Tensors in full generality:

- TM bundle of vectors or contravariant tensors of order 1

  T^*M bundle of co-vectors or covariant tensors of order 1

~~Basis for T_pM: $\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$~~

~~Each coord system defines a basis~~ throughout

- Given a coord system $\underset{\sim}{x}: \mathcal{U} \to \mathbb{R}^n$, $\underset{\sim}{x}$ determines a coordinate basis for TM, T^*M throughout $\mathcal{U}$:

$$T_pM = \mathrm{Span}\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$$

$$T_p^*M = \mathrm{Span}\left\{dx^1\big|_p, \dots, dx^n\big|_p\right\}.$$

- T_p^*M acts linearly on T_pM by $dx^i\left(a^j \frac{\partial}{\partial x^j}\right) = a^i$

  T_pM acts linearly on T_p^*M by $\frac{\partial}{\partial x^i}\left(a_j dx^j\right) = a_i$

Defn: A $\binom{k}{l}$-tensor T_p at p is an operator that acts linearly on k-copies of $T^*_p M$ and l-copies of $T_p M$

$$T_p : \underbrace{T_p^* M \times \cdots \times T_p^* M}_{k \text{ times}} \times \underbrace{T_p M \times \cdots \times T_p M}_{l\text{-copies}} \to \mathbb{R}$$

The set of all such operators is denoted

$$\mathcal{T}^k_l(p).$$

Example: $\left.\frac{\partial}{\partial x^i} \otimes dx^j \right|_p : T_p^* M \times T_p M \longrightarrow \mathbb{R}$

$$\left.\frac{\partial}{\partial x^i} \otimes dx^j \right|_p \left(a_\sigma dx^\sigma , b^\tau \frac{\partial}{\partial x^\tau} \right) = a_i b^j$$

picks out i comp   picks out j component

This is multi-linear: linear in each input slot.

$\otimes$ = Tensor

- <u>Thm</u> 1 : $\mathcal{T}^k_\ell(p)$ is a vector space of dimension $n^k n^\ell$, and a basis is the set of all tensor products:

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \right\}$$

a set of $n^{k+\ell}$ objects

$i_1 \cdots i_k\ j_1 \cdots j_\ell$ run $1 \cdots n$

where these operate <u>componentwise</u>

$$\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell} \left(\overset{\in T_p^*M}{\omega^1}, \ldots, \overset{\in T_p^*M}{\omega^k}, \overset{\in T_pM}{X_1}, \ldots, \overset{\in T_pM}{X_\ell} \right)$$

$$= \frac{\partial}{\partial x^{i_1}}(\omega^1) \cdots \frac{\partial}{\partial x^{i_k}}(\omega^k)\, dx^{j_1}(X_1) \cdots dx^{j_\ell}(X_\ell)$$

$$= a^1_{i_1} \cdots a^k_{i_k}\, b^{j_1}_1 \cdots b^{j_\ell}_\ell$$

$$\omega^i = a^i_\sigma dx^\sigma, \quad X_i = b_i^\sigma \frac{\partial}{\partial x^\sigma}$$

<u>Thm</u> ① $\left\{ dx^i \otimes \frac{\partial}{\partial x^j}\Big|_p \right\}_{i,j=1,\dots,n}$ is a basis for $\mathcal{J}^1_1(p)$.

<u>Thm</u> ② $\left\{ dx^{i_1} \otimes \cdots \otimes dx^{i_n} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_n}} \right\}$, $i_1, \dots, i_n, j_1, \dots, j_n = 1, \dots, n$ is a basis for $\mathcal{J}^k_\ell(p)$.

- I.e. ; more generally : let $V_1, \dots, V_n$ be finite vector spaces. We say L is a multilinear function on $V_1, \dots, V_n$ if

$$L : V_1 \times \cdots \times V_n \longrightarrow \mathbb{R}$$

and L is linear in each slot

$$L(v_1, \dots, v_{k-1}, a v_k + b \bar{v}_k, v_{k+1}, \dots, v_n)$$

$$= aL(v_1, \dots, v_k, \dots, v_n) + b(v_1, \dots, \bar{v}_k, \dots v_n).$$

$V_1 \times \cdots \times V_n = \{ (v_1, \dots, v_n) , \ v_i \in V_i \}$

<u>Thm</u>: The set of all L is a vector space of dimension $= d_1 \cdots d_n$ where

$$d_k = \dim V_k .$$

Pf. (Homework — complete the details)

To start: recall:

- $V_k^* \equiv$ dual space of $V_k \equiv$ the set of all linear functionals defined on V_k

- <u>Riez Representation Thm</u>: The dual space V^* of a finite dimensional vector space V is ~~isomorphic~~ also equal to d $\boxed{\cancel{\text{also } d}}$. Indeed, if $e_1, \ldots, e_m$ is a basis for V, then $\{L_{e_1}, \ldots, L_{e_m}\}$ is a basis for V^* where

$$L_{e_i}(a^j e_j) = a^i .$$

- ~~Facts~~ A basis for the multilinear functions on $\bar{V}_1 \times \cdots \times \bar{V}_n$ is

$$L^1_{e_{i_1}} \otimes \cdots \otimes L^n_{e_{i_n}}$$

where

$$L^1_{e_{i_1}} \otimes \cdots \otimes L^n_{e_{i_n}} (v_1, \ldots, v_n)$$

$$= L^1_{e_{i_1}}(v_1) \cdots L^n_{e_{i_n}}(v_n)$$

FIP for homework

- A $\binom{k}{l}$-tensor $T_p \in \mathcal{T}^k_l(p)$ can be expressed in terms of the basis for $\mathcal{T}^k_l(p)$ determined by the $\underset{\sim}{x}$-coords —

$$T_p = \underbrace{T^{i_1 \cdots i_k}_{j_1 \cdots j_l}}_{} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}$$

