Tensors in full generality:

- $T^1M$ bundle of vectors or contravariant tensors of order 1
- $T^*M$ bundle of co-vectors or covariant tensors of order 1

Each coord system defines a basis throughout.

Given a coord system $\mathbf{x}: U \rightarrow \mathbb{R}^n$, $\mathbf{x}$ determines a coordinate basis for $T_pM$ and $T^*_pM$ throughout $U$:

$T_pM = \text{Span} \left\{ \frac{\partial}{\partial x^i} \right\}$

$T^*_pM = \text{Span} \left\{ dx^i \right\}$

$T^*_pM$ acts linearly on $T_pM$ by $dx^i (a^j \frac{\partial}{\partial x^j}) = a^i$

$T_pM$ acts linearly on $T^*_pM$ by $\frac{\partial}{\partial x^i} (a; dx^j) = a_j$
\[ \text{Defn: A (}^{(a)} \text{-tensor } T_p \text{ at } p \text{ is an operator that acts linearly on } h \text{-copies of } T^*_p M \text{ and } l \text{-copies of } T_p M \]

\[ T : T^*_p M \times \cdots \times T^*_p M \times T_p M \times \cdots \times T_p M \rightarrow \mathbb{R} \]

\[ \text{in } h \text{ times} \]

\[ \text{in } l \text{-copies} \]

The set of all such operators is denoted \( Y^h_M(p) \).

\[ \text{Example: } \frac{\partial}{\partial x^i} \otimes dx^j \bigg|_p : T^*_p M \times T_p M \rightarrow \mathbb{R} \]

\[ \frac{d}{dx^i} \otimes dx^j \bigg|_p \left( a_i dx^i, b_j \frac{\partial}{\partial x^j} \right) = a_i b_j \]

This is \underline{multi-linear}: linear in each input slot.

\[ \otimes = \text{Tensor} \]
Thm 0: $Y^k(e)$ is a vector space of dimension $n^k$, and a basis is the set of all tensor products:

\[
\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_n}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_e} \right\}
\]

where these operate component-wise

\[
\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_n}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_e}(w^1, \ldots, w^k, x_1, \ldots, x_e)
\]

\[
= \frac{\partial}{\partial x^{i_1}}(w^1) \cdots \frac{\partial}{\partial x^{i_n}}(w^k) dx^{i_1}(x_1) \cdots dx^{i_n}(x_e)
\]

\[
= a_{i_1}^1 \cdots a_{i_n}^k \cdot b_1^{i_1} \cdots b_e^{i_n}
\]

\[
\omega^i = a^i_0 dx^0, \quad X_i = b_i^0 \frac{\partial}{\partial x^0}
\]
Thm 0 \{ dx^i \otimes \frac{\partial}{\partial x^i} \}_{i,j=1,\ldots,n} \text{ is a basis for } \mathcal{D}^r (\mathbb{R}^n).

Thm 1 \{ dx^i \otimes \cdots \otimes dx^i \otimes \frac{\partial}{\partial x^i} \otimes \cdots \otimes \frac{\partial}{\partial x^m} \}_{i_1,\ldots,i_m, j_1,\ldots,j_m = 1,\ldots,n} \text{ is a basis for } \mathcal{D}_e (\mathbb{R}^n).

---

*I.e.: more generally: let \( V_1, \ldots, V_n \) be finite vector spaces. We say \( L \) is a multilinear function on \( V_1, \ldots, V_n \) if

\[
L : V_1 \times \cdots \times V_n \rightarrow \mathbb{R}
\]

and \( L \) is linear in each slot

\[
L (v_1, \ldots, v_{k-1}, \alpha v_k + b \overline{v_k}, \ldots, V_n)
= \alpha L(v_1, \ldots, v_k, \ldots, V_n) + b \overline{L(v_1, \ldots, V_k, \ldots, V_n)}.
\]

\( \forall \alpha, b \in \mathbb{R}, \forall v_j \in V_j \).
Thm: The set of all $L$ is a vector space of dimension $= d_1 \ldots d_n$ where $d_n = \dim V_n$.

Def. (Homework - complete the details)

To start: recall:

- $V_n^* = \text{dual space of } V_n = \text{the set of all linear functionals defined on } V_n$

Riesz Representation Thm: The dual space $V^*$ of a finite-dimensional vector space $V$ is isomorphic to $d^d$. Indeed, if $e_1, \ldots, e_m$ is a basis for $V$, then $\{Le_1, \ldots, Le_m\}$ is a basis for $V^*$ where $Le_i(a, e_j) = a^i_j$. 

