Mechanics in curved spacetime:

Freefall: $\frac{dx^i}{d\tau} = u^i \quad \nabla_u u = 0$

$\therefore$ A force diverts a particle from freefall:

- In a fixed background metric g_{ij},

4-momentum $p^i = m u^i$

4-force $f^i \qquad \nabla_u p = f$

generalizes: $\qquad \frac{d}{dt}(mv) = f \qquad$ (Newton)

- Said Differently: $p = mu \Rightarrow$

$$f = \nabla_u p = m \nabla_u u = m \underbrace{\nabla_u \frac{dx^i}{d\tau}}_{\text{4-acceleration}}$$

$$(\nabla_u u)^\ell = \underbrace{u^i u^\ell_{,i}}_{} + \Gamma^\ell_{ik} u^k u^i = \frac{d^2 u^\ell}{d\tau^2} + \Gamma^\ell_{ik} u^i u^k$$

$$\underbrace{u^i \frac{\partial}{\partial x^i}}_{\frac{d}{d\tau}} \frac{dx^\ell}{d\tau} = \frac{d^2 u^\ell}{d\tau^2}$$

is the 4-acceleration

- <u>Note</u>: $u(\tau)$ a unit vector (τ is the arclength parameterization of $x(\tau)$) expect

$\langle u, \nabla_u u \rangle = 0$ "4-acceleration $\perp$ 4-velocity"

<u>check</u>: $0 = \nabla_u \langle u, u \rangle = \langle \nabla_u u, u \rangle + \langle u, \nabla_u u \rangle$

$= 2 \langle u, \nabla_u u \rangle = 0$ ✓

- <u>Consider</u>: Stress tensor for <u>dust</u>:

$$T^{ij} = \rho u^i u^j$$

moving in fixed background metric g_{ij}.

Conservation energy-momentum:

(*) $\operatorname{div} T = 0 = T^i_{\ j;i}$ $\quad j = 0, \dots, 3$.

<u>Claim</u>: (*) $\Rightarrow$ dust particles move along geodesics of g.

☒ Newtonian Limit & Determination of K:

$$G = KT \tag{1}$$

- Assume stress tensor for "dust"

$$T^{ij} = \rho u^i u^j \qquad u^i = \frac{dx^i}{d\tau} \qquad c^2 d\tau^2 = g_{ij}\, dx^i dx^j$$

- assume all velocities are small: (in x-coords)

$$\frac{|\underset{\sim}{V}|}{c} << 1 \qquad \left(\left|\frac{V}{c}\right| = O(\varepsilon^2) \right) \tag{2}$$

- Assume $g_{ij} = \eta_{ij} + \varepsilon h_{ij}$ in x-coords

$\equiv$ "g nearly Minkowskian in x-coords" $\varepsilon << 1$.

$$c^2 d\tau^2 = \underbrace{-d(x^0)^2 + d\underset{\sim}{x}^2}_{-\left\{1 - \frac{|V|^2}{c^2}\right\} c^2 dt^2} + \varepsilon h_{ij}\, dx^i dx^j$$

$$d\tau = \sqrt{1 - \frac{|V|^2}{c^2} - \varepsilon h_{ij} \frac{1}{c^2} \frac{dx^i}{dt} \frac{dx^j}{dt}}\; dt$$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{|V|^2}{c^2}}} + O(\varepsilon)$$

on any particle trajectory.

$$= 1 + O(\varepsilon) + O\left(\left|\frac{V}{c}\right|^2\right) \tag{3}$$

- Thus: $T^{ij} = \rho \left(\frac{dt}{d\tau}\right)^2 \frac{dx^i}{dt}\frac{dx^j}{dt}$

$$\Rightarrow T^{00} = \rho c^2 + O(c)$$
$$T^{0j} = O(c)$$
$$T_{ij} = O(1)$$

For the asymptotics, assume $\rho = O(\varepsilon)$. (4)

- Now: in $G = \kappa T$, ignore all terms $O(\varepsilon^2)$ or $O\left(\frac{|v|}{c}\right) \equiv O(\varepsilon^2)$ and look for the geodesics g (the trajectories of the dust particles) in x coordinates in order to cf with Newton's trajectories:

