# The co-tangent Bundle:


- Given a coord. system on $\mathcal{M}$

$$\underset{\sim}{x} : U_{\underset{\sim}{x}} \longrightarrow \mathbb{R}^n,$$

We call $\frac{\partial}{\partial x^i}$ the coordinate vector fields on $\mathcal{M}$. I.e., at each $p \in \mathcal{M}$, $\left.\frac{\partial}{\partial x^i}\right|_p$ is the operator on functions $f : \mathcal{M} \to \mathbb{R}$ that evaluates

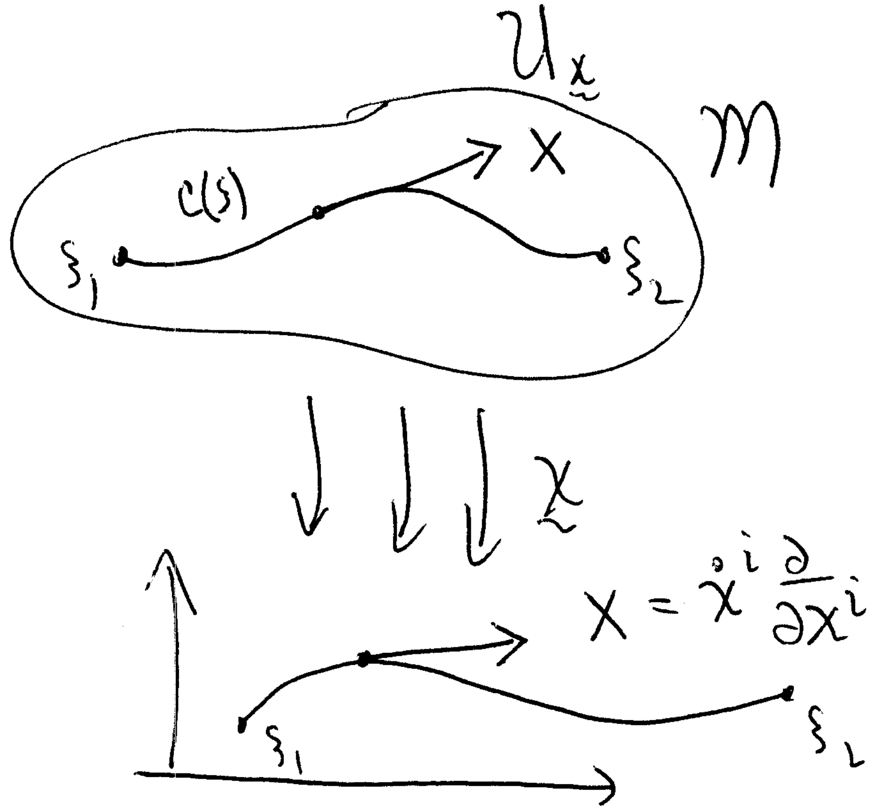
$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \frac{\partial}{\partial x^i} f \circ \underset{\sim}{x}$$

- ~~Defn: the differential $dx^i|_p$ op~~

- It is useful to have a gadjet that takes a vector $X \in T_p\mathcal{M}$ and evaluates the "infinitessimal" change in the coord $x^i$ corresponding to a motion in $\mathcal{M}$ approximated by $X$

Picture:

$X = \text{"}\frac{dc}{d\xi}\text{"}$




$$\dot{x}^i \equiv i\text{-component of } X \equiv \frac{dx^i}{d\xi}$$

Thus: $dx^i = \dot{x}^i \, d\xi$

$$\Delta x^i \Big]_{\xi_1}^{\xi_2} = \int_{\xi_1}^{\xi_2} \dot{x}^i \, d\xi = \int_{\xi_1}^{\xi_2} dx^i$$

Slightly more abstract:

Defn: $dx^i\big|_p$ is an operator on $T_pM$ that evaluates the $i$-component of a vector:

$$dx^i\big|_p \left(a^i \frac{\partial}{\partial x^i}\Big|_p\right) = a^i(p)$$

<u>Defn</u>: $dx^i$ is then the smoothly varying operator that restricts at $p$ to a linear operator on tangent vectors:

$$dx^i \left(a^j \frac{\partial}{\partial x^j}\right) = a^i \quad \leftarrow \text{at each } p.$$