$L e_i (a^j e_j) = a^i_j$. 

\[ L e_i (a^j e_j) = a^i_j. \]
A basis for the multilinear functions on $V_1 \times \cdots \times V_n$ is

$$L_{ei_1} \otimes \cdots \otimes L_{ei_n}$$

where

$$L_{ei_1} \otimes \cdots \otimes L_{ei_n} (v_1 \otimes \cdots \otimes v_n) = L_{ei_1}^1 (v_i) \cdots L_{ei_n}^n (v_n)$$

Flip for homework
A \((\hat{e})\)-tensor \(T_{\alpha \beta} \in \mathcal{Y}^k_\hat{e}(\mathfrak{g})\) can be expressed in terms of the basis for \(\mathcal{Y}^k_\hat{e}(\mathfrak{g})\) determined by the \(x\)-coordinates:

\[
T_{\alpha \beta} = T_{i_1 \ldots i_n} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_n}} \otimes dx^{\hat{1}} \otimes \ldots \otimes dx^{\hat{k}}
\]

"The components of \(T\) in the coord system \(\hat{x}\)"

**Theorem:** Under change of basis \(x \rightarrow y\), the components transform by

\[
T_{\alpha_1 \ldots \alpha_n}^{\beta_1 \ldots \beta_n} = T_{i_1 \ldots i_n}^{\beta_1 \ldots \beta_n} \frac{\partial x^{i_1}}{\partial y^{\beta_1}} \cdot \ldots \cdot \frac{\partial x^{i_n}}{\partial y^{\beta_n}}
\]

Sum repeated up-down indices.

\(x\) is the fundamental tensor transformation law FIP.
A \((k^1)\)-tensor field is an assignment of a \((k^2)\)-tensor \(T_p \in \mathcal{F}_k^k(M)\) to each \(p \in M\). For \(p \in \mathcal{U}_x\), \(T_p\) can be expressed in terms of the \(x\)-coordinate basis for \(\mathcal{F}_k^k(M)\):

\[ T_p = T^{i_1 \ldots i_n}_{i_1 \ldots i_n} \, dx^{i_1} \otimes \ldots \otimes dx^{i_n} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_n}} \]

"Sum repeated up-down indices \(i_0, \ldots \rrbracket \)

**Defn:** \(T^{i_1 \ldots i_n}_{i_1 \ldots i_n}(p)\) are the \(x\)-components of the tensor \(T\) at \(p\).

**Q:** How do you describe "smoothly varying" tensor fields? 

**Ans:** say \(T^{i_1 \ldots i_n}_{i_1 \ldots i_n}(p)\) smoothly vary as func of \(p\).
Examples

1. Consider a \((1, 1)\)-tensor field \(T\). Then over \(dx^j\)

\[
T = T^i_j \, dx^i \otimes \frac{\partial}{\partial x^j}
\]

This defines a linear transformation of the tangent space \(T_M\) as follows:
(I.e., view it as operating on the \textit{first} slot)

\[
T(x, -) = T^i_j \, dx^i \otimes \frac{\partial}{\partial x^j}(x, -)
\]

If \(X = a^i \frac{\partial}{\partial x^i}\), then

\[
T(x, -) = T^i_j \, dx^i \otimes \frac{\partial}{\partial x^j}(a^i \frac{\partial}{\partial x^i}, -)
= a^i T^i_j \frac{\partial}{\partial x^i} \equiv \text{a vector at } p \text{ with } x\text{-coords}
\]

\[
\Gamma : \text{X} = a^i \frac{\partial}{\partial x^i} \rightarrow a^i T^i_j \frac{\partial}{\partial x^i}
\]
Define: $J^k_x = U J^k_x(p)$ is the vector field on $M$. The bundle of $(k,2)$-tensors over $M$. The "local trivialization" that define the smoothness properties of $J^k_x$ are given by:

$$\phi_x : U J^k_x(p) \rightarrow \mathbb{R}^n \times \mathbb{R}^{nk}$$

$$T_p \rightarrow (x(p), T^i_{\alpha_1...\alpha_k}(p))$$

Real #s at each $p$ & specification of indices.
Conclude: $T_p$ defines a linear transformation of $T_p M$ by mapping the $x$-components of vectors by

$$T_p: \mathbf{a} \xrightarrow{\delta x_j} b^i \delta x^i$$

$$b^i = T_{ij}^k a^k$$

hence

$$b = (b^1, \ldots, b^n)^T$$

$$a = (a^1, \ldots, a^n)^T$$

$$b = T a$$

Conclude: $(1) - \text{tensors}$ describe linear transformations of the tangent space.
Conclude: for every \( (\cdot) \)-tensor field \( T \) and \( p \in M \), \( \exists \) a coord. system defined in a nbhd of \( p \) in which

\[ T^i_j(p) = D + N \]

