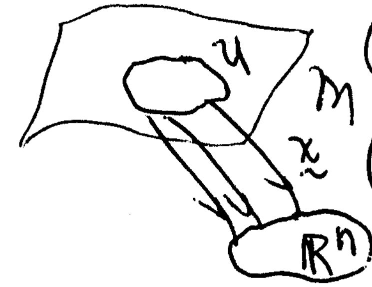
$\boxed{\text{Manifolds}}$ (I)



▫ <u>Manifolds</u>: An n-manifold is a metric space that is locally homeomorphic to $\mathbb{R}^n$

(*) <u>Precisely</u>: $\forall\ p \in M$ $\exists$ a nbhd $\mathcal{U} \ni p$ and a 1-1 onto mapping $\underset{\sim}{x}: \mathcal{U} \to \mathbb{R}^n$ such that the open sets in $\mathcal{U}$ are <u>exacty</u> premiage of the open sets in $\mathbb{R}$.

<u>Defn</u>: $\{P_k\} \to P_0$ in M if P_k is eventually within any open set containing P_0 as $k \to \infty$.

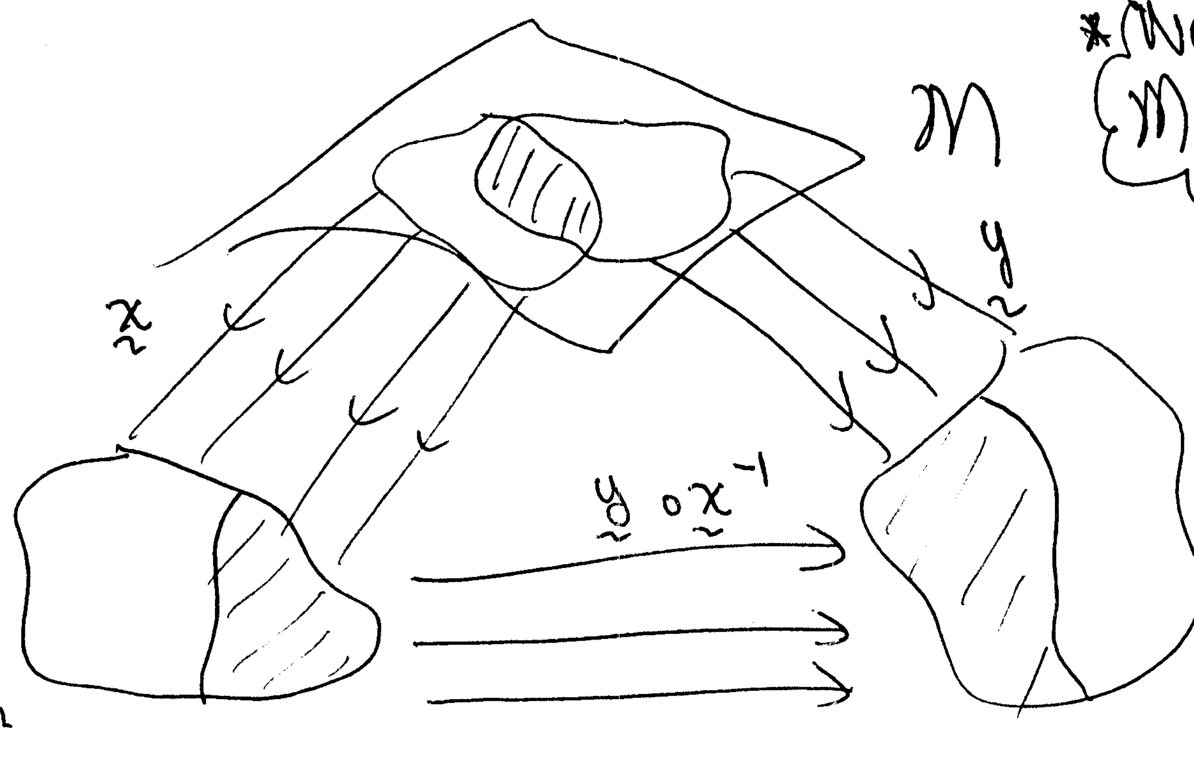
By (*), $\{P_k\} \to P_0$ iff $\underset{\sim}{x}(P_k) \longrightarrow \underset{\sim}{x}(P_0)$ in $\mathbb{R}^n$ $\Rightarrow$ " The coordinate charts determine the convergence properties of the manifold "

