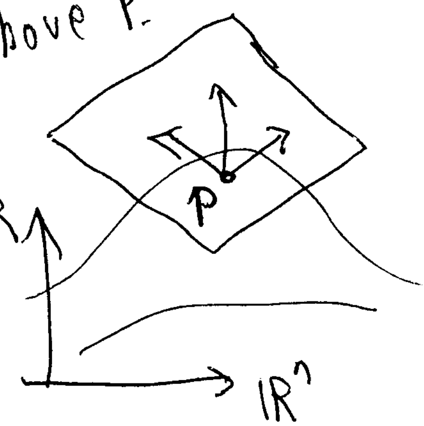
Math 280
S98 II

The Tangent Bundle TM

- $M \equiv M^n$ a k-dimensional manifold
- $T_p M \equiv$ the tangent space to M at pt $p \in$

$T_p M \equiv$ vector space $\mathbb{R}^n \equiv$ Fiber above p.



Note: you can add & scalar multiply elements of $T_p M$:

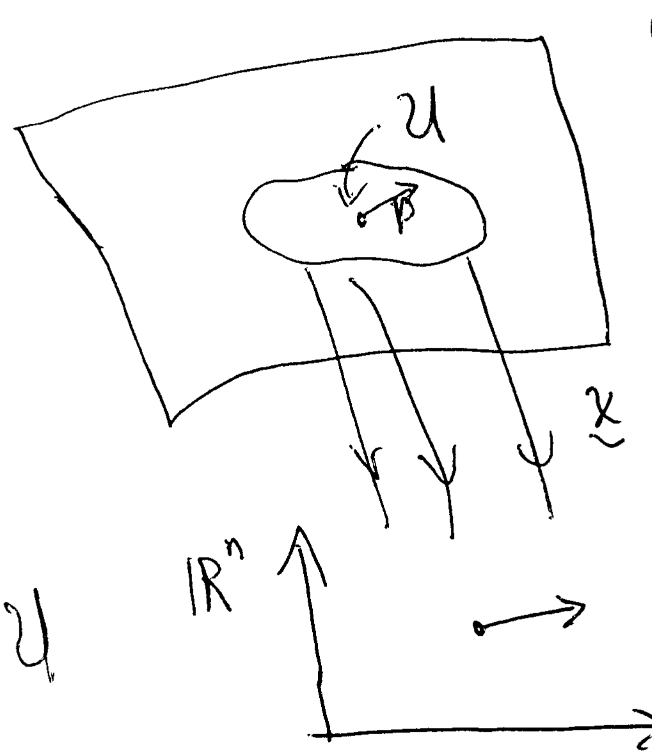
$$a^i \frac{\partial}{\partial x^i}\Big|_p + b^i \frac{\partial}{\partial x^i}\Big|_p = (a^i + b^i) \frac{\partial}{\partial x^i}\Big|_p$$

sum $i = 1, \dots, n$ $\quad a^i\ b^i$ constants.

- Defn: a vector field is an assignment of a tangent vector to each point $p \in M$

Q: How can we tell whether the vector field is smooth?

- Within a given coord chart, the vector field can be written:



$$X \equiv X(p) = a^i(p) \frac{\partial}{\partial x^i} \quad p \in U$$

implicitly assumed to be at point p

<u>Defn</u>: we say X is a <u>smooth</u> vector field if the coordinate functions $a^i(p)$ are smooth functions

Using this notion of smoothness, we can make M together with all its tangent spaces into one big manifold called the <u>Tangent Bundle</u> TM

<u>Defn</u>: $T\mathcal{M} = \{(p, X_p),\ p \in \mathcal{M},\ X_p \in T_p\mathcal{M}\}$ endowed with the following <u>coordinate charts</u> that define the <u>open sets</u> in $T\mathcal{M}$:

For every coordinate chart $\underset{\sim}{x} : \mathcal{M} \to \mathbb{R}^n$, let

$$\phi_{\underset{\sim}{x}} : T\mathcal{U} \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$
$$(p, X_p) \longrightarrow (\underset{\sim}{x}(p), \underset{\sim}{a})$$

where: $X_p = a^i \frac{\partial}{\partial x^i}\Big|_p$, $\underset{\sim}{a} = (a^1, \ldots, a^n)$.

<u>Summary</u>: "each coordinate chart on $\mathcal{M}$ induces a coordinate chart on $T\mathcal{M}$ obtained by taking the $\underset{\sim}{x}$components of the vectors in $T_p\mathcal{M}$ as coordinates on $T_p\mathcal{M}$."

Thus: If $X = a^i \frac{\partial}{\partial x^i} = b^\alpha \frac{\partial}{\partial y^\alpha}$, then the smoothness of X is determined by the smoothness of the mapping $a \longrightarrow b$ defined on the overlaps.

