Manifolds: An $n$-manifold is a metric space that is locally homeomorphic to $\mathbb{R}^n$.

Precisely: $\forall p \in M \exists$ a nbhd $U \ni p$ and a $1,$-onto mapping $\chi: U \rightarrow \mathbb{R}^n$ such that the open sets in $U$ are exactly preimages of the open sets in $\mathbb{R}^n$.

Defn: $\{p_k\} \rightarrow p_0$ in $M$ if $p_k$ is eventually within any open set containing $p_0$ as $k \rightarrow \infty$.

By (*), $\{p_k\} \rightarrow p_0$ iff $\chi(p_k) \rightarrow \chi(p_0)$ in $\mathbb{R}^n$.

"The coordinate charts determine the convergence properties of the manifold."

Note: Since open sets in $\mathbb{R}^n$ are homeomorphic to $\mathbb{R}^n$, it doesn't matter whether $\chi$ maps to $\mathbb{R}^n$ or an open set in $\mathbb{R}^n$. 
* Conclude: a manifold $M$ consists of a set of points together with all the coordinate charts $\{x_α\}$ that cover it.

**Defn:** $M$ is said to be $C^k$ if, on the overlays of the coordinate charts, the coordinate maps compose to make $C^k$ maps $\mathbb{R}^n \to \mathbb{R}^n$.

**I.e.:** $y \circ x^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is $C^k$. Assume $k \geq 0$.

**Defn:** $f: M \to \mathbb{R}$ is $C^k$ if $f \circ x$ is $C^k$. 

(We assume $M$ is $C^\infty$.)
Note: there is no natural notion of length on a manifold — for this we need the additional structure of a Riemannian Metric.

We now see how far you can go without a metric.

Examples: 1. Sphere: \( \{ x^2 + y^2 + z^2 = 1 \} = S^2 \)

Two natural coordinate systems — "stereographic projection"

Map \( P \mapsto x(P) \). Chart covers whole of \( S^2 \)

Two different charts cover the sphere (Not one)

H.W. find an formula for \( x_N \) and \( x_S \) and show that \( x_N \circ x_S^{-1} \) is \( C^0 \) on \( \mathbb{R}^2 / \mathbb{S}^1 \times S^1 \mapsto \mathbb{R}^2 / \mathbb{S}^1 \times S^1 \).
We find map: \((x, y, z) \mapsto (u, v)\)

\[\overrightarrow{NP} = t \overrightarrow{P \cdot x(p)}\]

\[(x, y, z-1) = t(u-x, v-y, 0-2)\]

\[t = \frac{x}{u-x} = \frac{y}{v-y} = \frac{z-1}{-2}\]

\[\frac{x}{u-x} = \frac{z-1}{-2} \Rightarrow u-x = \frac{xz}{1-\frac{z}{2}}\]

\[\frac{y}{v-y} = \frac{z-1}{-2} \Rightarrow v-y = \frac{yt}{1-\frac{z}{2}}\]

\[X_N: (u(x, y, z), v(x, y, z)) = (x, y) \left(1 + \frac{z}{1-\frac{z}{2}}\right) = \frac{(x, y)}{1 - \frac{z}{2}}\]

\[X_S: (u(x, y, z), v(x, y, z)) = \frac{(x, y)}{t + 1 + \frac{z}{2}}\]
$\alpha = x + iy$

$W_2 = \frac{x+iy}{1+z}$

$W_1 = \frac{1-z}{1+z}$

$W_2 = 1-x^2-y^2 = 1-\lambda x^2 = 1-\frac{1-\frac{\lambda}{\bar{\lambda}}}{1+\frac{\lambda}{\bar{\lambda}}}$

$z_1 = \frac{1-\lambda}{1+\lambda}$

Reflection about unit circle for $w_1 \neq 0$

Smooth mapping for $w_1 = 0$

$\bar{z} = \frac{1-\lambda}{1+\lambda}$

$z_2 = \frac{1-\lambda}{1+\lambda}$

Conclusion: $w_1 \neq 0 \Rightarrow \bar{w}_1 \neq \frac{1-\lambda}{1+\lambda}$
Ex 2: Mobius Strip \(0 \leq x \leq 1, \ 0 < y < 1\) with opposite pts identified.

