The cotangent Bundle:

Given a coordinate system on $M$

$$\chi : U \chi \rightarrow \mathbb{R}^n,$$

We call $\frac{\partial}{\partial x^i}$ the coordinate vector fields on $M$. I.e., at each $p \in M$, $\frac{\partial}{\partial x^i} |_p$ is the operator on functions $f : M \rightarrow \mathbb{R}$ that evaluates

$$\frac{\partial}{\partial x^i} |_p (f) = \frac{\partial}{\partial x^i} f \circ \chi.$$

Define the differential $\frac{\partial x^i}{\partial p} |_p$.

It is useful to have a good (local) that takes a vector $X \in T_pM$ and evaluates the "infinitesimal" change in the coordinate $x^i$ corresponding to a motion in $M$ approximated by $X$. 

\[ X = \frac{dc}{ds} \]

\[ X = \chi \frac{\partial}{\partial x^i} \]

\[ \dot{x}^i = i\text{-component of } X = \frac{dx^i}{ds} \]

Thus:

\[ dx^i = \dot{x}^i \, ds \]

\[ \Delta x^i \bigg|_{s_1}^{s_2} = \int_{s_1}^{s_2} \dot{x}^i \, ds = \int_{s_1}^{s_2} dx^i \]

Slightly more abstract:

\[ \text{Defn: } dx^i \bigg|_p \text{ is an operator on } T_p M \text{ that evaluates the } i\text{-component of a vector:} \]

\[ dx^i \bigg|_p (\alpha^i \frac{\partial}{\partial x^i} \bigg|_p ) = \alpha^i (p) \]
**Defn:** $dx^i$ is then the smoothly varying operator that restricts at $p$ to a linear operator on tangent vectors:

$$dx^i \left( \frac{\partial}{\partial x^j} \right) = a^i_j \quad \text{at each } p.$$

**Thus:**

$$\dot{x}^i = \frac{dx^i}{d\xi} \quad \text{on a curve}$$

$$\dot{x}^i d\xi = \left[ dx^i \right]$$

**For us:**

$$dx^i(X) = dx^i \left( \dot{x}^j \frac{\partial}{\partial x^j} \right)$$

$$= \dot{x}^i$$

So

$$dx^i = dx^i(X) \, d\xi$$

coordinate increment of the operator along the curve with tangent vector $X$. 
Thus we can calculate or express the coordinate increment along a curve in an invariant coordinate sense:

\[ \Delta x^i = \int \Delta x^i(X) \, ds \quad x = \frac{dc}{ds} \]

\[ \frac{dc}{ds} \]

an operator on TM defined indepent of coords

Said dist: "\( \Delta x^i \) is the coord expression for something defined indepent of coords"
Thm: \( \{dx^1, \ldots, dx^n\} \) define a basis for the set of linear operators on \( T_pM \) at each \( p \in M \). FIP (HW). Hint: Riet Rep. Thm (x).

Defn: \( w = a_i \, dx^i \implies w(p) = a_i(p) \, dx^i \big|_p \) is called a \underline{1-form}, or \underline{covector field}.

Thm: if \( w = a_i \, dx^i = b_a \, dy^a \) then

\[ a_i = \frac{\partial y^a}{\partial x^i} \, b_a \]

and

\[ dx^i = \frac{\partial x^i}{\partial y^a} \, dy^a. \]

"1-forms make clear the transformation property of 'coordinate increments',"!

Eg: \( dx^i = dx^i(\hat{c}) \, ds = \frac{\partial x^i}{\partial y^a} \, dy^a(\hat{c}) \, ds \)
Proof of Thm: \[ a_i \, dx^i = b_\alpha \, dy^\alpha \]

\[ \Rightarrow \quad a_i \, dx^i (X) = b_\alpha \, dy^\alpha (X) \]

A vector field \( X \). If \( X = a^i \frac{\partial}{\partial x^i} \), then

\[ a_i \, dx^i (X) = a_i \, dx^i (\bar{a}^i \frac{\partial}{\partial x^i}) = a_i \bar{a}^i \]

\[ X = b^\alpha \frac{\partial}{\partial y^\alpha} \quad \Rightarrow \]

\[ b_\alpha \, dy^\alpha (X) = b_\alpha \, dy^\alpha (\bar{B}^\beta \frac{\partial}{\partial y^\beta}) = b_\beta \bar{B}^\alpha \]