"The components of T in the coord system $\underset{\sim}{x}$"

<u>Theorem</u>: Under change of basis $\underset{\sim}{x} \to \underset{\sim}{y}$, The components transform by

$$(*) \quad \boxed{\bar{T}^{\alpha_1 \cdots \alpha_k}_{\beta_1 \cdots \beta_l} = T^{i_1 \cdots i_k}_{j_1 \cdots j_l} \frac{\partial x^{j_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{j_l}}{\partial y^{\beta_l}} \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{i_k}}}$$

Sum repeated up-down indices

(*) is the fundamental tensor transformation law FIP

- A $\binom{k}{\ell}$-tensor field is an assignment of a $\binom{k}{\ell}$-tensor $T_p \in \mathcal{T}^k_\ell(p)$ to each $p \in M$.

For $p \in \mathcal{U}_{\underset{\sim}{x}}$, T_p can be expressed in terms of the $\underset{\sim}{x}$-coord basis for $\mathcal{T}^k_\ell(p)$:

$$T_p = T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}}$$

"Sum repeated up-down indices i_σ, j_τ"

<u>Defn</u>: $T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k}(p)$ are the $\underset{\sim}{x}$-components of the tensor T at p.

Q: How do you describe "smoothly varying" tensor fields?

Ans: say $T^{j_1 \cdots j_\ell}_{i_1 \cdots i_k}(p)$ smoothly vary as fn's of p.

Examples

① Consider a $\binom{1}{1}$-tensor field T. Then over $\mathcal{U}_{\underset{\sim}{x}}$,

$$T = T^i_j \, dx^j \otimes \frac{\partial}{\partial x^i}$$

for each p, this defines a linear transformation of the tangent space T_pM as follows:
(I.e., view it as operating on the first slot)

$$T(X, -) = T^i_j \, dx^j \otimes \frac{\partial}{\partial x^i}(X, -)$$

If $X = a^i \frac{\partial}{\partial x^i}$, then

$$T(X, -) = T^i_j \, dx^j \otimes \frac{\partial}{\partial x^i}\left(a^\sigma \frac{\partial}{\partial x^\sigma}, -\right)$$

$$= a^j T^i_j \frac{\partial}{\partial x^i} \equiv \text{a vector at } p \text{ with } \underset{\sim}{x}\text{-comp's } a^j T^i_j \checkmark$$

$T : X = a^i \frac{\partial}{\partial x^i} \longmapsto a^j T^i_j \frac{\partial}{\partial x^i}$

<u>Defn</u>: $\mathcal{J}^k_\ell \equiv \bigcup_{p \in M} \mathcal{J}^k_\ell(p)$ is the vector bundle of $\binom{k}{\ell}$-tensors over M. The "local trivializations" that define the smoothness properties of $\mathcal{J}^k_\ell$ are given by

$$\phi_{\underset{\sim}{x}} : \bigcup_{p \in \mathcal{U}_{\underset{\sim}{x}}} \mathcal{J}^k_\ell(p) \longrightarrow \mathbb{R}^n \times \mathbb{R}^{(n^{k+\ell})}$$

$$T_p \longmapsto \left(\underset{\sim}{x}(p),\ T^{j_1 \cdots j_k}_{i_1 \cdots i_k}(p)\right)$$

real #'s at each p & specification of indices

Conclude: T_p defines a linear transformation of $T_p\mathcal{M}$ by mapping the x-components of vectors by

$$T_p : a^j \frac{\partial}{\partial x^j} \longmapsto b^i \frac{\partial}{\partial x^i}$$

$$\boxed{b^i = T^i_j \, a^j} \Leftarrow \text{Matrix multiplication}$$

I.e. if $\underset{\sim}{b} = (b^1, \dots, b^n)^{tr}$

$\underset{\sim}{a} = (a^1, \dots, a^n)^{tr}$

$$\underset{\sim}{b} = T \underset{\sim}{a} \qquad T \equiv T^{i \leftarrow \text{row}}_{j \leftarrow \text{column}}$$

~~Conclude: $\binom{1}{1}$-tens~~

Conclude: $\binom{1}{1}$-tensors describe linear transformations of the tangent space

- Conclude: for every $\binom{1}{1}$-tensor field T and $p \in M$, $\exists$ a coord system defined in a nbhd of p in which

$$T^i_j(p) = D + N$$

(FIP You need to show that every change of basis at p can be realized by some change of coords)

- Theorem: the eigenvalues of $T^i_j(p)$ are coordinate independent objects

Proof: Recall:

$$C(\lambda) \equiv \det|\lambda I - M| \equiv \text{characteristic polyn of } M$$

- The roots of c are exactly the eigenvalues of M, repeated according to multiplicity

- $$\det|\lambda I - A^{-1}MA| = \det|A^{-1}(\lambda I - M)A|$$

$$= \cancel{\det(A^{-1})} \det|\lambda I - M| \cancel{\det(A)} = \det|\lambda I - M|$$

Conclude: the coefficients of the characteristic polynomial of a $\binom{1}{1}$-tensor are independent of coordinates; i.e.

$$C(\lambda) = \det|T^i_j(p) - \lambda I^i_j|$$

is a polynomial in λ determined indep't of coordinates.