<u>Thus</u>:
$$\dot{x}^i = \frac{dx^i}{d\xi} \quad \text{on a curve}$$

$$\dot{x}^i \, d\xi = \text{"} dx^i \text{"}$$

For us:
$$dx^i(X) = dx^i\left(\dot{x}^j \frac{\partial}{\partial x^j}\right)$$
$$= \dot{x}^i$$

So
$$\text{"} dx^i \text{"} = \underbrace{dx^i(X)}_{\text{the operator}} \, d\xi$$

coordinate increment along the curve with tangent vector $X$

Thus: we can calculate or express the coordinate increment along a curve in an <u>invariant</u> coord-indep<sup></sup>t sense:

$$\Delta x^i = \int_{\xi_1}^{\xi_2} \underbrace{dx^i(X)}_{\substack{\text{an operator on } TM \\ \text{defined indep't of coords}}} d\xi \qquad X = \frac{dc}{d\xi}$$

Said diff: "$dx^i$ is the coord expression for something defined indep't of coords"

<u>Thm</u>: $\{dx^1, \dots, dx^n\}$ define a basis for the set of linear operators on $T_pM$ at each $p \in M$. FIP (HW). Hint: Riez Rep. Thm (*nei pg)

<u>Defn</u>: $\omega = a_i\, dx^i \iff \omega(p) = a_i(p)\, dx^i\big|_p$ is called a <u>1-form</u>, or <u>covector field</u>

<u>Thm</u>: if $\omega = a_i\, dx^i = b_\alpha\, dy^\alpha$, then

$$a_i = \frac{\partial y^\alpha}{\partial x^i} b_\alpha$$

and

$$dx^i = \frac{\partial x^i}{\partial y^\alpha} dy^\alpha.$$

"1-forms make clear the transformation properties of '<u>coordinate increments</u>' "

increment in $x^i$-coord computed in $\underset{\sim}{y}$-coords

Eg: $\underset{\wedge}{\text{"}dx^i\text{"}} = dx^i(\dot{c})\, d\xi = \frac{\partial x^i}{\partial y^\alpha} dy^\alpha(\dot{c})\, d\xi$

HW. argue that this makes sense

[illegible]

Pf of Thm: $a_i\, dx^i = b_\alpha\, dy^\alpha$

$\Rightarrow \quad a_i\, dx^i(X) = b_\alpha\, dy^\alpha(X)$

$\forall$ vector field $X$. If $X = \bar{a}^i \frac{\partial}{\partial x^i}$, then

$$a_i\, dx^i(X) = a_i\, dx^i\left(\bar{a}^j \frac{\partial}{\partial x^j}\right) = a_i \bar{a}^i$$

$X = \bar{b}^\alpha \frac{\partial}{\partial y^\alpha} \quad \Rightarrow$

$$b_\alpha\, dy^\alpha(X) = b_\alpha\, dy^\alpha\left(\bar{b}^\beta \frac{\partial}{\partial y^\beta}\right) = b_\alpha \bar{b}^\alpha$$

But $\bar{b}^\alpha = \frac{\partial y^\alpha}{\partial x^i} \bar{a}^i \Leftarrow$ "trans. law for vectors"

$$\Rightarrow \quad \underbrace{a_i}\, \bar{a}^i = \underbrace{b_\alpha \frac{\partial y^\alpha}{\partial x^i}}\, \bar{a}^i$$

$$\Rightarrow \quad \boxed{a_i = b_\alpha \frac{\partial y^\alpha}{\partial x^i}}$$

*Use:

Riez Representation thm: all linear fnals on $\vec{a} \in \mathbb{R}^n$ are given by $L = \vec{V} \cdot \vec{a}$ for some $\vec{V} \in \mathbb{R}^n \Rightarrow (\mathbb{R}^n)^* \approx \mathbb{R}^n$

$$dx^i\left(\bar{a}^j \frac{\partial}{\partial x^j}\right) = \bar{a}^i = \bar{b}^\alpha \frac{\partial x^i}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha} dy^\alpha\left(\bar{b}^\alpha \frac{\partial}{\partial y^\alpha}\right)$$

$$= dx^i(X)$$

Same vector!

$$= \frac{\partial x^i}{\partial y^\alpha} dy^\alpha(X)$$

$$\Rightarrow \boxed{dx^i = \frac{\partial x^i}{\partial y^\alpha} dy^\alpha} \checkmark$$