But \[ \bar{B}^\alpha = 2y^\alpha \bar{a}^i \] \( \Rightarrow \) "trans. law for vectors."

\[ \Rightarrow \quad a_i \bar{a}^i = b_\alpha 
\frac{\partial y^\alpha}{\partial x^i} \bar{a}^i \]

\[ \Rightarrow \]

\[ a_i = b_\alpha \frac{\partial y^\alpha}{\partial x^i} \]

Use:

Lie Representation: all linear forms over \( \mathbb{R}^n \) are given by \( L = \nabla \cdot \hat{a} \) for some \( \hat{a} \in \mathbb{R}^n \) \( \hat{a} \in (\mathbb{R}^n)^* \approx \mathbb{R}^n \)
\[ \begin{align*}
\frac{dx^i}{\partial x^j} (x) &= \frac{dx^i}{\partial y^a} (b^a \frac{\partial \alpha}{\partial y^a}) \\
\text{Same vector!} \\
\end{align*} \]

\[ dx^i = \frac{\partial x^i}{\partial y^a} dy^a \]

**Defn:** The cotangent bundle \( T^* M \) is the vector bundle over \( M \) with fibres'
Defn: the vector space \( \text{Span} \{dx'|_p, \ldots, dx^n'|_p\} \)

is denoted \( T^*_p M \) = \( \text{Span} \{dx'|_p, \ldots, dx^n'|_p\} \)

Defn: the cotangent bundle, denoted \( T^* M \), is the vector bundle obtained by taking \( a_i dx^i|_p + b_i dx^i|_p = (a_i + b_i)dx^i|_p \) as the vector space operation on \( T^*_p M \), and the local trivialization are given by

\[
\varphi_x : T^*_x M \rightarrow U_x \times \mathbb{R}^n
\]

\[
\varphi_x : a_i dx^i|_p \mapsto (p, a)
\]

Conclude: "the coord. coeff. of the 1-forms give meaning to a smoothly varying covector."
**Defn**: An object that is defined indept. of coordinates is said to be defined 

**Invariantly**.

- A scalar 
  
  \[ f: M \to \mathbb{R} \]
  
  defined indept. of coordinates.

- A tangent vector \( X_p \), thought of as an operator on scalars, is invariantly defined

  \[ X_p : f \mapsto \mathbb{R} \]

- A vector \( \omega_p \), thought of as an operator on \( T_p M \) is invariantly defined

  \[ \omega_p : T_p M \to \mathbb{R} \]

  \[ \omega_p = a_i \, dx_i \] (coord. expression for \( \omega_p \)).
Vector fields and covector fields $\omega$ are invariantly defined as operations
\[ \omega|_x = a_i \, dx^i \quad \text{for } \omega \]

\[ u_x \]

Vector field: smooth mapping from $M \to \mathbb{T}M$
\[ p \mapsto X_p \in T_p M \]

1-form: smooth mapping from $M \to T^\ast M$
\[ p \mapsto \omega_p \in T^\ast_p M \]
Co-vectors keep track of hyperplanes — specifically: If $w = a_i \, dx^i \in T^*_p M$, then the $x$-component $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ gives the $x$-coordinate normal (based on the $x$-coordinate dot product) to the hyperplane spanned by the $(n-1)$-dimensional span of $X \in T_p M$ s.t. $w(X) = 0$.

**Proof.** $w$ is a linear transformation

$$w : T_p M \rightarrow \mathbb{R}$$

$\Rightarrow \dim \ker w = n-1$. Let $X_1, \ldots, X_n$ in $T_p M$ be a basis for $\ker w$, so

$$w(X_i) = 0, \quad i = 1, \ldots, n-1$$

Then $w(X_i) = 0$ in every coordinate representation of $w$ of $X_i$. That is
Fix $x$-coordinates:  
$\omega = a_j \, dx^j \in T^*_p M$

$X_i = b_i \, \frac{\partial}{\partial x^j} \in T_p M$

$\omega(X_i) = a_j \, b_i = a_x \cdot b_i = 0 \Leftrightarrow a_x \perp b_i$

$\Rightarrow a_x$ is the vector normal to $\mathcal{S}(X_1, X_2)$

in $x$-coordinates as claimed $\Rightarrow$ true in every coordinate system.