<u>Note</u>: since open sets in $\mathbb{R}^n$ are homeomorphic to $\mathbb{R}^n$, it doesn't matter whether $\underset{\sim}{x}$ maps to $\mathbb{R}^n$ or an open set in $\mathbb{R}^n$

- Conclude: a manifold $\mathcal{M}_n$ consists of a set of points together with all the coordinate charts $\{\underset{\sim}{x}_\alpha\}_{\alpha \in \Lambda}$ that cover it.

Defn: $\mathcal{M}$ is said to be C^k if, on the overlaps of the coordinate charts, the coordinate maps compose to make C^k maps $\mathbb{R}^n \to \mathbb{R}^n$

* We assume $\mathcal{M}$ is C^∞



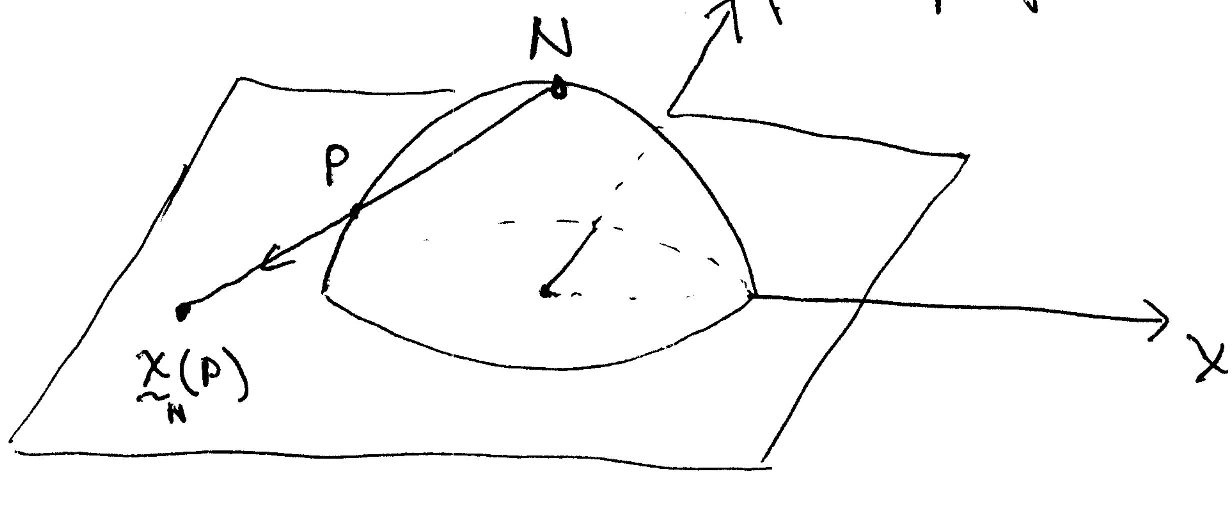
I.e., $\underset{\sim}{y} \circ \underset{\sim}{x}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is C^k. Assume $k = \infty$

Defn: $f : \mathcal{M} \to \mathbb{R}$ is C^k if $f \circ \underset{\sim}{x}$ is C^k

- Note: there is no natural notion of length on a manifold – for this we need the additional structure of a Riemannian Metric
- We now see how far you can go without a metric.