Conclude: If $T \in \mathcal{T}^1_1$, then $\forall p$,

$$\det |\lambda I^i_j - T^i_j| = \prod_{i=1}^{n} (\lambda - \lambda_i) = C(\lambda)$$

is defined indep't of coord's. Thus the eigenvalues λ_i of a $\binom{1}{1}$-tensor are invariants.

Theorem (Algebra): If

$$C(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + b_1 \lambda^{n-1} + \cdots + b_n,$$

then b_k are invariants of T_p and

(1) b_k = sum of products of e-vals taken k at a time (including multiplicity)

(2) b_k = sum of the principl minor det's of T^i_j of order k

Defn: A princ. minor det of order k is the det of a matrix obtained by deleting the same $n-k$ rows & cols of T^i_j

(12B)

<u>Defn</u>: A principle minor det. of order k for an $n\times n$ matrix M is the det. of a $k\times k$ matrix obtained from M by deleting $n-k$ rows & colms of M, s.t. the <u>same</u> rows as colms are deleted.

<u>Corollary</u>: $\boxed{b_1 = \text{trace } T^i_j(p) \text{ is invariant}}$

$b_n = \det T^i_j(p)$ is invariant

(FIP).

Alternate:

- Thm: The trace of $T_p \equiv T^{i}_{\ i}(p) \equiv \sum_{i=1}^{n} T^{i}_{\ i}(p)$ is a coordinate indept object

Pf: ① Fact: trace (AB) = trace BA $\forall$ $n\times n$ matrices A, B

$$\therefore \quad tr(A^{-1}MA) = tr(MA^{-1}A) = tr M$$

② Directly by summation convention:

$$\bar{T}^{\alpha}_{\ \cdot\beta} = T^{i}_{\ j} \frac{\partial x^j}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^i}$$

$$tr\, \bar{T}^{\alpha}_{\ \beta} = \bar{T}^{\alpha}_{\ \alpha} = T^{i}_{\ j} \underbrace{\frac{\partial x^j}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^i}}_{\delta^{j}_{\ i}} = T^{i}_{\ j}\, \delta^{j}_{\ i} = T^{j}_{\ j} \checkmark$$

$$\delta^{j}_{\ i} \equiv \text{identity matrix} \equiv \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Note:

$$\delta^{i}_{\ j} \frac{\partial x^j}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^i} = \underbrace{\frac{\partial x^i}{\partial y^\beta} \frac{\partial y^\alpha}{\partial x^i}}_{J^{-1}\cdot J} = \delta^{\alpha}_{\ \beta} \checkmark$$

Conclude: The identity δ^i_j is a $\binom{1}{1}$-tensor indep. of coords.

- More generally: let $T^{i_1 \cdots i_k}_{j_1 \cdots j_\ell}$ components of $T \in \mathcal{T}^k_\ell$. Then

$$T^{\sigma\, i_2 \cdots i_k}_{\sigma\, j_2 \cdots j_\ell} = \text{"contraction of } T \text{ on the } i_1\, j_1 \text{ indices"}$$

are the components of a tensor in $\mathcal{T}^{k-1}_{\ell-1}$.

- *Homework*: Show that *symmetry* in the components of a $\binom{1}{1}$ tensor T *is not* a coordinate independent property of T; i.e., $T^i{}_j = T^j{}_i \not\Rightarrow \bar{T}^\alpha{}_\beta = \bar{T}^\beta{}_\alpha$

Soln: $(A^{-1}TA)^t \stackrel{?}{=} A^t T^t A^{-t}$ not always if

$A \neq A^t$, $A^{-1} = A^{-t}$

I.e., $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $T = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$A^{-1}TA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \end{bmatrix}} = \begin{bmatrix} \lambda_1 & \lambda_1 - \lambda_2 \\ 0 & \lambda_2 \end{bmatrix}$$

T symmetric, $\bar{T}$ not ✓

- **Example**: let $T = T_{ij}\, dx^i \otimes dx^j \in \mathcal{J}_{11}$

I.e., at each p, $T_{ij}(p)$ are the components of a $\binom{0}{2}$-tensor.

T_p defines a bi-linear form on T_pM; i.e.

$$T_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$

$$T_p(X_p, Y_p) = T_{ij}\, a^i b^j \Big|_p$$

$$X_p = a^i \frac{\partial}{\partial x^i}\Big|_p \quad Y_p = b^j \frac{\partial}{\partial x^j}\Big|_p$$

"E.g., an inner product on T_pM defines a $\binom{0}{2}$-tensor at p."