(F1P you need to show that every change of basis at \( p \) can be realized by some change of coords)

- **Theorem**: the eigenvalues of \( T^i_j(p) \) are coordinally independent objects
Proof: Recall:

\[ C(\lambda) = \det |\lambda I - M| = \text{characteristic polynomial of } M \]

- The roots of \( C \) are exactly the eigenvalues of \( M \), repeated according to multiplicity.
- \[ \det |\lambda I - A^{-1}MA| = \det |A^{-1}(\lambda I - M)A| \]
  \[ = \det(A) \det |\lambda I - M| \det(A) = \det |\lambda I - M| \]

Conclude: the coefficients of the characteristic polynomial of a (\( \ell \))-tensor are independent of coordinates, i.e.

\[ C(\lambda) = \det |T_{j}^{i}(x) - \lambda I_{j}^{i}| \]

is a polynomial in \( \lambda \) determined independent of coordinates.
Conclude: If $T \in \mathcal{Y}_1$, then for
\[
\det | \lambda I_j^i - T_j^i | = \prod_{i=1}^{n} (\lambda - \lambda_i) = C(\lambda)
\]
is defined indep't of coord's. Thus the eigenvalues $\lambda_i$ of a $(1)$-tensor are invariants.

**Theorem (Algebra):** If
\[
C(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + b_1 \lambda^{n-1} + \cdots + b_n
\]
then $b_k$ are invariants of $T_p$ and

1. $b_k = \text{sum of products of e-vals taken } k \text{ at a time (including multiplicity)}$
2. $b_k = \text{sum of the principal minor det's of } T^i_j \text{ of order } k$

**Defn:** A principal minor det of order $k$ is the det of a matrix obtained by deleting $n-k$ rows & cols of $T$. 
Definition: A principle minor det. of order \( k \) for an \( n \times n \) matrix \( M \) is the det. of a \( k \times k \) matrix obtained from \( M \) by deleting \( n-k \) rows \& columns of \( M \), s.t. the same rows as columns are deleted.

Corollary: 

\[ b_1 = \text{tr} \left( T_{i,j}^2 (p) \right) \text{ is invariant} \]

\[ b_n = \det T_{i,j}^2 (p) \text{ is invariant} \]

(FIP)
Alternate:  

\[ \text{Thm: The trace of } T_p = T_i^j(p) = \sum_{i=1}^{n} T_i^j(p) \]

is a coordinate independent object.

**Proof:**

1. **Fact:** trace \((AB) = \text{trace } BA \) for any matrices \(A, B\).

\[ \therefore \quad \text{tr}(A^{-1}MA) = \text{tr}(MA^{-1}A) = \text{tr } A \]

2. **Directly by summation convention:**

\[ T^x_{\beta} = T_i^j \frac{\partial x^j}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^i} \]

\[ \text{tr } T^x_{\beta} = T^x_{\alpha} = T_i^j \frac{\partial x^j}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^i} = T_i^j \delta^i_{\alpha} \]

\[ \delta^i_{\alpha} = \text{identity matrix} \]

\[ = \begin{cases} 0 & \text{if } \alpha \neq i \\ 1 & \text{if } \alpha = i \end{cases} \]

Note:

\[ \delta^i_{\alpha} \frac{\partial x^j}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^i} = \frac{\partial x^i}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^i} = \delta^x_{\alpha} \]
Conclude: The identity $\delta^i_j$ is a $(1,1)$-tensor independent of coordinates.

More generally: Let $T^{i_1 \cdots i_k}_{\delta_1 \cdots \delta_k}$ components of $T \in \mathcal{V}^k$. Then

$$T^{i_1 \cdots i_m}_{\delta_1 \cdots \delta_k} = \text{"contraction of } T \text{ on the } i_1 \delta_1 \text{ indices"}$$

are the components of a tensor in $\mathcal{V}^{k-1}$.