Newton: accelerations of particles $a^i = \frac{d^2x^i}{dt^2}$ are given by

$$\nabla \Phi = -a \qquad (5)$$

where Φ = grav. potential satisfies

$$\Delta \Phi = 4\pi G_0 \rho$$ $\Leftrightarrow$ "continuum version of inverse square force law"

(6)

- In order to express T in terms of dimensionless small quantity $\frac{|\underline{v}|}{c}$, write

$$G^{ij} = \frac{K_0}{c^2} T^{ij}$$

$$\frac{1}{c^2} T^{ij} = \begin{bmatrix} \rho & \rho O(\frac{v}{c}) \\ \rho O(\frac{v}{c}) & \rho O(\frac{v}{c})^2 \end{bmatrix} \qquad \rho = O(\varepsilon) \tag{7}$$

Now assume dimensionless $\varepsilon \ll 1$, assume

$$\begin{aligned} h_{ij} &= O(1) \\ h_{ij,k} &= O(1) \\ h_{ij,kl} &= O(1) \end{aligned} \tag{8}$$

and ignore all terms of order ε^2 or higher.

Assume deriv's of h wrt $x^0 = ct$ are higher order:

$$\frac{\partial h_{ij}}{\partial x^0} = \frac{1}{c} \frac{\partial h_{ij}}{\partial t} = O(\varepsilon) \tag{9}$$

"slowly varying grav. fields."

- Under these assumptions, we look for the geodesics of g: to leading order in ε, assuming $\left|\frac{v}{c}\right| = O(\varepsilon^2)$:

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau}$$

$$\frac{1}{c^2}\frac{d^2x^i}{d\tau^2} = -\cancel{c^2}\,\Gamma^i_{jk}\frac{dx^j}{dc\tau}\frac{dx^k}{dc\tau} \qquad (10)$$

$$\Rightarrow \quad \frac{dx^j}{dc\tau} = O\left(\frac{|v|}{c}\right) \text{ unless } j = k = 0$$

<u>Now</u>: $\Gamma^i_{jk} = \frac{1}{2}g^{i\sigma}\left\{-g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j}\right\}$

$$= \frac{1}{2}\eta^{i\sigma}\left\{-h_{jk,\sigma} + h_{\sigma j,k} + h_{k\sigma,j}\right\}\varepsilon + O(\varepsilon^2)$$

no sum on i

$$= \frac{1}{2}\eta^{ii}\left\{-h_{jk,i} + h_{ij,k} + h_{ki,j}\right\}\varepsilon + O(\varepsilon^2)$$

Thus, by (10), since $\Gamma^i_{jk} = O(\varepsilon)$,

$$\Gamma^i_{jk}\frac{dx^j}{dc\tau}\frac{dx^k}{dc\tau} = O(\varepsilon^2) \qquad (11)$$

unless $j = k = 0$, in which case we obtain

$$\frac{1}{c^2}\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00} + O(\varepsilon^2) \tag{12}$$

and

$$\Gamma^i_{00} = \tfrac{1}{2}\eta^{ii}\{-h_{00,i} + h_{i0,0} + h_{0i,0}\}\varepsilon$$

$$= -\tfrac{1}{2}\eta^{ii} h_{00,i}\,\varepsilon + O(\varepsilon^2) \qquad i \neq 0 \tag{13}$$

$$= O(\varepsilon^2) \qquad i = 0.$$

<u>Conclude</u>: to leading order in ε, the geodesics are given by

$$\frac{d^2x^i}{dt^2} = \tfrac{1}{2}c^2\{+h_{00,i}\}\varepsilon \tag{14}$$

Comparing this with Newtons Law

$$\frac{d^2x^i}{dt^2} = -\Phi_{,i} \tag{15}$$

we obtain correspondence with

$$\frac{c^2}{2} h_{00}\, \varepsilon = -\Phi$$

$$\varepsilon h_{00} = -\frac{2}{c^2}\Phi$$

$$\boxed{g_{00} = -1 + \varepsilon h_{00} = -1 - \frac{2}{c^2}\Phi} \qquad (16)$$

gives correspondence between Einstein metric geodesics ($\varepsilon \ll 1$) and Newtonian particle trajectories. That is, the geodesics of g agree