Examples: ① Sphere: $\{x^2+y^2+z^2=1\} \equiv S^2$

Two natural coordinate systems – "stereographic projection"

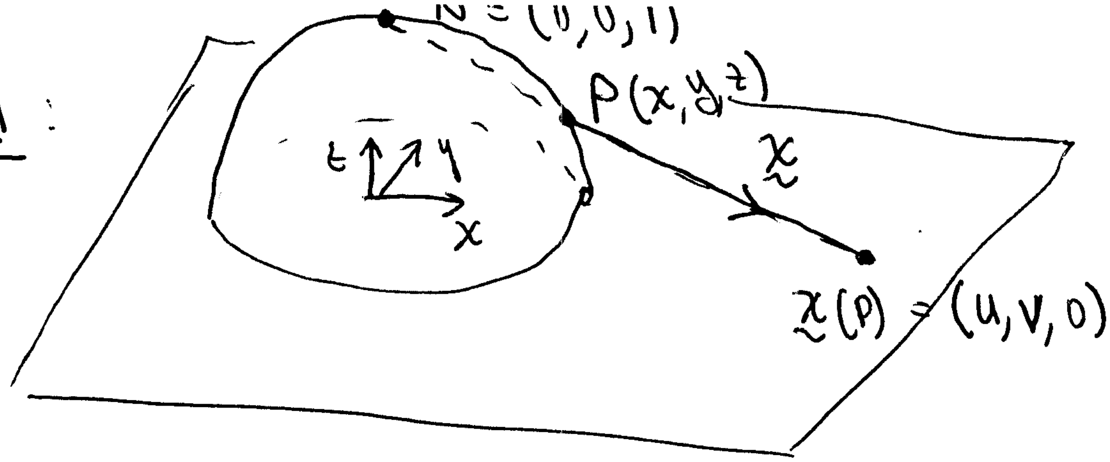


Map $P \longmapsto \underset{\sim}{x}(P)$. Chart covers whole of S^2

Two different charts cover the sphere (Not one)

H.W. find a (very simple) formula for $\underset{\sim}{x}_N$ and $\underset{\sim}{x}_S$ and show that $\underset{\sim}{x}_N \circ \underset{\sim}{x}_S^{-1}$ is C^∞ on $\mathbb{R}^2/\{N,S\} \longrightarrow \mathbb{R}^2/\{N,S\}$

Soln :



We find map : $(x,y,z) \longmapsto (u,v)$

$$\overrightarrow{NP} = t\,\overrightarrow{P\cdot \underset{\sim}{x}(p)}$$

$$(x,y,z-1) = t\,(u-x,\, v-y,\, 0-z)$$

$$t = \frac{x}{u-x} = \frac{y}{v-y} = \frac{z-1}{-z}$$

$$\frac{x}{u-x} = \frac{z-1}{-z} \Rightarrow u-x = \frac{xz}{1-z} \qquad \boxed{u = x + \frac{xz}{1-z}}$$

$$\frac{y}{v-y} = \frac{z-1}{-z} \Rightarrow v-y = \frac{yz}{1-z} \qquad \boxed{v = y + \frac{yz}{1-z}}$$

$$\underset{\sim}{x}_N : (u(x,y,z),\, v(x,y,z)) = (x,y)\left(1 + \frac{z}{1-z}\right) = \frac{(x,y)}{1-z}$$

$$\underset{\sim}{x}_S : (u(x,y,z),\, v(x,y,z)) = \frac{(x,y)}{+1+z}$$

(3C)

$$W_1 = \frac{x+iy}{1-z} \qquad W_2 = \frac{x+iy}{1+z} \qquad \alpha = x+iy$$

$$W_1 = \frac{\alpha}{1-z} \qquad \alpha = (1-z)W_1$$

$$\boxed{W_2 = \frac{1}{1+z}\alpha = \frac{1-z}{1+z}W_1}$$

$$z^2 = 1-x^2-y^2 = 1-|\alpha|^2 = 1-(1-z)^2|W_1|^2$$

$$\frac{1-z^2}{(1-z)^2} = |W_1|^2 \Rightarrow |W_1|^2 = \frac{1+z}{1-z} \Rightarrow \frac{1-z}{1+z} = \frac{1}{|W_1|^2}$$

$\Rightarrow \boxed{W_2 = \dfrac{W_1}{|W_1|^2}}$ $\Leftrightarrow$ reflection about unit circle