Homework: Show that symmetry in the components of a $(1,1)$ tensor $T$ is not a coordinate independent property of $T_{ij}$ i.e., $T^i_j = T^j_i \neq T^\alpha_\beta = T^\beta_\alpha$. 
Solution: 
\[ (A^{-1}TA)^T = A^T T^t A^{-T} \] 
not always if 
\[ A \neq A^T, \quad A^{-1} = A^{-T} \]

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} \lambda_1, 0 \\ 0, \lambda_2 \end{bmatrix} \]

\[ A^{-1}TA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1, 0 \\ 0, \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1, \lambda_1 \\ 0, \lambda_2 \end{bmatrix} \]

\[ T \text{ symmetric}, \quad \bar{T} \text{ not } \]
Example: \( T = T_{ij} \, dx^i \otimes dx^j \in T^*_p \)  

I.e., at each \( p \), \( T_{ij}(p) \) are the components of a \((2)\)-tensor.

\( T_p \) defines a \underline{bi-linear} form on \( T_p M \); i.e.,

\[ T_p : T_p M \times T_p M \rightarrow \mathbb{R} \]

\[ T_p (X_p, Y_p) = T_{ij} \left. \frac{\partial}{\partial x^i}(X_p) \right|_p \left. \frac{\partial}{\partial x^j}(Y_p) \right|_p \]

\[ X_p = \left. \frac{\partial}{\partial x^i}(X_p) \right|_p \quad Y_p = \left. \frac{\partial}{\partial x^j}(Y_p) \right|_p \]

"E.g., an inner product on \( T_p M \) defines a \((2)\)-tensor at \( p \)."

Under change of coordinates:

\[ T_{AB} = T_{ij} \left. \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \right|_p \Rightarrow T = J^T T J \quad J = \left[ \frac{\partial x^i}{\partial y^b} \right] \]
Def: A \mathbf{Y}_n \text{ is symmetric if}

\[ T_{ij} = T_{ji} \]

Thm: Symmetry is a coordinate independent property of (\(2\))-tensors.

Proof: \[ T_{ab} = T_{ij} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \]

\[ = T_{ji} \frac{\partial x^i}{\partial y^b} \frac{\partial x^j}{\partial y^a} = T_{ij} \frac{\partial x^j}{\partial y^b} \frac{\partial x^i}{\partial y^a} \]

\[ = T_{ba} \]

"i.e., assume \( T_{ij} = T_{ji} \) \( \Rightarrow \) \( T_{ab} = T_{ba} \)"

\(16\)
**Defn**: A symmetric, non-degenerate $(0,2)$-tensor defined on all of $M$ is called a **metric Riemannian metric** on $M$.

I.e., it defines an inner product on each $T_p M$, $p \in M$.

**Thm**: A non-degenerate is an invariant property of the $(2)$ or $(1)$ tensor.

P.f. $\det J^t T J = \det J^t\det T \det J$

$= \det T$

$\therefore \det T \neq 0 \iff \det T \neq 0$

We always assume our coordinate charts satisfy $\det J \neq 0 \neq \det J^{-1}$.
Let $g$ denote a metric on $M$

$g \in \mathcal{Y}_2^0(M)$.

In coordinates

$g = g_{ij} \, dx^i \otimes dx^j$

For $X_p, Y_p \in T_p M$, define

$\langle X_p, Y_p \rangle_p = g(X, Y) \big|_p$

$= g_{ij}(p) \, dx^i \otimes dx^j \left( a^2 \left. \frac{\partial}{\partial x^i} \right|_p b \right. \left. \frac{\partial}{\partial x^j} \right|_p$

$= g_{ij} \left. a^2 b^j \right|_p = a^{tr} g \, b$

$a = \left( \begin{array}{c} a^1 \\ \vdots \\ a^m \end{array} \right), \quad b = \left( \begin{array}{c} b^1 \\ \vdots \\ b^m \end{array} \right), \quad g = g_{ij}$
• Drop the $P$ and write

$$\langle x, y \rangle = g(x, y) = \delta_{ij} a^i b^j$$

where the components of the vector fields $X$ & $Y$ and metric $g$ are always assumed to be computed at the same point $p \in M$.

• **Theorem (Algebra)**: Let $\langle \cdot, \cdot \rangle$ denote a non-degenerate symmetric bi-linear form defined on a vector space $V^n$. Then one can always construct an orthonormal basis via the Gramm-Schmidt procedure; i.e., $E \{ e_i \}_{i=1}^n$, such that

$$\langle e_i, e_j \rangle = s_i \delta_{ij} \in \text{sum}$$

where $s_i = \pm 1$. Moreover, the number
of $s_i < 0$ and the number of $s_i > 0$ are the same for every such orthonormal basis. Thus $\sum s_i$ is independent of basis.

