Example: Conservation Laws

Compressible Euler Equations:

\[ \begin{align*}
\rho_t + \text{div} (\rho u) &= 0 \\
(\rho u^i)_t + \text{div} (\rho u^2 u + p e^i) &= 0 \\
E_t + \text{div} (S(E+p)) &= 0
\end{align*} \]

In 1-d:

\[ \begin{align*}
\begin{pmatrix} 
\rho_t \\
\rho u_t \\
E_t
\end{pmatrix} + \begin{pmatrix} 
\rho u \\
\rho u^2 + p \\
(E+p)u_x
\end{pmatrix} &= 0
\end{align*} \]

Cons. Mass
Cons. Momentum
Cons. Energy

\[ E = \rho e + \frac{1}{2} \rho u^2 \]

\( \rho \) = density
\( u \) = velocity
\( E \) = energy
\( \frac{E}{\text{vol}} \) = specific internal energy
\( \frac{E}{\text{mass}} \) = energy per mass
In general - an equation of state required to get 3 equations in 3 unknowns:

\[ e = e(v, s) \quad v = \frac{1}{\rho} \]

\[ p = e_v(v, s) \quad s = \text{specific entropy} \]

Unknowns: \((\rho, v, s)\)

OR

\[ e = e(p, s) \]

Unknowns: \((s, \rho, p)\)

General nxn system of conservation laws:

\[ \dot{U}_t + F(U)_x = 0 \]

\( F - \text{nonlinear Euler: } U = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} \)

\( U = \text{"the conserved quantities"} \)
\[ U_t + A(U) \cdot U_x = 0 \quad A(U) = \frac{\partial F}{\partial U} \quad (x) \]

Under change of dependent variables:

\[ V = \Phi(U) \]

eg.
\[ \left( \begin{array}{c} \psi \\ \phi \end{array} \right) = \Phi \left( \begin{array}{c} \psi \\ \phi \end{array} \right) \]

Then

\[ V_t = \frac{\partial V^d}{\partial U^j} \frac{\partial U^j}{\partial t} \]

\[ V_x = \frac{\partial V^d}{\partial U^j} \frac{\partial U^j}{\partial x} \]
Conservation Law:

\[ V_t^i + \frac{\partial F^i}{\partial U^j} U_x^j = 0 \]

Multiply by \( \frac{\partial V^d}{\partial U^i} \):

\[ \frac{\partial V^d}{\partial U^i} V_t^i + \frac{\partial V^x}{\partial U^i} \frac{\partial F^i}{\partial U^j} \frac{\partial U^\sigma}{\partial U^r} \frac{\partial U^t}{\partial U^j} U_x^i \]

\[ \bar{V}^\lambda_\tau \]

\[ \bar{V}^\tau_\tau \]

\[ J^{-1} A J \]

\[ \bar{V}^\lambda_\tau + \frac{\partial V^d}{\partial U^i} \frac{\partial F^i}{\partial U^j} \frac{\partial U^\sigma}{\partial U^r} \frac{\partial U^t}{\partial U^j} V_x^\tau \]

\[ \bar{A}^\lambda_\tau \]
Conclude: under change of dependent variable, $U_t + A(\tilde{\nu}) U_x$ transforms like:

$U^2_{\nu_t} U^2_{\nu_x}$ transform like a vector:

$$\tilde{V}^a = \frac{\partial V^a}{\partial U^i} U_t$$

$A = A^i_j$ transforms like a (1,1)-tensor:

$$\tilde{A}^a_b = A^i_j \frac{\partial U^j}{\partial U^b} \frac{\partial V^a}{\partial U^i}$$

Conclude: For any choice of state variables $\nu$,

Conclude: The eigenvalues of $A$ can be computed from the equation $V_t + \tilde{A} \cdot \dot{V}_x$ for any state variables $V$. 
The right eigenvectors of $A$ transform like vectors:

$$ A \mathbf{R} = \lambda \mathbf{R} $$

$$ \Rightarrow \quad A \tilde{\mathbf{R}} = \lambda \tilde{\mathbf{R}} $$

where

$$ \tilde{\mathbf{R}}^k = R^i \left[ \frac{\partial \mathbf{U}^k}{\partial U^i} \right] $$

In this example, $U$ & $V$ represent coordinates on the "manifold" of dependent variables.