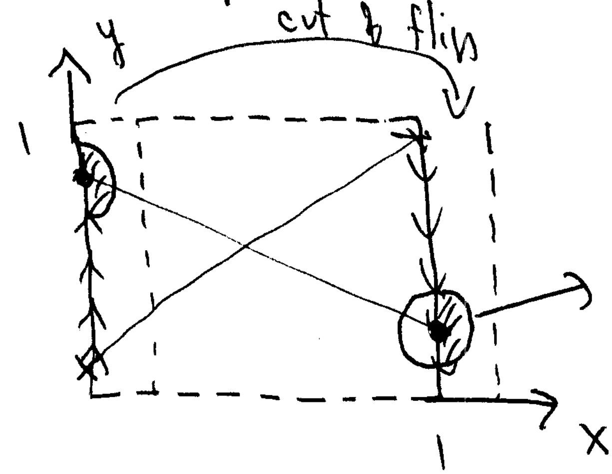
smooth mapping for $W_1 \neq 0$ ✓

Conclude: $\chi_S \circ \chi_N^{-1} : W_1 \mapsto W_2$

$W_2 = \dfrac{W_1}{|W_1|^2}$ is C^∞ $W_1 \neq 0$

(not analytic! $z \mapsto \frac{z}{|z|}$ not analytic)

<u>Ex</u> 2 Mobius Strip $0 \le x \le 1$, $0 < y < 1$ with opposite pts identified:

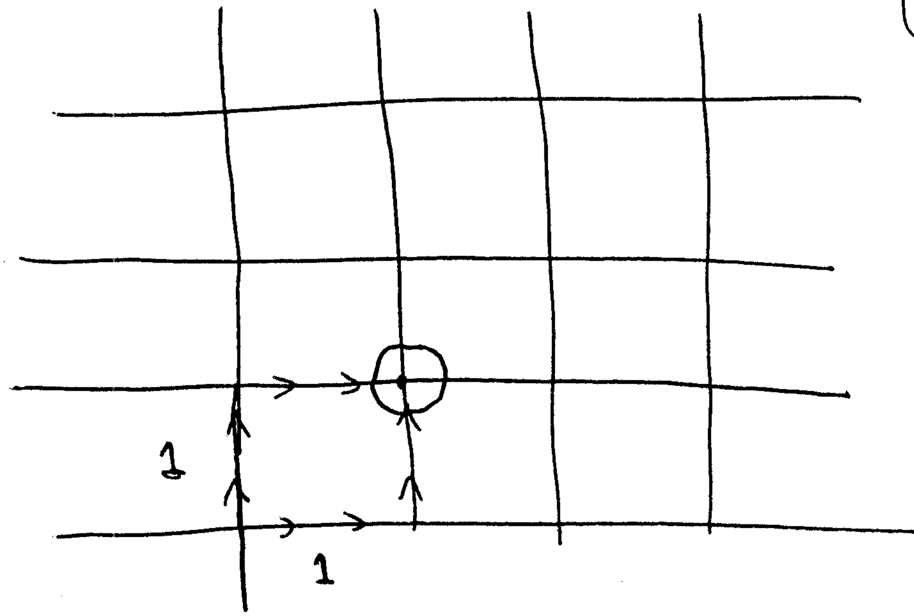


~~the identity mapping~~ the mapping from this nbd to underlying pts in $\mathbb{R}^2$ define a coord patch

<u>Note</u>: the <u>boundary</u> is S^2

Ex 3 Torus

"The unit square with opposite points identified"



To be precise -

$(x,y) \in \mathbb{R}^2$ $\quad (x_1, y_1) \sim (x_2, y_2)$ if $(x_2, y_2) - (x_1, y_1) \in \mathbb{Z}_2$

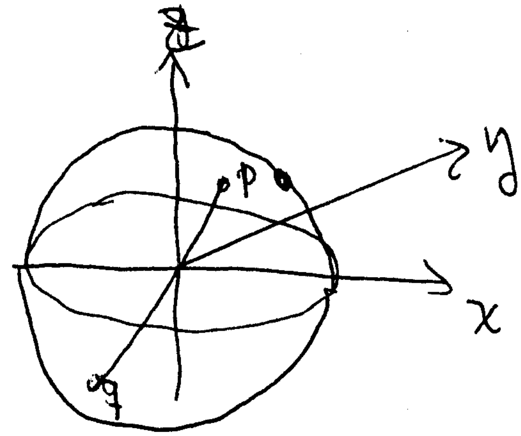
$T = \mathbb{R}^2 / \mathbb{Z}^2$ defines a set.