**Corollary:** Let $g$ be a metric on $M$. Then $\exists$ a nbhd of each $p \in M$ such that

$$g_{ij}(p) = \begin{bmatrix} s_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & s_n \end{bmatrix}$$

$s_i = \pm 1$. Moreover, if the signature of $\langle \cdot, \cdot \rangle_p$ is $k$, $0 \leq k \leq n$, then we can arrange it so that

$s_1 = \ldots = s_a = -1$

$s_a+1 = \ldots = s_n = +1$

$$a = \frac{n-k}{2}$$

i.e., $-a + (n-a) = -2a = k-n$

$$a = \frac{n-k}{2}$$
Proof: Fix $P$. Use the following fact:

**Lemmas:** Let $X_1, \ldots, X_n$ be linearly independent vectors in $T_pM$ (i.e., a basis for $T_pM$). Then $\exists$ a coordinate system

$$\mathcal{X} : U_x \rightarrow \mathbb{R}$$

$$p \in U_x$$

such that

$$X_i^p = \frac{\partial}{\partial x^i}.$$
Proof: Let $y_\sim = M(x - x_0) + y_0$. 

$\tilde{x} = (x', \ldots, x^n)$ a coordinate system, $p \in \mathfrak{U}_x$.

$M = \text{const } n \times n \text{ matrix }$. 

$\tilde{x}(\phi) = \tilde{x}_0$.

We find $M$ so that in $y_\sim$-words,

$$X_i \mid_p = \frac{\partial}{\partial y_i} \bigg|_p$$

But $\Box \Rightarrow \frac{\partial y^d}{\partial x^i} = M = \text{const } \in \mathfrak{J}$

$$X_i = a_i \frac{\partial}{\partial x^o} = b_i \frac{\partial}{\partial y^x}$$

$$b_i = \frac{\partial y^d}{\partial x^o} a^o$$

Want: $b_i = \delta_i \Rightarrow \frac{\partial y^d}{\partial x^o} a^o = \delta^o \quad i = 1, \ldots, n$
Thus:

$$\frac{\partial y^k}{\partial x^i} \delta_i^j = \delta_j^k, \quad i = 1, \ldots, n$$

$$J \left[ \frac{\partial g_1}{\partial x^i} \ldots \frac{\partial g_n}{\partial x^i} \right] = \text{id}$$

$$\Rightarrow \text{choose } J = M = \left[ \begin{array}{cccc}
\frac{\partial g_1}{\partial x^i} & \cdots & \frac{\partial g_n}{\partial x^i}
\end{array} \right] \text{ suffices}$$
Defn: If \( g \) is a smooth metric of signature \( n \) defined throughout \( M \), then \( g \) is called a Riemannian metric. If sign of \( g \) is \( n-2 \), it is called a Lorentzian metric or Minkowskian metric.

Thm: If \( g \) is a Riemannian metric, then

\[ \langle X, X \rangle > 0 \quad \text{iff} \quad X \neq 0. \quad \text{(at each)} \]

Pt. \[ \langle X, X \rangle = g_{ij} a^i a^j \quad \text{at each } p \in M, \]

\[ X = \sum a^i \partial \]

But \( \forall p \in M \), \( \exists \) coordinates \( y^i \) in which \( g_{ab} = \delta_{ab} \)

\[ \langle X, X \rangle_p = \sum_{a=1}^{n} (a^i \partial_a)^2 > 0 \]

\[ \text{iff } X \neq 0. \quad X = a^i \frac{\partial}{\partial y^i} \]
For a Lorentzian metric, \( \langle X, X \rangle_p = 0 \) is possible — in this case we say \( X_p \) is null at \( p \).

Let \( g \) be a metric.

Notation: \( g^{ij} = (g_{ij})^{-1} \)

Thm: \( g^{ij} \) transforms like a \( (2, 0) \) tensor. (FIP)

Pf. \( g^{ij} = (g_{ij})^{-1} \implies g^{ij} g_{ij} = \delta^i_j \)