- Under change of coordinates:

$$\bar{T}_{\alpha\beta} = T_{\underset{\text{row}}{i}\,\underset{\text{colm}}{j}} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^{j \leftarrow \text{row}}}{\partial \underset{\text{colm}}{y^\beta}} \iff \bar{T} = J^{tr}\, T\, J \qquad J = \frac{\partial x^i}{\partial y^\beta}$$

<u>Defn</u>: $T \in \mathcal{Y}_{11}$ is symmetric if

$$T_{ij} = T_{ji}$$

<u>Thm</u> ① Symmetry is a <u>coordinate</u> <u>indept</u> property of $\binom{0}{2}$-tensors

<u>Proof</u>:
$$\bar{T}_{\alpha\beta} = T_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$$
$$= T_{ji} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} = T_{ji} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^i}{\partial y^\alpha}$$
$$= \bar{T}_{\beta\alpha}$$

"i.e., assume $T_{ij} = T_{ji} \Rightarrow \bar{T}_{\alpha\beta} = \bar{T}_{\beta\alpha}$"

• **Defn**: T is <u>non-degenerate</u> if $\det T_{ij} \neq 0$.

Thm 2 T <u>non-degenerate</u> is an invariant property of $\binom{0}{2}$ or $\binom{1}{1}$ tensor.

P.f. $\det J^t T J = \det J^t \det T \det J$

$= \det \overline{T}$

$\therefore \det \overline{T} \neq 0$ iff $\det T \neq 0$

[We always assume our coord charts satisfy $\det J \neq 0 \neq \det J^{-1}$]

Defn: a symmetric, non-degenerate $\binom{0}{2}$-tensor defined on all of M is called a ~~metric~~ Riemannian <u>metric</u> on M

I.e., it defines an inner product on each T_pM, $p \in M$.

- Let g denote a metric on M

$$g \in \mathcal{Y}^0_2(M).$$

In coordinates

$$g = g_{ij}\, dx^i \otimes dx^j$$

For $X_p, Y_p \in T_pM$, define

$X = a^i \frac{\partial}{\partial x^i}$ $Y = b^j \frac{\partial}{\partial x^j}$

$$\langle X_p, Y_p \rangle_p = g(X, Y)\big|_p$$

$$= g_{ij}(p)\, dx^i \otimes dx^j \left(a^\sigma \frac{\partial}{\partial x^\sigma}\Big|_p, b^\tau \frac{\partial}{\partial x^\tau}\Big|_p \right)$$

$$= g_{ij}\, a^i b^j \big|_p \equiv \underset{\sim}{a}^{tr} \underset{\approx}{g}\, \underset{\sim}{b}$$

$$\underset{\sim}{a} = \begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix} b^1 \\ \vdots \\ b^n \end{pmatrix} \quad \underset{\approx}{g} = g_{ij}$$

colm

- Drop the p and write

$$\langle X, Y\rangle = g(X,Y) = g_{ij}\, a^i b^j$$

where the components of the vector fields X & Y and metric g are always assumed to be computed at the same $p \in M$.

SKIP TO PG 21

- Theorem (Algebra): Let $\langle\ ,\ \rangle$ denote a non-degenerate symmetric bi-linear form defined on a vector space V^n. Then one can always construct an ortho-normal basis via the Gram-Schmidt procedure; ie., $\exists\ \{e_i\}_{i=1}^{n}$ such that

$$\langle e_i, e_j\rangle = s_i\, \delta_{ij} \quad \leftarrow \text{NO SUM}$$

where $s_i = \pm 1$. Moreover, the number of $s_i < 0$ and the number of $s_i > 0$ are ~~both indepen~~ the <u>same</u> for every such orthonormal basis. Thus $\Sigma s_i =$ signature of the bilinear form is indep of basis.

<u>Corollary</u>: Let g be a metric on M. Then $\exists$ a nbhd of each $p \in M$ such that

$$g_{ij}(p) = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix}$$

$s_i = \pm 1$. Moreover, if the signature of $\langle\ ,\ \rangle_p$ is k, $0 \le k \le n$, then we can arrange it so that

$$s_1 = \cdots = s_a = -1$$

$$s_{a+1} = \cdots = s_n = +1$$

$$a = \frac{n-k}{2}$$

I.e., $-a + (n-a) = k$

$-2a = k - n$

$a = \frac{n-k}{2}$

<u>Proof</u>: FIP: Use the following fact:

SKIP

<u>LEMMA</u>: Let $X_1, \dots, X_n$ be linearly indept vectors in T_pM (ie., a basis for T_pM). Then $\exists$ a coord system

$$\underset{\sim}{x} : \mathcal{U}_{\underset{\sim}{x}} \longrightarrow \mathbb{R}$$

$$p \in \mathcal{U}_{\underset{\sim}{x}}$$

such that

$$X_i\Big|_p = \frac{\partial}{\partial x^i}\Big|_p .$$

<u>Proof</u>

<u>Homework.</u>

Proof: Let $\underset{\sim}{y} = M(\underset{\sim}{x} - \underset{\sim}{x}_0) + \underset{\sim}{y}_0$ (*)

$\underset{\sim}{x} = (x^1, \dots, x^n)$ a coordinate system, $p \in U_{\underset{\sim}{x}}$