- ~~It remains to determine K in $G = KT$.~~
~~Ignoring ε^2 terms, & setting $\frac{|v|}{c} = O(\varepsilon^2)$,~~

$$\cancel{G^{ij} = K_0 \frac{1}{\varepsilon^2} T^{ij} = K_0 \begin{bmatrix} \rho & O(\varepsilon^4) \\ O(\varepsilon^2) & O(\varepsilon^4) \end{bmatrix}}$$

~~$\Rightarrow$ the only~~

with Newtonian trajectories in limit $\varepsilon \to 0$ if we identify $g_{00} = -1 - \frac{2}{c^2}\Phi \iff \Phi = -\frac{c^2}{2}(1 + g_{00})$

- It remains to determine κ in $G=\kappa T$, and to show that, to leading order in ε,

$$G = \kappa T \tag{17}$$

constrains $\Phi = \frac{c^2}{2}(g_{00}+1)$ to satisfy the Newtonian counterpart

$$\Delta \Phi = 4\pi G_0 \rho \, . \tag{18}$$

To this end:

$$G_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij} = \kappa T_{ij} \tag{19}$$

$$G^{\sigma}{}_{\sigma} = R^{\sigma}{}_{\sigma} - \tfrac{1}{2} R \cdot 4 = \kappa T^{\sigma}{}_{\sigma} \qquad T = T^{\sigma}{}_{\sigma}$$

$$-R = \kappa T$$

putting this in (19)

$$R_{ij} + \tfrac{1}{2} \kappa T g_{ij} = \kappa T_{ij}$$

or

$$R_{ij} = \kappa \left(T_{ij} - \tfrac{1}{2} T g_{ij} \right) \tag{20}$$

We show that the R_{00} ~~equation~~ eqn in (20) reduces to $\Delta \Phi = 4\pi G_0 \rho$:

$$R_{00} = R^i_{0i0} = \Gamma^i_{00,i} - \underset{\uparrow \atop O(\varepsilon^2)}{\Gamma^i_{0i,0}} + \underbrace{\Gamma\Gamma - \Gamma\Gamma}_{O(\varepsilon^2)} \tag{21}$$

$$\Gamma^i_{00,i} = \frac{1}{2} g^{i\tau} \{ -g_{00,\tau i} + \underbrace{g_{\tau 0,0i} + g_{0\tau,0i}}_{O(\varepsilon^2)} \}$$

$$= \frac{1}{2} \eta^{i\tau} \{ -h_{00,\tau i} \} \varepsilon + O(\varepsilon^2)$$

$$= \frac{1}{2} \varepsilon \sum_{i=1}^{3} - h_{00,ii} = -\frac{1}{2} \varepsilon \Delta h_{00} \tag{22}$$

Using $\varepsilon h_{00} = -\frac{2}{c^2} \Phi$, we obtain

$$R_{00} = \Gamma^i_{00,i} = + \frac{1}{c^2} \Delta \Phi = c^2 K \frac{1}{c^2} \{ T_{00} - \frac{1}{2} T g_{00} \} \tag{23}$$

$c^{-2} T_{00} = \rho$, $c^{-2} T = -\rho$, $g_{00} = -1 + O(\varepsilon)$

$\Rightarrow \frac{1}{c^2} \{ \ \} = \frac{\rho}{2} + O(\varepsilon^2)$.

So (23) gives

$$\Delta \Phi = +\frac{c^4 K}{2}\rho + O(\varepsilon^2)$$

$$\therefore \quad \frac{c^4 K}{2} = 4\pi G_0 \quad \Rightarrow \quad \boxed{K = \frac{8\pi G_0}{c^4}}$$

Note ① $\Delta \Phi = \frac{c^4 K}{2}\rho$ is the $O(\varepsilon)$ term in this expansion. because $\Phi = -2^{-1}c^2 \varepsilon h_{00}$ & $\rho = O(\varepsilon)$.