Conclude: The vector of components normal to a hyperplane in a given coordinate system will transform covariantly, not contravariantly.

i.e. $(\tilde{N}_x)_j = (\tilde{N}_y)_j \frac{\partial y_i}{\partial x^j}$

Ref. $(N_x)_1, \ldots, (N_x)_n$ are components of the covector $(N_x)_i \, dx^i$. \checkmark
Summary: "The normal to a hyperplane is a covector not a vector!"

Note: if we have a metric $g$

$$\langle x, y \rangle = g_{ij} x^i y^j$$

$x = x^i \frac{\partial}{\partial x^i} \circ T_p M$

$y = y^i \frac{\partial}{\partial x^i} \circ T_p M$

then we can define a vector normal to a hyperplane by:

$$\langle N, x_i \rangle = 0 \quad i = 1, \ldots, n-1$$

where $\{x_1, \ldots, x_{n-1}\}$ span the hyperplane.
Example: Differential/Gradient of a fn

- Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) smooth

  - in \( x \)-words: \( y = f \circ x^i = f(x) \)
  - in \( y \)-words: \( y = f \circ y^{-1} = f(y) \)

Classically, it makes sense to differentiate \( f \) in direction of vector \( X \) at \( p \):

\[
\nabla_X f = X(f) = a_i \frac{\partial f}{\partial x^i} = a_i \frac{\partial f}{\partial x^i}:
\]

- Define \( X = a_i \frac{\partial}{\partial x^i} \)

- Define: \( df = \frac{\partial f}{\partial x^i} \) is the differential of \( f \)
Claim: df so defined is a 1-form $df \in T^*_pM$.

Pf.

\[
\frac{\partial f}{\partial y^x} dy^x = \frac{\partial f}{\partial x^i} e_y^x \frac{\partial x^i}{\partial y^x} dx^i = \frac{\partial f}{\partial x^i} dx^i
\]

\[
\delta i = \text{id}
\]

\[
\Rightarrow \delta i = \text{id}
\]

Conclude: The components of the gradient of $f$ transform covariantly not contravariant.

\[
\Rightarrow \text{"The gradient of a function is a covector, not a vector"}
\]

Without a metric: df makes sense, not $\nabla f$.
We need a metric to define a vector $\nabla f$ at each $p \in M$.

I.e., if we have $g$, $\langle x, y \rangle = g_{ij} x^i y^j$.

Define the vector $\nabla f$ by:

$$\langle \nabla f, y \rangle = df(y)$$

Riesz Representation — every linear functional can be represented by a vector through the inner product

$$\langle \nabla f, - \rangle : T_p M \to \mathbb{R}$$

$$df(-) : T_p M \to \mathbb{R}$$
Homework: Prove that if $g_{ij}$ is defined in each coordinate system so that (at $p_0$)

$$g_{ij} a^i b^j = \langle x, y \rangle = \bar{g} \bar{a} \bar{b}$$

$\uparrow$

$\bar{x}$-coords $\bar{y}$-coords: $X = \bar{a} \frac{\partial}{\partial \bar{x} \bar{a}} \bar{y}$

$X = a^i \frac{\partial}{\partial x^i}, \quad Y = b^j \frac{\partial}{\partial x^j}$

then $g_{ij}$ must transform by:

$$g_{ij} \frac{\partial x^i}{\partial \bar{x} \bar{a}} \frac{\partial x^j}{\partial \bar{y} \bar{b}} = \bar{g} \bar{a} \bar{b}$$

"$g$ is a $(0,2)$ tensor"

Show this is equivalent to

in matrix form: $J = \frac{\partial x^i}{\partial \bar{x} \bar{a}} \text{ row}$, $g = g_{ij} \text{ colm}$

$$J^T \bar{g} J = \bar{g}$$

$n \times n$ $n \times n$ $n \times n$ $n \times n$