~~Defn: The cotangent bundle $T^*M$ is the vector bundle over $M$ with fibre's~~

Defn: the vector space Span$\{dx^1|_p, \ldots, dx^n|_p\}$ is denoted

$$T_p^* M = \text{Span}\,\{dx^1|_p, \ldots, dx^n|_p\}$$

Defn: the cotangent bundle, denoted $T^*M$, is the vector bundle obtained by taking $a_i dx^i|_p + b_i dx^i|_p = (a_i + b_i) dx^i|_p$ as the vector space operation on $T_p^*M$, and the local trivializations are given by

$$\phi_{\underset{\sim}{x}} : T^*M\big|_{U_{\underset{\sim}{x}}} \longrightarrow U_{\underset{\sim}{x}} \times \mathbb{R}^n$$

$$a_i dx^i|_p \longmapsto (p, \underset{\sim}{a})$$

Conclude: "the coord. coeff's of the 1-forms give meaning to a smoothly varying covector.

☒ <u>Defn</u>: An object that is defined indept. of coordinates is said to be defined <u>invariantly</u>

- A scalar fn on a manifold is ~~an~~ <u>invariantly defined</u>

$$f : M \longrightarrow \mathbb{R}$$

defined indept of coords ✓

- A tangent vector $X_p$, thought of as an operator on scalars, is <u>invariantly defined</u>

$$X_p : \quad f \longmapsto \mathbb{R}$$

$$X_p = a^i \frac{\partial}{\partial x^i}\Big|_p$$ ⇐ coord expression for an invariant object.

- A <u>covector</u> $\omega_p$, thought of as an operator on $T_p M$ is invariantly defined

$$\omega_p : T_p M \longrightarrow \mathbb{R} \qquad \omega_p = a_i \, dx^i\Big|_p$$ ⇐ coord. expression for $\omega_p$

- Vector fields and covector fields ⟺ 1-forms are invariantly defined as operators

$$\omega\Big|_{\mathcal{U}_{\underset{\sim}{x}}} = a_i\, dx^i \quad \leftarrow \quad \underset{\sim}{x}\text{-coord expression for } \omega$$

$$X\Big|_{\mathcal{U}_{\underset{\sim}{x}}} = a^i \frac{\partial}{\partial x^i} \quad \leftarrow \quad \underset{\sim}{x}\text{-coord expression for } \omega.$$

Vector field: smooth mapping from $M \to TM$

$$p \longmapsto X_p \in T_pM$$

1-form: smooth mapping from $M \longrightarrow T^*M$

$$p \longmapsto \omega_p \in T_p^*M$$

☐ Co-vectors keep track of hyperplanes — Specifically: If $\omega = a_i dx^i \in T_p^*M$, then the $\underset{\sim}{x}$-components $\underset{\sim}{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ gives the $\underset{\sim}{x}$ <u>coordinate</u> <u>normal</u> (based on the $\underset{\sim}{x}$-coord dot product) to the hyperplane spanned by the $(n-1)$-dimensional span of $X \in T_pM$ st $\omega(X) = 0$.

<u>Proof</u>: $\omega$ is a linear transformation

$$\omega : T_pM \longrightarrow \mathbb{R}$$

$\Rightarrow \dim \operatorname{Ker} \omega = n-1$. Let $X_1, \dots, X_n$ in $T_pM$ be a basis for $\ker \omega$, so

$$\omega(X_i) = 0 \quad , i = 1, \dots, n-1$$

Then $\omega(X_i) = 0$ in every coord representation of $\omega$ & $X_i$. That is

Fix $\underset{\sim}{x}$-coords: $\omega = a_j\, dx^j \in T_p^* M$

$X_i = b_i^j \frac{\partial}{\partial x^j} \in T_p M$

$\omega(X_i) = a_j b_i^j = \underset{\sim}{a}_{\underset{\sim}{x}} \cdot \underset{\sim}{b}_i = 0 \iff \underset{\sim}{a}_{\underset{\sim}{x}} \perp \underset{\sim}{b}_i$

$\Rightarrow \underset{\sim}{a}_{\underset{\sim}{x}}$ is the coord normal to $Sp\{X_1 \cdots X_n\}$ in $\underset{\sim}{x}$-coordinates as claimed. $\Rightarrow$ true in every coord system $\blacksquare$

<u>Conclude</u>: The vector of components normal to a hyperplane in a given coord. syst will transform covariantly not contravariantly

i.e. $(\vec{N}_{\underset{\sim}{x}})_i = (\vec{N}_{\underset{\sim}{y}})_\alpha \frac{\partial y^\alpha}{\partial y^i}$

Pf. $((N_{\underset{\sim}{x}})_1, \ldots, (N_{\underset{\sim}{x}})_n)$ are components of the covector $(N_{\underset{\sim}{x}})_i\, dx^i$ ✓

<u>Summary</u>: "The normal to a hyperplane is a covector <u>not</u> a vector!"