M = const $n \times n$ matrix. $\underset{\sim}{x}(p) = \underset{\sim}{x}_0$

We find M so that in $\underset{\sim}{y}$-coords,

$$X_i\Big|_p = \frac{\partial}{\partial y^i}\Big|_p$$

But (*) $\Rightarrow$ $\dfrac{\partial y^\alpha}{\partial x^i} = M = \text{const} \equiv J$

$$X_i = a_i^\sigma \frac{\partial}{\partial x^\sigma} = b_i^\alpha \frac{\partial}{\partial y^\alpha}$$

$$b_i^\alpha = \frac{\partial y^\alpha}{\partial x^\sigma} a^\sigma$$

Want: $b_i^\alpha = \delta_i^\alpha \Rightarrow \dfrac{\partial y^\alpha}{\partial x^\sigma} a_i^\sigma = \delta_i^\alpha$

$i = 1, \dots, n$

Thus:

$$\frac{\partial y^\alpha}{\partial x^\sigma} \, a^\sigma_i = \delta^\alpha_i \qquad i = 1, \dots, n$$

$$\Leftrightarrow \qquad J \left[\underset{\sim}{a}_1 \cdots \underset{\sim}{a}_n \right] = \mathrm{id}$$

$\Rightarrow$ choose $J = M = \left[\underset{\sim}{a}_1 \cdots \underset{\sim}{a}_n \right]^{-1}$ suffices ✓

<u>Defn</u>: If g is a smooth metric of signature n defined throughout M, then g is called a Riemannian metric.

If sign of g is $n-2$, it is called a Lorentzian metric or Minkowskian metric.

<u>Thm</u>: If g is a Riemannian metric, then

F/P $$\langle X, X\rangle > 0 \text{ iff } X \neq 0. \quad \left(\begin{smallmatrix}\text{at each}\\ p\in M\end{smallmatrix}\right)$$

Pf. $\langle X, X\rangle = g_{ij}\, a^i a^j$ at each $p \in M$,

$$X = a^i \frac{\partial}{\partial x^i}.$$

But $\forall\, p \in M$, $\exists$ coords $\underset{\sim}{y}$ in which $g_{\alpha\beta} = \delta_{\alpha\beta}$

$$\langle X, X\rangle_p = \delta_{\alpha\beta}\, a^\alpha a^\beta = \sum_{\sigma=1}^{n} (a^\sigma)^2 > 0$$

iff $X \neq 0$. $X = a^\alpha \frac{\partial}{\partial y^\alpha}$ ✓

For a Lorentzian metric, $\langle X, X \rangle_p = 0$ is possible – in this case we say X_p is <u>null</u> at p.

- Let g be a metric.

Notation: $g^{ij} = (g_{ij})^{-1}$

<u>Thm</u>: g^{ij} transforms like a $\binom{2}{0}$ tensor (FIP)

Pf. $g^{ij} = (g_{ij})^{-1} \Leftrightarrow g_{i\sigma} g^{\sigma j} = \delta_i^j$

We show $g^{\alpha\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}$ is the inverse of $g_{\alpha\beta}$. Indeed

$$\boxed{g^{ij} \underbrace{\frac{\partial y^\sigma}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j} g_{\sigma\beta}}_{g_{ij}} = g}$$

$$g^{ij} \frac{\partial y^\sigma}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j} g_{\sigma\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^j} \underbrace{\frac{\partial y^\sigma}{\partial x^i} g_{\sigma\tau} \frac{\partial y^\tau}{\partial x^k}}_{g_{ik}} \underbrace{\frac{\partial x^k}{\partial y^\beta}}_{\delta^\tau_\beta = id}$$

$$= \underbrace{g^{ij} g_{ik}}_{\delta^j_k} \frac{\partial y^\alpha}{\partial x^j} \frac{\partial x^k}{\partial y^\beta} = \delta^j_k \frac{\partial y^\alpha}{\partial x^j} \frac{\partial x^k}{\partial y^\beta}$$

$\Rightarrow j = k$

(assume $g^{ij} = (g_{ij})^{-1}$)

$$= \frac{\partial y^\alpha}{\partial x^k} \frac{\partial x^k}{\partial y^\beta} = \delta^\alpha_\beta \checkmark$$

- A metric g defines an inner product $\langle \, , \rangle_p$ at each $p \in M$, and this defines a canonical mapping (isomorphism)

$$T_p M \leftrightarrow T_p^* M$$

$$X_p \longmapsto \omega^p \quad \text{where } \omega^p(Y_p) = \langle X_p, Y_p \rangle$$

$$\boxed{\omega_X = \langle X, - \rangle}$$

Q: How do components of X_p map to components of ω^p?

Ans: $X_p = a^i \frac{\partial}{\partial x^i}$ $\quad Y_p = b^j \frac{\partial}{\partial x^j}$ then

$$\omega^p(Y_p) = \langle X_p, Y_p \rangle = \underbrace{g_{ij} a^i}_{a_j} b^j = a_j b^j$$

$$\Rightarrow \boxed{\omega^p = a_j \, dx^j}$$

Conclude: a metric gives a natural identification

$$X_p = a^i \frac{\partial}{\partial x^i} \Longleftrightarrow a_i \, dx^i = \omega^p$$

$$\boxed{a_i = g_{i\sigma} a^\sigma}$$

Defn: We say a^σ has been obtained from a_i by raising the index by the metric.