We show \( g^{\alpha \beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \) is the inverse of \( g_{\alpha \beta} \). Indeed
A metric $g$ defines an inner product $\langle \cdot, \cdot \rangle_p$ at each $p \in M$, and this defines a canonical mapping (isomorphism)

$$T_p M \leftrightarrow T^*_p M$$

$$X_p \mapsto \omega^p \text{ where } \omega^p(Y_p) = \langle X_p, Y_p \rangle$$

$$\omega_x = \langle X, - \rangle$$

**Q:** How do components of $X_p$ map to components of $\omega^p$?

**Ans:**

$$X_p = a^i \frac{\partial}{\partial x^i}, \quad Y_p = b^j \frac{\partial}{\partial x^j} \quad \text{then}$$

$$\omega^p(Y_p) = \langle X_p, Y_p \rangle = g_{ij} a^i b^j = a^i b^j$$

$$w^p = a^i dx^i$$
Conclude: a metric gives a natural identification

\[ X_p = a^i \frac{\partial}{\partial x^i} \leftrightarrow a^i \, dx^i = \omega^p \]

\[ a^- = a^i \cdot g^i_0 \cdot a^0 \]

Defn: We say \( a^- \) has been obtained from \( a^i \) by raising the index by the metric.

Conversely:

\[ g^{ij} = (g_{ij})^{-1} \]

\[ g^{i0} \cdot a^- = a^i \]
More generally: We can define the raising/lowering of an index by a metric in an arbitrary tensor — 

e.g. \( R = R^i_{\ j\ k\ l} \ dx^i \otimes dx^j \otimes dx^k \otimes dx^l \)

Components: \( R^i_{\ jke} \)

Lower the index \( i \): \( R^i_{\ jke} = R^0_{\ jke} \)

Lower the index \( j \): \( R^i_{\ jke} \ dx^i \otimes dx^j \otimes dx^k \otimes dx^0 = 0 \)

Thm: Raising a lowering an index by the metric is a tensor operation.
Recall: A linear transformation is "self-adjoint" if the corresponding transformation matrix is symmetric.

Q: If symmetry is not a covariant property of (1)-tensors that represent linear trans. of the vector space $T_p M$, then how can "self-adjoint" be invariant?

Ans: Defn: A linear transformation $T : V ightarrow V$ of a vector space $V$ is self-adjoint relative to an inner product $\langle \cdot, \cdot \rangle$ on $V$ if

$$\langle v_1, TV_2 \rangle = \langle TV_1, v_2 \rangle$$

for all $v_1, v_2 \in V$. 
For a \((1,1)\)-tensor \(T\) on \(M\) 

\[ g \Rightarrow \langle \cdot , \cdot \rangle \text{ on } T_p M, \]

this means the self-adjoint at each \(p\) mean:

\[ \langle X_p, T(Y_p) \rangle = \langle T(X_p), Y_p \rangle \]

\[ T = T^i_j \frac{\partial}{\partial x^i} \otimes dx^j \in \mathfrak{g}^* \]

\[ T(X) = T^i_j a^j \frac{\partial}{\partial x^i} \]

\[ T: a^i \frac{\partial}{\partial x^i} \mapsto T^i_j a^j \frac{\partial}{\partial x^i} \]

Thus:

\[ \langle X_p, T(Y_p) \rangle = \langle T(X_p), Y_p \rangle \]
Thm: Let $T_{i_1 \ldots i_n}^{o_1 \ldots o_m}$ be compts. of $T \in \mathfrak{CJ}$. Then symmetry or antisymmetry will any two upper or any two lower indices is a covariant (coordinate indep.) effect property. FIP

Eq: $T_{i_1 \ldots i_n}^{o_1 \ldots o_m} \overset{\partial x^i}{\partial y^d} \overset{\partial x^j}{\partial y^a} \ldots = T_{d_1 \ldots d_m}^{a_1 \ldots a_n}$

Or: base it on the invaraid properties:

$T(\ldots x_i, y_j \ldots) = T(\ldots y_j, x_i \ldots)$

$\forall x_i, y_j \Rightarrow$ symmetry of corresponding indices.
A metric defines the notion of length on a manifold:

\[ L = \int_{s_1}^{s_2} \sqrt{\langle X, X \rangle} \, ds \]

\[ X = \frac{\partial}{\partial x^i} \quad \langle X, X \rangle = g_{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \]

in the \( z \)-coordinate system.

