Example: Conservation Laws

Compressible Euler Equations:

$$\rho_t + \operatorname{div}(\rho u) = 0$$

$$(\rho u^i)_t + \operatorname{div}(\rho u^i u + p e^i) = 0$$

$$E_t + \operatorname{div}(u(E+p)) = 0$$

In 1-d:

$$\begin{pmatrix} \rho_t \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = 0 \qquad \begin{array}{l} \text{cons. mass} \\ \text{cons. mom} \\ \text{cons. energy} \end{array}$$

$$E = \rho e + \tfrac{1}{2}\rho u^2$$

ρ = density $\quad$ e = specific internal energy $= \dfrac{\text{energy}}{\text{mass}}$

u = velocity

$E = \dfrac{\text{energy}}{\text{vol}}$

In general - an equation of state required to get 3 equations in 3 unknowns:

Eg. $e = e(v, s)$ $\quad v = \frac{1}{\rho}$

$p = e_v(v, s)$ $\quad s \equiv$ specific entropy

Unknowns: (ρ, u, s)

OR

$e = e(p, \rho)$

Unknowns (ρ, u, p)

General $n \times n$ system of conservation laws:

$$U_t + F(U)_x = 0$$

F - nonlinear $\quad$ Euler: $U = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}$

$U \equiv$ "the conserved quantities"

(9) $$\bar{U}_t + A(\bar{U})\cdot\bar{U}_x = 0 \qquad A(\bar{U}) = \frac{\partial F^i}{\partial \bar{U}^j} \qquad (c)$$

$(n\times 1) \quad n\times n \quad n\times 1$

Under change of <u>dependent</u> <u>variables</u>:

$$\bar{V} = \phi(\bar{U})$$

eg.,
$$\begin{pmatrix} \rho \\ u \\ S \end{pmatrix} = \phi\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}.$$

Then

$$\bar{V}_t^\alpha = \frac{\partial \bar{V}^\alpha}{\partial \bar{U}^j}\, U_t^j$$

$$\bar{V}_x^\alpha = \frac{\partial \bar{V}^\alpha}{\partial \bar{U}^j}\, \bar{U}_x^j$$

Conservation Law:

$$\bar U_t^i + \frac{\partial F^i}{\partial \bar U^j}\bar U_x^j = 0$$

mult by $\frac{\partial \bar V^\alpha}{\partial \bar U^i}$:

$$\underbrace{\frac{\partial \bar V^\alpha}{\partial \bar U^i}\bar U_t^i}_{\bar V_t^\alpha} + \underbrace{\frac{\partial \bar V^\alpha}{\partial \bar U^i}\frac{\partial F^i}{\partial \bar U^\sigma}\frac{\partial \bar U^\sigma}{\partial \bar V^\tau}}_{J^{-1}AJ}\underbrace{\frac{\partial \bar V^\tau}{\partial \bar U^j}\bar U_x^j}_{\bar V_x^\tau}$$

$$\delta^\sigma_j$$

$$\bar V_t^\alpha + \underbrace{\frac{\partial \bar V^\alpha}{\partial \bar U^i}\frac{\partial F^i}{\partial \bar U^\sigma}\frac{\partial \bar U^\sigma}{\partial \bar V^\tau}}_{\bar A^\alpha_\tau} V_x^\tau$$

Conclude: under change of dependent variable, $U_t + A(\bar U) U_x$

transforms like:

$\bar U^i_t, \bar U^i_x$ transform like a vectors: $\boxed{\bar V^\alpha_t = \frac{\partial V^\alpha}{\partial \bar U^i} U^i_t}$

$A \equiv A^i_j$ transforms like a $\binom{1}{1}$-tensor:

$$\boxed{\bar A^\alpha_\beta = A^i_j \frac{\partial \bar U^j}{\partial V^\beta} \frac{\partial V^\alpha}{\partial \bar U^i}}$$

~~Conclude: For any choice of state variables V,~~

Conclude: The eigenvalues of A can be computed from the equation $V_t + \bar A \cdot V_x$ for any state variables V.

- The right eigenvectors of A transform like vectors:

$$AR = \lambda R$$

$$\Rightarrow \quad \bar{A}\bar{R} = \lambda \bar{R}$$

where

$$\boxed{\bar{R}^\alpha = R^i \frac{\partial \bar{V}^\alpha}{\partial \bar{U}^i}}$$

In this example, $\bar{U}$ & $\bar{V}$ represent coordinates on the "manifold" of dependent variables

---

- ~~The left eigenvectors of A transform like (components) of co-vectors:~~

$$\cancel{LA = \lambda A}$$

- the left eigenvectors of A transform like components of co-vectors:

$$LA = \lambda L \quad (L_i A^i_j = \lambda L_j)$$

$$\Rightarrow \quad \bar{L}\bar{A} = \lambda \bar{L} \quad (\bar{L}_\alpha A^\alpha_\beta = \lambda L_\beta)$$

where

$$\boxed{\bar{L}_\alpha = \frac{\partial U^i}{\partial \bar{V}^\alpha} L_i}$$

<u>Moreover</u>: $LA = \lambda L$ has a soln iff

$$\det|\lambda I - A| = 0$$

$\Rightarrow$ iff λ is a root of char polyn $\Rightarrow$ right/left eigenvalues are the same.