Conversely: $g^{ij} = (g_{ij})^{-1} \Rightarrow$

$$\boxed{g^{i\sigma} a_\sigma = a^i}$$

<u>More generally</u>: We can define the raising / lowering of an indice by a metric in an arbitrary tensor –

eg. $R = R^{i}{}_{jkl} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l \in \mathcal{T}^1_3$

Components: $R^{i}{}_{jkl}$

Lower the index i: $R_{ijkl} = R^{\sigma}{}_{jkl}\, g_{\sigma i}$

$$R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l \in \mathcal{T}^0_4$$

<u>Thm</u>: Raising a lowering indices by the metric is a <u>tensor</u> operation.

<u>Recall</u>: A linear transformation is "self-adjoint" if the corresponding transformation matrix is symmetric.

Q: if symmetry is not a covariant property of $\binom{1}{1}$-tensors that repre. linear trans. of the vector space T_pM, then how can "self-adjoint" be invariant?

<u>Ans</u>: <u>Defn</u>: a linear transformation T on a vector space V is self adjoint rel to an inner product $\langle\ ,\ \rangle$ on V if

$$\langle v_1, Tv_2\rangle = \langle Tv_1, v_2\rangle$$

$\forall\ v_1, v_1 \in V$.

For a $\binom{1}{1}$-tensor T on M & metric $g \Rightarrow \langle\, ,\, \rangle_p$ on T_pM, ~~this means the~~ self-adjoint at each p means

$$(*) \qquad \langle X_p, T(Y_p)\rangle = \langle T(X_p), Y_p\rangle$$

$$T = T^i_j \frac{\partial}{\partial x^i} \otimes dx^j \in \mathcal{J}^1_1$$

$$T(X) = T^i_j a^j \frac{\partial}{\partial x^i}$$

$$T : a^i \frac{\partial}{\partial x^i} \longmapsto T^i_j a^j \frac{\partial}{\partial x^i}$$

$$a^i \longmapsto T^i_j a^j \quad \text{(componentwise) (in } \underset{\sim}{x}\text{-coord) (at } p)$$

Thus:

$$(*) \Leftrightarrow \underbrace{g_{\sigma\tau} a^\sigma T^\tau_j b^j}_{\langle X_p, T(Y_p)\rangle} = \underbrace{g_{\tau\sigma} T^\tau_j a^j b^\sigma}_{\langle T(X_p), Y_p\rangle}$$

Thm: Let $T^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ be compt's of $T \in \mathcal{J}^k_l$.
Then symmetry or antisymmetry wrt any two upper or any two lower indices is a covariant (coordinate indep't) ~~stat~~ property.

FIP

Eg: $$T_{\cdots ij \cdots}^{\cdots\cdots} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \cdots = T_{\cdots \alpha\beta \cdots}^{\cdots\cdots}$$

$$\parallel$$

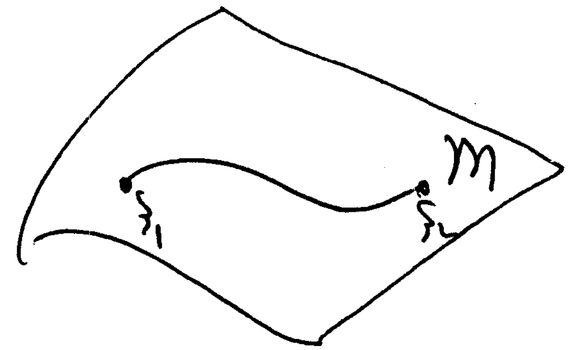
$$T_{\cdots ji}^{\cdots} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \cdots = T_{\cdots \beta\alpha \cdots}^{\cdots\cdots}$$

Or: base it on the invariant properties:

$$T(\cdots X, Y \cdots) = T(\cdots Y, X \cdots)$$

$\forall X, Y \Rightarrow$ symmetry of corresponding indices.

- A metric defines the notion of <u>length</u> on a manifold:



$$L = \int_{\xi_1}^{\xi_2} \sqrt{\langle X, X \rangle}\, d\xi$$

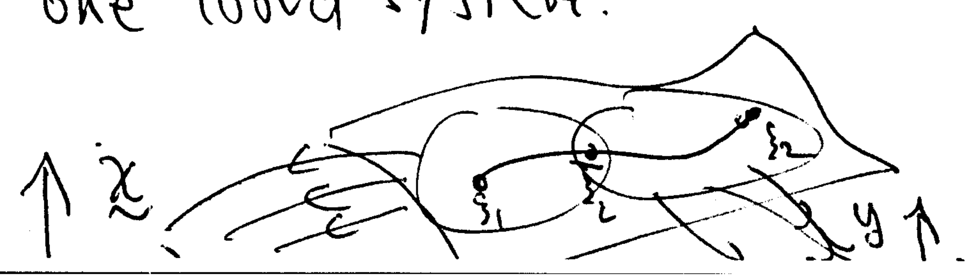
$$X = \dot{x}^i \frac{\partial}{\partial x^i} \qquad \langle X, X \rangle = g_{ij}\, \dot{x}^i \dot{x}^j \qquad (*)$$

in the $\underset{\sim}{x}$-coordinate system.

