{\text{"Gauss Thm"}}{=} \int_{\boxed{4}} \operatorname{div} \rho u \, dV$$

$\Leftrightarrow$ $\operatorname{div} \rho u = 0$ continuity equation.

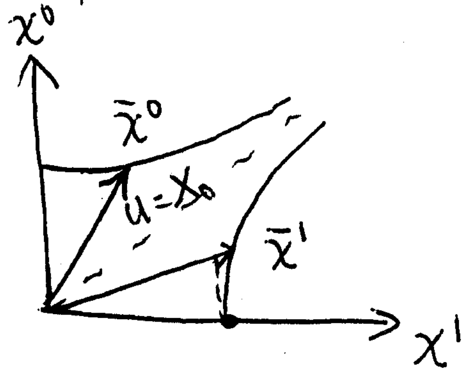
Gauss Thm: $\displaystyle\int_{\partial\Omega} X^i d\Sigma_i = \int_{\Omega} X^i{}_{;i}\, \varepsilon$ $\boxed{\text{SKIP}}$

$\varepsilon = dx^0 \wedge \cdots \wedge dx^3 \equiv$ vol form

$\varepsilon \lrcorner n = \varepsilon(n, -, -, -) \equiv d\Sigma_i$ 3-form.

$\Rightarrow$ can localize global cons. law: Problem w. stress tensor.

Energy & Momentum Flux Vectors:



Moving x frame sees contracted length $\Rightarrow$ increase in mass density by factor γ.

But $p^0 = mu^0 = m\gamma \Rightarrow$ increase in energy by another factor γ. $\Rightarrow$ i.e. "m" $\xRightarrow{\text{cont lim}}$ "$\rho\gamma = \frac{\text{rest mass}}{\text{3-vol}}$"

so "mu^0" $\Rightarrow$ "$\rho\gamma\cdot\gamma = \frac{\text{energy}}{\text{3-vol}}$"

Energy flux vector: $\rho\gamma u = \rho u^0 u$

Momentum flux vector $(\rho u^i)u \equiv \rho u^i u$

Stress-Energy Tensor for fluid (no pressure - only the density contributes to energy & mom:)

$$T^{ij} = \rho u^i u^j$$

Cons of 4-momentum in L-frame $\equiv$

$$\operatorname{div} T = 0. \equiv T^{ij}{}_{,j} = 0$$