~~The coord patches are mappings from $\mathcal{U} \subset \mathbb{R}^2$~~
~~$\mathcal{U} \to \mathbb{R}^2$~~

For any $\mathcal{U}$ in $\mathbb{R}^2$ that is open and contains only one ele of equiv class of each pt., the identity mapping defines a coord system.

Ex 4 Projective space:

Let $p \sim q$ if p is opposite to q. Then Projective space is S^2 with opposite pts identified. Coord patches defined on open sets in S^2 that lie in one hemisphere.



☒ <u>Tangent</u> <u>Space</u>: Let $\mathcal{M} \equiv \mathcal{M}^n$ be an n-dim manifold:

Curve: $C: \xi \longmapsto C(\xi) \in \mathcal{M}$

$\xi \in [\xi_1, \xi_2] \in \mathbb{R}$

C cont, 1-1.

- We want to say

$$X_p = \frac{dc}{d\xi}\Big|_p$$

is tangent vector to C, at p, but this makes no sense.

- We could take the tangent vector down in $\mathbb{R}^n$ & call X the equiv class of such vectors in <u>all</u> coord systems. We proceed differently.

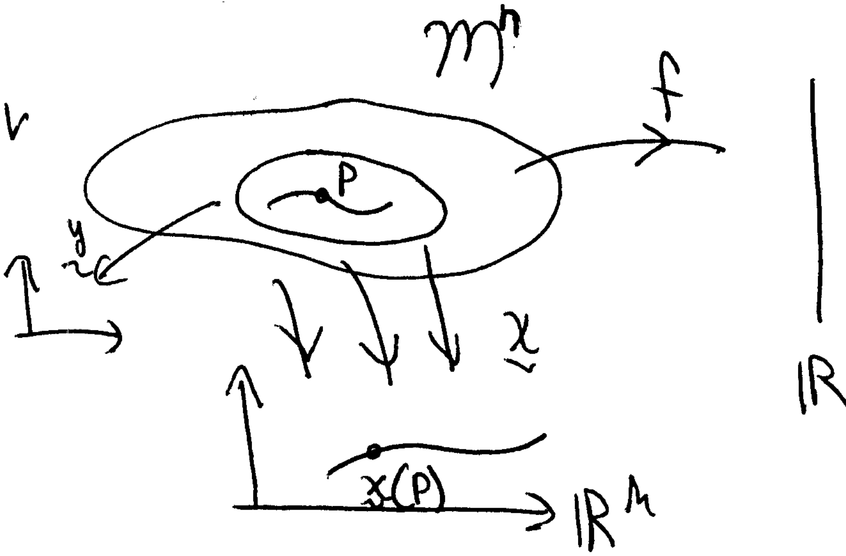
- Q: What do you use tangent vectors for?

Ans: to describe directional rates of change of functions

∴ Define X by how it "acts on functions"

- Let f be a scalar function:

$$f : \mathcal{M}^n \longrightarrow \mathbb{R}$$



We define

$$\frac{d}{d\xi} f(c(\xi)) \quad \underline{\text{well}} \ \underline{\text{defined}}$$

- Call: $f \circ \underset{\sim}{x}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\equiv$ "the function written in $\underset{\sim}{x}$-coordinates"

$f \circ \underset{\sim}{y}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\equiv$ "the fn written in $\underset{\sim}{y}$-coords."