<u>Thm</u>: L is indept of parameterization

<u>Note</u>: by this formula, you must break it up into pieces defined in one coord system:

we return to metrics later



☒ <u>Sylvester's</u> <u>Theorem</u>: (Law of Inertia)

Let g be any symmetric, non-degenerate $\binom{0}{2}$-tensor defined in a nbhd $\mathcal{U}$ of p.

$[g_{ij} = g_{ji}$ in every coord syst & $\det g_{ij} \neq 0]$

Then $\forall p \in \mathcal{U}$ $\exists$ a coord system y^α defined in a nbhd of p st

$$g_{\alpha\beta} = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix}, \quad \delta_m = \pm 1.$$

<u>Defn</u>: $\frac{\partial}{\partial y^\alpha}\Big|_p$ is an orthonormal basis for g @ p

$n_+ - n_- \equiv$ signature of g

$n_- \equiv$ # of -1's, $n_+ \equiv$ # of pos 1's.

<u>Thm</u>: Every o.n. basis has the same signature & n_+, n_- are constant in a nbhd of p.

(FIP)

- In fact, $\delta_i(p) = \frac{\lambda_i(p)}{|\lambda_i(p)|}$ where λ_i are the eigenvalues of the symmetric matrix

$$\underbrace{G^i_j(p) \equiv g_{ij}(p)}\ \text{[defined at } p \text{ in a given coord syst]}$$

If it holds in $\underset{\sim}{x}$-coords, won't hold in $\underset{\sim}{y}$-coords in general

<u>Proof</u>:

- Start with g_{ij} in $\underset{\sim}{x}$-coords, $p \in M$
- Define $G^i_j = g_{ij}$ "symmetric matrix @ p in $\underset{\sim}{x}$-words"
- G^i_j symmetric matrix in $\mathbb{R}^n$ $\Rightarrow$ real evals λ_α & ON basis of e-vectors $r^i_\alpha \in \mathbb{R}^n$ [O.N. wrt Euclidean coord metric]
- Define: $\boxed{\cancel{x}^i = r^i_\alpha(p) y^\alpha}$ $\Rightarrow$ defines $y^\alpha = y^\alpha(\underset{\sim}{x})$

$$\boxed{\frac{\partial}{\partial y^\alpha} = r^i_\alpha(p) \frac{\partial}{\partial x^i}} \qquad J = \frac{\partial x^i}{\partial y^\alpha} = r^i_\alpha(p)$$

Then:

$$\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \Leftrightarrow \bar{g} = J^T g J$$

But r^i_α O.N. $\Rightarrow J^T = J^{-1} \Rightarrow \bar{g} = J^{-1} g J = J^{-1} G J$

$$\therefore \quad \bar{g}_{\alpha\beta}(p) = \begin{bmatrix} \lambda_1(p) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(p) \end{bmatrix} = \Lambda$$

Thus: $$S^T \Lambda S = \begin{bmatrix} \frac{\lambda_1}{|\lambda_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_n}{|\lambda_n|} \end{bmatrix}, \quad S = \begin{bmatrix} \sqrt{|\lambda_1|}^{-1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{|\lambda_n|}^{-1} \end{bmatrix}$$

$$\therefore \quad S^T J^T g J S = \begin{bmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{bmatrix}$$

$$\boxed{y^\alpha = (JS)^\alpha_i \, x^i}$$ does it ✓

(at p, not in a nbhd, as e-vectors r^i_α are not in general coord vector fields)

☒ **Thm**: (Riemann Normal Coordinates)

Let g be a non-degenerate symmetric $\binom{0}{2}$-tensor on M (a metric). Then $\forall\, p \in M$ $\exists$ a coordinate system $\underset{\sim}{y}$ defined in a nbhd of p such that

$$g_{ij}(p) = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix} \equiv \eta_{ij}, \quad \delta_i = \pm 1 \qquad (a)$$

$$\frac{\partial}{\partial x^k} g_{ij}(p) \equiv g_{ij,k}(p) = 0. \qquad (b)$$

That is: by Taylor expanding g about p, this says $g_{ij}(\underset{\sim}{y}) = \eta_{ij}$ to within errors $O|\underset{\sim}{y} - \underset{\sim}{y}(p)|^2$

In Fact: you cannot make $g_{ij,kl}(p) = 0$ in general, & $R^i_{jkl}(p)$ is a tensorial measure of the 2nd derivatives of g at p.

Proof: Consider the (geodesic) equation

$$\ddot{\gamma}^k + \Gamma^k_{ij}\,\dot{\gamma}^i\dot{\gamma}^j = 0 \qquad (*)$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\sigma}\{-g_{ij,k} + g_{ki,j} + g_{jk,i}\} \quad \text{(known)}$$

- Note: (*) is a 2nd order autonomous ODE defined in $\underset{\sim}{x}$-coords whose solution curves

$$x^k = \gamma^k(t)$$

solve eqn (*).

- Thm: (geodesics) Solutions of (*) define the same curve in $\mathcal{M}$, indep't of coordinates. Call this curve $\gamma(t) \in \mathcal{M}$

Moreover, $\gamma(t)$ are the geodesics of g.
[Proof: later]

- We use (*) to define Riemann Normal Coordinates in a nbhd of p, & prove they satisfy (a), (b).