**Thm:** \( L \) is independent of parameterization.

**Note:** By this formula, you must break it up into pieces defined in one coordinate system.

We return to metric later.
Sylvester's Theorem: (Law of Inertia)

Let \( g \) be any symmetric, non-degenerate \((0,2)\)-tensor defined in a nbhd of \( P \).

\[ g_{ij} = g_{ji} \text{ in every coord syst } \& \det g_{ij} \neq 0 \]

Then \( \forall P \in U \) \( \exists \) a coord system \( y^a \) defined in a nbhd of \( P \) st

\[
\begin{bmatrix}
\delta_{ij} & \\
0 & \delta_n
\end{bmatrix}, \quad \delta_n = \pm 1.
\]

**Defn:** \( \frac{2}{\partial y^a} \bigg|_P \) is an orthonormal basis for \( g \) at \( P \)

\( n_+ - n_- = \text{signature of } g \)

\( n_- = \# \text{ of } -1's \), \( n_+ = \# \text{ of } +1's \).

**Thm:** Every orthonormal basis has the same signature.

\( n_+, n_- \) are constant in a nbhd of \( P \).

(FIP)
In fact, \( \delta^{-1}(\theta) = \frac{\lambda_i(\theta)}{|\lambda^2(\theta)|} \) where \( \lambda_i \) are the eigenvalues of the symmetric matrix

\[ G_{ij} = G_{ij}(\theta) \quad [\text{defined at } p \text{ in } \alpha] \]

If it holds in \( x \)-words, won't hold in \( y \)-words in general

**Proof:**

1. Start with \( G_{ij} \) in \( x \)-words, \( p \in M \)
2. Define \( G^i_j = G_{ij} \) "symmetric matrix @ \( p \) in \( x \)-words"
3. \( G^i_j \) symmetric matrix in \( \mathbb{R}^n \) \( \Rightarrow \) real evals \( \lambda_a \) on basis of e-vectors \( r^i_a \in \mathbb{R}^n \) [o.n. w.r.t Euclidean coord metric]
4. Define: \[
\frac{\partial}{\partial y^x} = r^i_a(\theta) \frac{\partial}{\partial x^i}; \quad J = \frac{\partial x^i}{\partial y^x} = r^i_a(\theta)
\]
Then:

\[ g_{\alpha \beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \quad \Rightarrow \quad \bar{g} = J^T g J \]

But \( r_\alpha \) on \( J^T = J^{-1} \) \( \Rightarrow \quad \bar{g} = J^{-1} g J \)

Thus:

\[ \frac{\partial g(p)}{\partial x^\beta} = \begin{bmatrix} \lambda_1(y) & 0 \\ 0 & \lambda_n(y) \end{bmatrix} = \Lambda \]

\[ S^T L S = \begin{bmatrix} \frac{\lambda_1}{\lambda_2} & 0 \\ 0 & \frac{\lambda_2}{\lambda_3} \end{bmatrix}, \quad S = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \]

\[ S^T J^T g J S = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \]

\[ y^\alpha = (JS)^\alpha \cdot x^i \]

\[ \text{does it?} \quad \begin{cases} \text{at } p, \text{ not in a nbhd, as} \\ \text{e-vectors } r^\alpha_i \text{ are not in } \text{general word vector fields} \end{cases} \]
Thm: (Riemann Normal Coordinates)

Let $g$ be a non-degenerate symmetric $(0, 2)$-tensor on $M$ (a metric). Then, for any $p \in M$, there exists a coordinate system $\gamma$ defined in a neighborhood of $p$ such that:

$$g_{ij}(p) = \begin{bmatrix} \delta_i \cdot \delta_j & 0 \\ 0 & \delta^i \cdot \delta^j \end{bmatrix} \equiv \gamma_i \cdot \gamma_j = \delta_{ij}$$

(1)

$$\frac{\partial}{\partial x^k} g_{ij}(p) = g_{ij, k}(p) = 0.$$  (2)

That is, by Taylor expanding $g$ about $p$, this says $g_{ij}(p) = \gamma_i \cdot \gamma_j$ to within errors $O(|x - \gamma(p)|^2)$.

In fact, you cannot make $g_{ij, k}(p) = 0$ in general, and $R_{ijk}^l(p)$ is a measure of the 2nd derivatives of $g$ at $p$. 
Proof: Consider the (geodesic) equation
\[ \ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0 \quad (\star) \]
\[ \Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left\{ -g_{ij,\ell} + g_{kij,\ell} + g_{\ell ji} \right\} \quad \text{(known)} \]

Note: \((\star)\) is a 2nd order autonomous ODE defined in \(x\)-coords whose solution curves
\[ x^k = x^k(t) \]
solve eqn \((\star)\).