Note ② Dimensional constants c, G_0 must come in because $[G] = \frac{1}{L^2}$ & $[T] = \frac{M}{L^3}\frac{L^2}{T^2} = [\rho c^2]$

Note ③ Choose units of time so $c = 1$ (measure time in meters) and mass so that $G_0 = 1$ (measure mass in meters) then $G = KT$ becomes G

$$\boxed{G = 8\pi T}$$

Note: $x^0 = ct$ defines time in meters (x^0) in terms of time in seconds

$a = G_0 \frac{M}{r^2}$ $\quad [a] = \frac{L}{T^2} \Rightarrow x^0 = ct \Rightarrow [a] = \frac{1}{L}$

$\Rightarrow \langle ar^2 \rangle = G_0 M$ $\quad \langle \ \rangle$ defines mass in meters in terms of M in grams $\Rightarrow G_0 = 1$ ✓

SKIP

Looking more carefully, we note that (8) makes no sense because $[h_{ij,k}] = \frac{1}{L}$, and hence $h_{ij,k} = O(1)$ is not a dimensionally correct statement. In place of (8), we assume

$$\frac{h_{ij,0}}{\|h_{ij,k}\|} = O(\varepsilon^2) \qquad (8)'$$

where $\|h_{ij,k}\| = \sup_{\substack{ijk \\ k\neq 0}} |h_{ij,k}|$. Then (8) expresses that time changes in h are small rel to spacial changes because $x^0 = ct$.

E.g., we could assume $h_{ij,0} = 0$, for a stationary grav. field.

Then (10) becomes

$$\frac{1}{c^2}\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{jk}\frac{dx^j}{dc\tau}\frac{dx^k}{dc\tau} \qquad (10)'$$

and

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma}\{-g_{jk,\sigma} + g_{\sigma j,k} + g_{k\sigma,j}\}$$

$$= \frac{1}{2}\eta^{ii}\{-h_{jk,i} + h_{ij,k} + h_{ki,j}\}(\varepsilon + O(\varepsilon^2))$$

which is now dimensionally correct. Now

Let $\|\Gamma\| = \sup_{ijk}|\Gamma^i_{jk}|$, $\Rightarrow$

$$\|\Gamma\| = \|\nabla h\| O(\varepsilon)$$

and thus (11) is replaced by

$$\Gamma^i_{jk}\frac{dx^j}{dc\tau}\frac{dx^k}{dc\tau} = \Gamma^i_{00}\frac{dx^0}{dc\tau}\frac{dx^0}{dc\tau} + \|\nabla h\| O(\varepsilon) \qquad (11)'$$

and (12) becomes

$$\frac{1}{c^2}\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00} + O(\|\nabla h\|)\varepsilon \qquad (12)'$$

and so $\frac{h_{ij,0}}{\|h_{ij,\mu}\|} = O(\varepsilon) \Rightarrow$

$$\Gamma^i_{00} = \frac{1}{2}\eta^{ii}\{-h_{00,i} + h_{i0,0} + h_{0i,0}\}\varepsilon$$

$$= -\frac{1}{2}\eta^{ii}h_{00,i}(1+O(\varepsilon)) \qquad (13)'$$

which is dimensionally correct, & plugging into (12') gives

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2 h_{00,i} + O(\|\nabla h\|)\varepsilon) \qquad (14)'$$

↑ required for indep't under rescalings of L.

cf with Newton $\Rightarrow$

$$\frac{d^2x^i}{dt^2} = -\Phi_{,i}$$

& taking leading order $\Rightarrow$

$$\frac{c^2}{2} h_{00} \varepsilon = -\Phi$$

$$\varepsilon h_{00} = -\frac{2}{c^2}\Phi$$

$$g_{00} = -1 + \varepsilon h_{00} = -1 - \frac{2}{c^2}\Phi$$

Dimensionally Correct.

Same idea leads to dimensionally correct argument for (17) – (23)