The left eigenvectors of $A$ transform like (components of) $\mathbf{R}$ vectors:

$$ L \mathbf{A} = \lambda L \mathbf{A} $$
The left eigenvectors of $A$ transform like components of co-vectors:

$$LA = \lambda L \quad (L_i A^i_j = \lambda L_j)$$

$$L\bar{A} = \lambda L \quad (L_\alpha A^\alpha_\beta = \lambda L_\beta)$$

where

$$L_\alpha = \frac{2U^i}{\partial v^d} L_i$$

Moreover: $LA = \lambda L$ has a soln iff

$$\det |\lambda I - A| = 0$$

$\Rightarrow \lambda$ is a root of char polyn $\Rightarrow$ right/left eigenvalues are the same.

Theorem:
Lemma: if \( LA = \lambda_1 L \), \( AR = \lambda_2 R \), then
\[
L - R = 0
\]

Pf.
\[
L - A R = \lambda_1 L - \lambda_2 R = \lambda_1 L - \lambda_2 R
\]
\[
= \lambda_1 (L - R) = \lambda_1 L - \lambda_2 R
\]
\[
\Rightarrow L - R = 0 \quad \text{unless} \quad \lambda_1 = \lambda_2.
\]

Definition: system is strictly hyperbolic if \( A \) has \( n \) real & distinct eigenvalues at each state value \( U \).

Oft quoted: "\( L^k \) is orthogonal to all the right eigenvectors \( R_e \), \( k \)th."
The eigenvalues and eigenvectors associated with system (c) are important as follows:

Construction: Let $R^i(U), \lambda(U)$ be the eigenvector-eigenvalue fields associated with matrix $A(U) = A^i_j(U)$. Let $U^i(x, t)$ be any smooth function constructed as a solution to the condition, i.e.,

$$\frac{d}{ds} U^i(s) = R^i(U(s)).$$

Let $U(s)$ denote an integral curve of the vector field $R^i$ defined on $U$-space; i.e.,
Let $\lambda(U(s))$ be the eigenvalue along the integral curve.

**Defn**: $\lambda$ is genuinely nonlinear if $\forall \lambda \cdot R_i \neq 0$.

I.e., $\lambda$ is monotone increasing or decreasing along $U(s)$.

Let $U(x,t)$ be any function satisfying:

- $U(x,t) = U(s)$ along lines in the $xt$-plane of speed $\frac{dx}{dt} = \lambda(U(s))$.

**Theorem**: $U(x,t)$ is a solution of (c). Such a cnm is called a $\lambda$-simply wave.
Assume $\nabla \cdot R_i \neq 0$ along integral curve $R$, so that we can take $U = U(\lambda)$ as parametrically $R$.

**Theorem:** Let $\lambda(x,t)$ denote any smooth solution of

$$\lambda_t + \lambda \lambda_x = 0.$$  

Then

$$U(\lambda(x,t))$$

is a smooth soln of

$$U_t + A(U)x = 0.$$  

**Cor.:** Since $\lambda$ is a scalar, this construction is valid in any set of variables $V$, so long as $V'(\lambda) = R$.
Proof:  \[ U_t = U' \lambda_t = R \lambda_t \]

\[ U_x = U' \lambda_x = R \lambda_x \]

Thus

\[ U_t + A(U) U_x = R \lambda_t + A R \lambda_x \]

\[ = R \lambda_t + R \lambda \lambda_x = R(\lambda_t + \lambda \lambda_x) \cdot 0 \]
Comment: $\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0$ is in conservation form. Not all systems of form

$$\mathbf{V}_t + \mathbf{A}(\mathbf{V})_x = 0$$

can be written in conservation form. In fact — it is the conservation form that determines the shock-waves.

Note: $\mathbf{L}_i$ is a left eigenvector of