Note: the boundary is \(S^2\)
Ex. 2 Torus

"The unit square with opposite points identified"

To be precise -

\( x, y \in \mathbb{R}^2 \quad (x_1, y_1) \sim (x_2, y_2) \text{ if } (x_2, y_2) - (x_1, y_1) \in \mathbb{Z}^2 \)

\[ T = \mathbb{R}^2 / \mathbb{Z}^2 \quad \text{defines a set} \]

The coord patches are mappings from \( U \subseteq \mathbb{R}^2 \)

\[ U \to \mathbb{R}^2 \]

For any \( U \) in \( \mathbb{R}^2 \) that is open and contains only one ele of equiv class of each pt., the identity mapping defines a coord system.
Ex (9) Projective space:
Let $p$ and $q$ be opposite to $q$. Then Projective space is $S^2$ with opposite pts identified. Coord patches defined on open sets in $S^2$ that lie in one hemisphere.
Tangent Space: Let $M^n = \mathbb{R}^n$ be an $n$-dimensional manifold. Curve: $C : \mathbb{R} \to M$

$$C(s) \in M$$

$s \in [s_1, s_2] \in \mathbb{R}$

C only 1-1.

- We want to say

$$X_p = \frac{dc}{ds} \bigg|_p$$

is tangent vector to $C_t$ but this makes no sense.

- We could take the tangent vector down in $\mathbb{R}^n$ and call $X$ the equivalence class of such vectors in all coordinate systems. We proceed differently.

Q: What do you use tangent vectors for? Ans: to describe rates of change of functions.

Define $X$ by how it "acts on functions."
Let $f$ be a scalar function:

$$f: M^n \rightarrow \mathbb{R}$$

We define

$$\frac{d}{ds} f(c(s)) \text{ well defined}$$

Call:

$$f \circ \chi^{-1}: \mathbb{R}^n \rightarrow \mathbb{R} = \text{ "the function written in } \chi \text{-coordinates"}$$

$$f \circ \gamma^{-1}: \mathbb{R}^n \rightarrow \mathbb{R} = \text{ "the function written in } \gamma \text{-coordinates"}$$

Then

$$\frac{df}{ds}(c(s)) = \frac{d}{ds} \left( f \circ \chi^{-1} \right)(\chi \circ c(s)) = \frac{\partial f}{\partial x_i} \dot{x}^i$$

$$\frac{df}{ds}(c(s)) = \frac{d}{ds} \left( f \circ \gamma^{-1} \right)(\gamma \circ c(s)) = \frac{\partial f}{\partial y_i} \dot{y}^i$$
Defn: The tangent vector to $C$ is the operator $X = \sum_i \frac{\partial x_i}{\partial y_\alpha} \frac{\partial}{\partial x_i}$ (operator on scalar funs). $x$-coord basis vector $\equiv e_i$ of the vector $X$.

Note: We must have $\frac{\partial x_i}{\partial x_i} = \delta^\alpha_{\beta} \frac{\partial y_\alpha}{\partial y_\beta}$ (* repeated above down indices from 1 to n)

Thm: in order for (*) to hold, the components $\dot{x}^i$ and coord. basis vectors $\frac{\partial}{\partial x_i}$ must satisfy the following coord transformation laws

$$\dot{x}^i = \frac{\partial x^i}{\partial y^\alpha} y^\alpha$$  (1)

$$\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha} \in \hom$$  (2)

$$\frac{\partial x^i}{\partial y^\alpha} = \frac{\partial y^\alpha}{\partial x^i} \chi \circ \chi^{-1} \quad \chi \circ \chi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$
\[ \nabla \cdot \frac{\partial}{\partial x^i} (f \circ x^{-1}) = \gamma^\alpha \frac{\partial}{\partial y^\alpha} (f \circ y^{-1}) \]

LHS = \[ \frac{\partial}{\partial x^i} (f \circ y^{-1}) \cdot \frac{\partial y^\alpha}{\partial x^i} \]

= \frac{\partial}{\partial x^i} (f \circ y^{-1}) \cdot \frac{\partial y^\alpha}{\partial x^i} = \gamma^\alpha \frac{\partial}{\partial y^\alpha} (f \circ y^{-1})

\[ \gamma^\alpha = \nabla \cdot \gamma^\alpha \frac{\partial y^\alpha}{\partial x^i} \]

(2) \[ \frac{\partial}{\partial x^i} (f \circ x^{-1}) = \frac{\partial}{\partial x^i} (f \circ y^{-1} \circ y \circ x^{-1}) \]

= \frac{\partial}{\partial y^\alpha} (f \circ y^{-1}) \cdot \frac{\partial y^\alpha}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial x^i}
Then

\[ L = \frac{\partial^2}{\partial x_i \partial x_j} \]

in a nbhd of \( P \).

Let \( L \) operate on all \( C^0 \) \( f \) defined in a nbhd of \( P \).

Assume \( L \) is a linear operator on function \( f \) at \( P \).

In fact, all linear operators on function \( f \) at \( P \) can be expressed as a linear combination of \( \frac{\partial}{\partial x_i} \) and \( \frac{\partial}{\partial x_j} \).

Then

\[ \text{span} \{ \frac{\partial}{\partial x_i} \} \text{ is a basis for the tangent space at } P \]