- <u>Note</u>: there is <u>no</u> <u>natural</u> notion of length within $T_p M$. I.e. no natural innerproduct

I.e. $X = a^i \frac{\partial}{\partial x^i}$ in $\underset{\sim}{x}$-coord's

$X = b^\alpha \frac{\partial}{\partial y^\alpha}$ in $\underset{\sim}{y}$-coords

$\underset{\sim}{a} = (a^1, \dots, a^n)$ components of X in $\underset{\sim}{x}$-coords

$\underset{\sim}{b} = (b^1, \dots, b^n)$ components of X in $\underset{\sim}{y}$-coord

There is no "natural" set of coords, so no natural length: $\underset{\sim}{a} \cdot \underset{\sim}{a} \neq \underset{\sim}{b} \cdot \underset{\sim}{b}$.

It turns out: defining a Riemannian metric is essentially equivalent to choosing a smoothly varying inner product on each $T_p M$ which is equivalent to choosing preferred coordinate systems in which $\underset{\sim}{a} \cdot \underset{\sim}{a} = 1$.

- The tangent bundle TM can be abstracted to something more general called a Vector bundle which can be abtracted to something still more general called a fibre Bundle

Picture: The whole thing is called E, a fibre bundle; special case: $E = TM$

- There is a base space B, [a manifold; or more generally, a topological space]
special case $B = M$

- Above each point $b \in B$ there is a fibre F_b; an "algebraic" space isomorphic to some typical fibre F.
  - When F_b is the tangent space to M at b, we have case of TM
  - When F_b is a vector space isomorphic to F, we have the case of a vector bundle

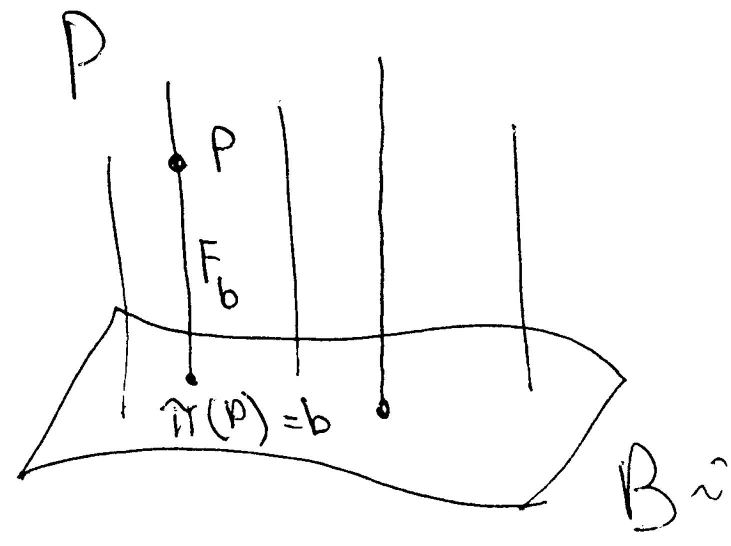
- When F_b is a space that can be acted on by a group G, we call it a fibre bundle.

Eg. T_pM can be acted on by the general linear group through a change of basis.

The defn. of fibre bundle is set up so as to describe in ~~the~~ a general way what it means to "smoothly vary" an element of the fibre as you vary the base point b.

PICTURE:

fibres together with base points make up all points in P



Defn: An n-d vector bundle is a 5-tuple

$(E, \pi, B, \oplus, \odot)$ st

① E & B are topological spaces (the total space and base space, resp.)

② $\pi : E \to B$ is continuous onto B

③ $\oplus$ and $\odot$ define vector addition and scalar multiplication in each fibre $\pi^{-1}(x)$, $x \in B$. ie.

$$\oplus : \bigcup_{p \in B} \pi^{-1}(p) \times \pi^{-1}(p) \longrightarrow E$$

$$\odot : \mathbb{R} \times E \to E$$

st

$$\oplus\left(\pi^{-1}(p) \times \pi^{-1}(p)\right) \subset \pi^{-1}(p)$$

$$\odot\left(\mathbb{R} \times \pi^{-1}(p)\right) \subset \pi^{-1}(p)$$

in such a way that each $\pi^{-1}(p)$ defines an n-d vector space over $\mathbb{R}$

4. Bundle is <u>locally trivial</u>

$\forall\, p \in B$, $\exists$ nbhd U of p and a homeomorphism (a bijective, bicontinuous map)

$$\phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n \qquad (*)$$

which is a vector space isomorphism from each $\pi^{-1}(q)$ onto $q \times \mathbb{R}^n$ $\forall\, q \in U$.

I.e. $\quad \phi(e) = (\pi(e), \hat{\phi}(e))$,

and if we restrict $\pi(e) = p$, then $\hat{\phi}$ is a vector space isomorphism from $\pi^{-1}(p) \to \mathbb{R}^n$

<u>Note</u>: the ϕ's give meaning to "smoothly varying sections"

A <u>section</u> is a map from $B \longrightarrow E$ st

$$p \longmapsto \pi^{-1}(p)$$

$\approx$ "a vector field"

<u>Conclude</u>: $T_p M$ is a vector space of dimension k, basis $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k} \right\}$.