Then

$$\frac{df}{d\xi}(c(\xi)) = \frac{d}{d\xi}(f \circ \underset{\sim}{x}^{-1})\underbrace{(\underset{\sim}{x} \circ c(\xi))}_{\text{curve in } \underset{\sim}{x}\text{-coords}} = \frac{\partial f}{\partial x^i} \dot{x}^i$$

$$\equiv \dot{x}^i \frac{\partial}{\partial x^i} f$$

$$\frac{df}{d\xi}(c(\xi)) = \frac{d}{d\xi}(f \circ \underset{\sim}{y}^{-1})(\underset{\sim}{y} \circ c(\xi)) = \frac{\partial f}{\partial y^i} \dot{y}^i$$

$$\equiv \dot{y}^i \frac{\partial}{\partial y^i} f$$

Defn: The tangent vector to $\mathcal{C}$ is the operator $X = \dot{x}^i \frac{\partial}{\partial x^i}$ (operates on scalar fn's)

$\dot{x}^i$: $\underset{\sim}{x}$- coord's of the vector $\mathbf{X}$.

$\frac{\partial}{\partial x^i}$: coord basis vector $\equiv e_i$

Note: We must have $\dot{x}^i \frac{\partial}{\partial x^i} = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}$ (*)

(sum repeated up-down indices from 1 to n)

Thm: in order for (*) to hold, the components $\dot{x}^i$ and coord. basis vectors $\frac{\partial}{\partial x^i}$ must satisfy the following coord transformation laws

$$\dot{x}^i = \frac{\partial x^i}{\partial y^\alpha} \dot{y}^\alpha \qquad (1)$$

$$\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \leftarrow \text{H.W.} \qquad (2)$$

$\frac{\partial x^i}{\partial y^\alpha} \equiv \frac{\partial}{\partial y^\alpha} \underset{\sim}{x} \circ \underset{\sim}{y}^{-1}$ $\quad \underset{\sim}{x} \circ \underset{\sim}{y}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$

(1) $$\dot{x}^i \frac{\partial}{\partial x^i}(f \circ \underset{\sim}{x}^{-1}) = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}(f \circ \underset{\sim}{y}^{-1})$$

$$\text{LHS} = \dot{x}^i \frac{\partial}{\partial x^i} f \circ (\underbrace{\underset{\sim}{y}^{-1}}_{\mathbb{R} \leftarrow \mathbb{R}^n} \circ \underbrace{\underset{\sim}{y} \circ \underset{\sim}{x}^{-1}}_{\mathbb{R}^n \leftarrow \mathbb{R}^n})$$

$$= \dot{x}^i \frac{\partial}{\partial y^\alpha}(f \circ \underset{\sim}{y}^{-1}) \cdot \frac{\partial \underset{\sim}{y}^\alpha}{\partial x^i}$$

$$= \dot{x}^i \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}(f \circ \underset{\sim}{y}^{-1}) = \dot{y}^\alpha \frac{\partial}{\partial y^\alpha}(f \circ \underset{\sim}{y}^{-1})$$

$$\Downarrow$$

$$\boxed{\dot{y}^\alpha = \dot{x}^i \frac{\partial y^\alpha}{\partial x^i}}$$

(2) $$\frac{\partial}{\partial x^i}(f \circ \underset{\sim}{x}^{-1}) = \frac{\partial}{\partial x^i}(f \circ \underset{\sim}{y}^{-1} \circ \underset{\sim}{y} \circ \underset{\sim}{x}^{-1})$$

$$= \frac{\partial}{\partial y^\alpha}(f \circ \underset{\sim}{y}^{-1}) \cdot \frac{\partial y^\alpha}{\partial x^i} \Rightarrow \boxed{\frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} = \frac{\partial}{\partial x^i}}$$

Defn: Span $\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$ is a basis for the tangent space of M^n at P

$$T_p M = \mathrm{Sp}\left\{ \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\}$$

In fact, all linear operators on functions at p can be expressed as a linear comb. of $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$; Precisely

Thm: Let L operate on all C^∞ fn's defined in a nbhd of $P \in M$. Assume L is a derivation at P; ie

$$L(fg) = L(f)\, g + f\, L(g).$$

Then

$$L = a^i \frac{\partial}{\partial x^i}\Big|_p$$

for some constants a^i. (P.f. FIP Spivak Pg 106)

— Optional —