Thm: (geodesics) Solutions of \((\star)\) define the same curve in \(M\), indep of coordinates.
Call this curve \(\gamma(t) \in M\)
Moreover, \(\gamma(t)\) are the geodesics of \(\mathcal{G}\).
[Proof: later]

We use \((\star)\) to define Riemann Normal coordinates in a nbhd of \(P\), & prove they satisfy (a), (b).
- Facts we need:
  - **Thm (ODE)**: If \( \xi(x(t)) \) of (*) defined in a nbhd of \( p \in M \), such that

\[
\begin{align*}
\xi_x(0) &= p \\
\frac{d}{dt} \xi_x(0) &= \dot{x}_x(p) = \xi \in T_p M
\end{align*}
\]

**Cor.** \( \xi_{sx}(t) = \xi_x(st) \), \( s \in \mathbb{R} \). (Homework)

[pf. Prove it directly from (*) in \( x \)-coordinates]

**Thm (ODE)**: If \( \{X_1, \ldots, X_n\} \subset T_p M \) is a basis

\[
\xi = \xi \sum X_i \in T_p M,
\]

then the mapping

\[
\begin{align*}
\xi^2 &: \rightarrow \xi_x(1) \\
0 &\rightarrow p
\end{align*}
\]

is 1-1 onto \& regular in a nbhd of \( p \).

pf. Write (*) as a 1st order system \& apply standard thms from ODE's.
Thus: Define a map from $T_p M \to M$:

$$T_p M \supset X \mapsto \chi_x (1)$$

Picture:

\[ \begin{array}{c}
\text{sx} \\
\gamma_x(s) \\
\chi_x(1) \\
\chi \in T_p M
\end{array} \]

"$\gamma_x(1)$ is the point 1-unit along the soln $\chi_x(t)$ starting at $p$ with velocity $\chi_p$"

- Now choose $\{X_1, \ldots, X_n\}$ an orthonormal basis at $p$.
- Define the coordinate system $Y$:

\[ \begin{array}{c}
y^i : y^i_x \mapsto \chi_x(1) \\
X = y^i X_i
\end{array} \]

"$y$ are the components of $X$ wrt $\{X_1, \ldots, X_n\}$.

Defined 1-1 regular in a nbhd of $p$ by Thm (ODE)"
Defn: \( y \) are the RNC's at \( p \) associated with basis \( \{ x_1, \ldots, x_n \} \)

- **Claim:** \( g_{ij}(p) = \delta_{ij} g_{ij,k}(p) = 0 \) in \( y \)-coords, if \( \{ x_i \} \) an orthonormal basis.

**Proof:** Choose \( x = s^i x_i \in T_x \mathcal{N} \), \( g \) consider the solution curve \( \gamma^k_x(t) \) of \( x \) in \( y \)-coords. Then by defn: \( s^k = \gamma^k_x(1) = y^k (\gamma_x(1)) \). Thus \( \gamma^k_x(1) = y^k (\gamma_x(1)) = y^k (\gamma_x'(1)) = \epsilon^k_x x^j \)

So \( \gamma^k_x(t) = t s^k \) \( \forall t, s^k \) fixed.

**Conclude:** \( \gamma^k_x(t) = 0 \) in \( y \)-coords.

\( \therefore \) by (**) \( \Gamma^h_{ij} \delta^i j = \Gamma^h_{ij} s^i s^j = 0 \)

\( \Rightarrow \Gamma^h_{ij} = 0. (\text{at } p) \ \forall \delta \in \mathbb{R}^n \)
Define:  \[ \Gamma_{ij,h} = \{-g_{ij,h} + g_{ki,j} + g_{jk,i}\} \]

Then \[ \Gamma_{ij}^h = 0 \iff \Gamma_{ij,h} = 0 \ (\forall i,j,k) \]

Lemma:  \[ \Gamma_{jk,i} + \Gamma_{ik,j} = g_{ij,h}(\pi) = 0 \]

Q: Since \( \dot{x}_h^t = 0, \ \dot{x}_h^k = \xi^h \) all along the geodesic, we have \( \Gamma_{ij}^h (x(t)) \xi^i \xi^j = 0 \), so why can't we conclude \( g_{ij,h} = 0 \) all along \( x(t) \), \( \xi \) hence in a nbhd of \( p \)?

Ans: at \( x(t), t \neq 0 \), we do not have \( \Gamma_{ij}^h (x(t)) \xi^i \xi^j = 0 \) \( \forall \xi, \ \text{only the} \ \xi^i = x_i^i (\pi) \).

Only at \( p = x(0) \) do we have \( \Gamma_{ij}^h \xi^i \xi^j = 0 \ \forall \xi \in \mathbb{R}^n \)