- Facts we need:
- <u>Thm</u> (ODE): $\exists !$ soln $\gamma_X(t)$ of (*), defined in a nbhd of $p \in M$, such that

$$\gamma_X(0) = p$$

$$\frac{d}{dt}\gamma_X(0) \equiv \dot{\gamma}_X(p) = X \in T_pM$$

<u>Cor</u>: $\gamma_{sX}(t) = \gamma_X(st)$, $s \in \mathbb{R}$. (Homework)

[Pf. Prove it directly from (*) in $\underset{\sim}{x}$-coordinates]

<u>Thm</u> (ODE): If $\{X_1, \ldots, X_n\} \subseteq T_pM$ is a basis,

$$X = \xi^i X_i \in T_pM,$$

then the mapping

$$\xi^i \longmapsto \gamma_X(1)$$

$$0 \longmapsto p$$

is 1-1 onto & regular in a nbhd of p.

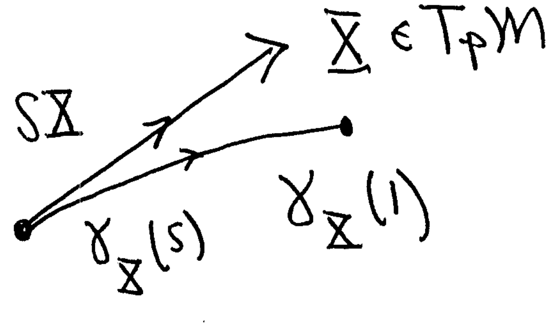
Pf. Write (*) as a 1st order system & apply standard thm's from ODE's.

- $\underline{\underline{\text{Thus}}}$: Define a map from $T_pM \to M$:

$$T_pM \ni \underline{X} \longmapsto \gamma_{\underline{X}}(1)$$

This is called the exponential map

Picture:



" $\gamma_{\underline{X}}(1)$ is the point 1-unit along the soln $\gamma_{\underline{X}}(t)$ starting at p with velocity $\underline{X}_p$ "

- Now choose $\{\underline{X}_1, \ldots, \underline{X}_n\}$ an O.N. basis @ p.
- Define the coordinate system $\underset{\sim}{y}$:

$$\underset{\sim}{y}^{-1}: y^i \longmapsto \gamma_{\underline{X}}(1)$$

$$\underline{X} = y^i \underline{X}_i$$

" $\underline{y}$ are the components of $\underline{X}$ wrt $\{\underline{X}_1 \ldots \underline{X}_n\}$ ".

Defined, 1-1 regular in a nbhd of p by Thm (ODE)

5

Defn: $\underset{\sim}{y}$ are the RNC's at p associated with basis $\{X_1, \dots, X_n\}$

- **Claim:** $g_{ij}(p) = \delta_{ij}$, $g_{ij,k}(p) = 0$ in y-coords, if $\{X_i\}$ an ON. basis.

Proof: Choose $X = \xi^i X_i \in T_x M$, & consider the solution curve $\gamma_X^k(t)$ of (*) in $\underset{\sim}{y}$-coords. Then by defn: $\xi^k = \gamma_X^k(1) = y^k(\gamma_X(1))$. Thus

$$\gamma_X^k(t) = y^k(\gamma_X(t)) = y^k(\gamma_{tX}(1)) = t\xi^k$$

So

$$\gamma_X^k(t) = t\,\xi^k \qquad \forall t,\ \xi^h \text{ fixed}$$

Conclude. $\ddot{\gamma}_X^k(t) \equiv 0$ in $\underset{\sim}{y}$-coords.

$\therefore$ by (*), $\Gamma_{ij}^k \dot{\gamma}^i \dot{\gamma}^j = \Gamma_{ij}^k \xi^i \xi^j = 0$

$\Rightarrow \Gamma_{ij}^k = 0$. (at p) $\forall \xi \in \mathbb{R}^n$

- <u>Define</u>: $\Gamma_{ij,k} = \{-g_{ij,k} + g_{ki,j} + g_{jk,i}\}$

$$= 2 g_{k\sigma} \Gamma^{\sigma}_{ij}.$$

Then $\Gamma^{k}_{ij} = 0$ iff $\Gamma_{ij,k} = 0$ $(\forall\, i\, j\, k)$

<u>Lemma</u>: $\Gamma_{ik,j} + \Gamma_{jk,i} = g_{ij,k}(p) = 0$ ✓

Q: Since $\ddot{\gamma}^{k}_{X} = 0$, $\dot{\gamma}^{k}_{X} = \xi^{k}$ all along the geodesic, we have $\Gamma^{k}_{ij}(\gamma_{X}(t))\, \xi^{i} \xi^{j} = 0$, so why can't we conclude $g_{ij,k} = 0$ all along $\gamma_{X}(t)$, & hence in a nbhd of p?

Ans: at $\gamma_{X}(t)$, $t \neq 0$, we do not have $\Gamma^{k}_{ij}(\gamma_{X}(t))\, \xi^{i} \xi^{j} = 0$ $\forall \xi$, <u>only</u> the $\xi^{i} = \dot{\gamma}^{i}_{X}(t)$.

Only @ $p = \gamma_{X}(0)$ do we have $\Gamma^{k}_{ij} \xi^{i} \xi^{j} = 0$ $\forall\, \underset{\sim}{\xi} \in \mathbb{R}^{n}$