<u>Note</u>: if we have a metric $g$,

$$\langle X, Y \rangle = g_{ij}\, x^i x^j \qquad X = x^i \frac{\partial}{\partial x^i} \in T_pM$$

$$Y = y^i \frac{\partial}{\partial x^i} \in T_pM$$

then we can define a <u>vector</u> <u>normal</u> to a <u>hyperplane</u> by:

$$\langle N, X_i \rangle = 0 \quad i = 1, \ldots, n-1$$

where $\{X_1, \ldots, X_{n-1}\}$ span the hyperplane.

**Example**: Differential / Gradient of a fn

- Let $f : \mathcal{M}^n \to \mathbb{R}$ smooth
- $f$ in $\underset{\sim}{x}$ coords: $y = f \circ \underline{x}^{-1} = f(\underset{\sim}{x})$

  in $\underset{\sim}{y}$ coords: $y = f \circ \underline{y}^{-1} = \bar{f}(\underset{\sim}{y})$

Classically, it makes sense to differentiate $f$ in direction of vector $X$ at $p$:

$$\nabla_X f \underset{\uparrow}{\equiv} X(f) = a^i \frac{\partial}{\partial x^i} f = a^i \underset{\uparrow}{\frac{\partial f}{\partial x^i}}$$

defn of $X$

classically the gradient of $f$

turn it around

$$a^i \frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial x^i} dx^i (X)$$

$$X = a^i \frac{\partial}{\partial x^i}$$

- Defn: $df = \frac{\partial f}{\partial x^i} dx^i$ is the differential of $f$

Claim: $df$ so defined is a 1-form $df \in T_p^*M$:

Pf.

$$\frac{\partial f}{\partial y^\alpha}\, dy^\alpha = \overbrace{\frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial y^\alpha}}^{\text{chn rule}}\overbrace{\frac{\partial y^\alpha}{\partial x^j}\, dx^j}^{dy^\alpha} = \frac{\partial f}{\partial x^i}\, dx^i$$

$$\underbrace{\frac{\partial x^i}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^j}}_{\delta^i_j \equiv \mathrm{id}} \qquad \boxed{i = j}$$

<u>Conclude</u>: The components of the gradiant of $f$ transform covariantly not contravariantly

$\Leftrightarrow$ "The gradient of a function is a covector not a vector"

Without a metric: $df$ makes sense not $\nabla f$

- We need a metric to define a vector $\nabla f$ at each $p \in M$.

I.e., if we have $g$, $\langle X, Y \rangle = g_{ij} X^i X^j$

Define the vector $\nabla f$ by:

$$\langle \nabla f, Y \rangle = df(Y)$$

Riesz Representation – every linear fn'al can be represented by a vector thru the inner product

$$\langle \nabla f, - \rangle : T_p M \to \mathbb{R}$$

$$df(-) : T_p M \to \mathbb{R}$$

Homework: Prove that if $g_{ij}$ is defined in each coordinate system so that (at $p \in M$)

$$g_{ij} a^i b^j = \langle X, Y \rangle = \bar{g}_{\alpha\beta} \bar{a}^\alpha \bar{b}^\beta$$

x-coords $\quad$ y coords : $X = \bar{a}^\alpha \frac{\partial}{\partial y^\alpha}$

$X = a^i \frac{\partial}{\partial x^i}$, $Y = b^j \frac{\partial}{\partial x^j}$ $\qquad$ $Y = \bar{b}^\beta \frac{\partial}{\partial y^\beta}$

then $g_{ij}$ must transform by:

$$\boxed{g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} = \bar{g}_{\alpha\beta}}$$

"$g$ is a $\binom{0}{2}$ tensor"

Show this is equivalent to $\boxed{\underset{n\times n}{J^T} \, \underset{n\times n}{g} \, \underset{n\times n}{J} = \underset{n\times n}{\bar{g}}}$

in matrix form: $J = \frac{\partial x^{i \leftarrow \text{row}}}{\partial y^{\beta \leftarrow \text{col}}}$, $g = g_{ij}$ ← colm, row