~~theorem~~:

<u>Lemma</u>: if $LA = \lambda_1 L$, $AR = \lambda_2 R$

$\lambda_1 \neq \lambda_2$, then

$$L_\sigma R^\sigma = 0$$

Pf.

$$L_\sigma A^\sigma{}_\tau R^\tau = \lambda_1 L_\tau R^\tau$$

$$= \lambda_2 L_\tau R^\tau$$

$\Rightarrow$ $L_\tau R^\tau = 0$ unless $\lambda_1 = \lambda_2$. ✓

<u>Defn</u>: system is <u>strictly hyperbolic</u> if A has n real & distinct e-values at each state value $\bar{U}$.

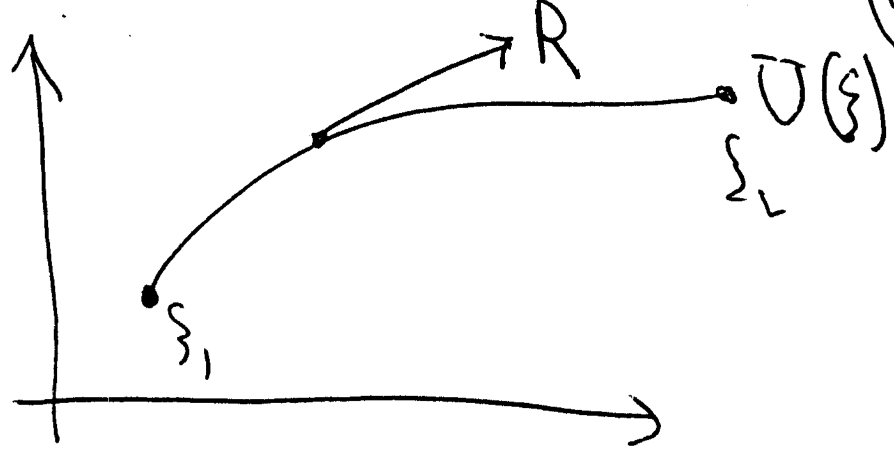
oft quoted: "L^k is orthogonal to all the right eigenvectors R_ℓ, $\ell \neq k$"

The eigenvalues & eigenvectors associated with system (c) are important as follows:

<u>Construction</u>: Let $R^i(\bar{U})$, $\lambda(\bar{U})$ be the eigenvector – eigenvalue fields associated with matrix $A(\bar{U}) \equiv A^i_j(\bar{U})$. Let $\bar{U}^i(x,t)$ be any smooth function ~~that satisfies the condition~~ constructed as follows: Let $\bar{U}(\xi)$ denote an integral curve of the vector field R^i defined on $\bar{U}$-space; i.e.,

$$\frac{d}{d\xi}\bar{U}^i(\xi) = R^i(\bar{U}(\xi)).$$

Picture:



Let $\lambda(\bar{U}(\xi))$ be the eigenvalue along the integral curve.

Defn: λ is genuinely nonlinear if $\nabla\lambda \cdot R_i \neq 0$

I.e., λ is monotone increasing or decreasing along $\bar{U}(\xi)$.

Let $\bar{U}(x,t)$ be any function satisfying:

$$U(x,t) \equiv \bar{U}(\xi) \quad \text{along lines in the } xt\text{-plane of speed } \frac{dx}{dt} = \lambda(\bar{U}(\xi))$$

$\xi = \xi(x,t)$ a smooth fn

Theorem: $\bar{U}(x,t)$ is a solution of (c). Such a soln is called a λ-simple wave.

Assume $\nabla\lambda \cdot R_i \neq 0$ along integral curve R, so that we can take $U = U(\lambda)$ as parameterizing R.

<u>Theorem</u>: Let $\lambda(x,t)$ denote any smooth solution of

$$\lambda_t + \lambda \lambda_x = 0.$$

Then

$$U(\lambda(x,t))$$

is a smooth soln of

$$U_t + A(U)_x = 0.$$

<u>Cor</u>: Since λ is a <u>scalar</u>, this construction is valid in any set of variables V, so long as

$$V'(\lambda) = R$$

Proof:

$$\bar{U}_t = U' \lambda_t = R \lambda_t$$

$$\bar{U}_x = U' \lambda_x = R \lambda_x$$

Thus

$$\bar{U}_t + A(\bar{U}) \bar{U}_x = R\lambda_t + A R \lambda_x$$

$$= R\lambda_t + R \lambda \lambda_x = R(\lambda_t + \lambda \lambda_x) = 0$$

<u>Comment</u>: $\bar{U}_t + F(\bar{U})_x = 0$ is in <u>conservation form</u>. Not all systems of form

$$\bar{V}_t + A(\bar{V})\bar{V}_x = 0$$

can be written in conservation form — In fact — it is the conservation form that determines the shock-waves.

<u>Note</u>: L_i is a <u>left</u> eigenvector of