Introduction/Special Relativity SHORT

Recall: Einsteins Theory of Gravity takes the assumption that the gravitational field is given by a symmetric, non-degenerate bilinear form g defined on spacetime. A coordinate system $\underset{\sim}{x} \equiv (x^0, x^1, x^2, x^3)$ given on spacetime determines the components $g_{ij}(\underset{\sim}{x})$, a symmetric non-deg matrix at each $\underset{\sim}{x}$. This determines the differential **ds** $\equiv$ arclength

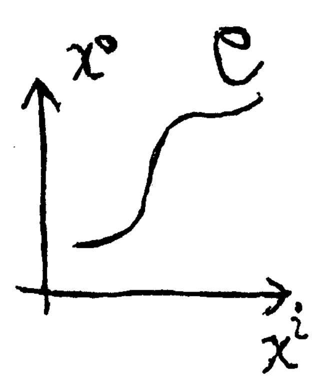
$$ds^2 = \pm g_{ij}\, dx^i dx^j .$$

- Given a curve $\underset{\sim}{x}(\xi)$ on spacetime, the g-length of the curve is given by

$$ds = \sqrt{\pm g_{ij}\, dx^i dx^j} = \sqrt{\pm g_{ij}\, \dot{x}^i \dot{x}^j}\; d\xi$$

since

$$dx^i = \dot{x}^i d\xi$$



$$\Rightarrow \qquad \Delta s = \int_{\xi_1}^{\xi_2} \sqrt{\pm g_{ij}\, \dot{x}^i \dot{x}^j}\; d\xi$$

<u>Assumption</u> (1): $ds = c\,d\tau$ where $d\tau$ is the proper time change (aging time) for an observer traversing $\mathcal{C}$.

E.g., if $[x^i]$ = meters, $[t]$ = seconds, $[x^0 = ct]$ meters

By taking $ds = c\,d\tau$, $x^0 = ct$, x^0 and ds have units of meters $\Rightarrow$ space & time have dim. of length.

<u>Assumption</u> (2): (timelike) Paths of minimal or critical length $\equiv$ geodesics of g are the free fall paths

<u>Assumption</u> (3): Non-rotating frames are ∥-transported by Connection for g along free-fall paths. (Diff Geom)

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Assumption of Special Relativity: $\exists$ a global coordinate system $\underset{\sim}{x}$ such that

$$g_{ij} = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & & 1 \end{bmatrix} \equiv \eta_{ij}$$

everywhere. $\Rightarrow$ "spacetime is flat"

$\underset{\sim}{x} \equiv$ Lorentz frame is orthonormal frame for g.

Special Relativity: Assume $g_{ij} = \eta_{ij} = \mathrm{diag}(-1,1,1,1)$ in $\underset{\sim}{x}$-coordinates. E.g., if we assume

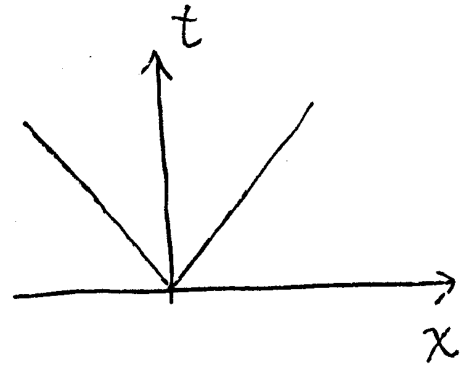
$$ds^2 = \eta_{ij}\, dx^i dx^j = -d(x^0)^2 + d(x^1)^2 + \cdots + d(x^3)^2$$

gives the metric in meters, then $x^0 = ct$, t in seconds, $s = c\tau$, τ proper time in sec's.

$\Rightarrow$ proper time change in second for observer traversing $c(s)$ is

$$\Delta\tau = \frac{1}{c}\int_{\xi_1}^{\xi_2} \sqrt{-\{-(\dot{x}^0)^2 + (\dot{x}^1)^2 + \cdots + (\dot{x}^3)^2\}}\, d\xi$$

Defn: a vector $X = a^i \frac{\partial}{\partial x^i}$ is



(1) timelike if $\langle X, X\rangle < 0$

(2) lightlike if $\langle X, X\rangle = 0$

(3) spacelike if $\langle X, X\rangle > 0$.

Assumption: The tangent vector to the "world line" or curve associated with any particle satisfies $\frac{dc}{d\xi}$ is timelike

"All particles move with speed $< c$"

Assumption ①: All particles move with speed $< c$ in the $\underset{\sim}{x}$-coordinates

Precisely: $\underset{\approx}{x}(t)$ the world line of a particle

$\Rightarrow \ \dot{x}^i \dfrac{\partial}{\partial x^i}$ is timelike. FIP

Assumption ②: The tangent vector to light rays is Lightlike. $\Leftrightarrow$ "light rays travel w. speed c"

~~For each o-N frame can identify vectors with points in space~~

We can use the o-N frame $\underset{\sim}{x}$ to identify vectors in $T_{\underset{\sim}{0}}M^4$ with points in the space: "identify components with pts in the spacetime"

$$X \equiv X^i \frac{\partial}{\partial x^i}\Big|_{\underset{\sim}{0}} \longleftrightarrow P = \Phi(X) : x^i(P) = X^i$$

Under this identification, we can interpret

$$ds(X) \equiv \sqrt{\left| -(X^0)^2 + \cdots + (X^3)^2 \right|},$$

as follows

This is the exponential map FIP

<u>X timelike</u>: $ds(X) = c\,d\tau$ is the proper time change between events $P_0 = \Phi(\underset{\sim}{0})$ and $P_1 = \Phi(X)$ as measured by observer moving with velocity vector X; Ie, if

$$\underset{\sim}{x} \circ c(\xi) \equiv \underset{\sim}{x}(\xi) \text{ satisfies } \dot{\underset{\sim}{x}}(\xi) = X, \quad \underset{\sim}{x}(0) = 0$$

then $x^i(\xi) = X^i \xi \Rightarrow$

$$c\Delta\tau = \int_0^1 \sqrt{-(X^0)^2 + (X^1)^2 + \cdots + (X^3)^2}\, d\xi = ds(X)$$

Note: if $d\xi \equiv ds$, (arclength param. of $c(\xi)$), then

$$ds = |X|\, ds \Rightarrow |X| = 1.$$

$\left[\text{i.e., } ds = g_{ij}\dot{x}^i\dot{x}^j d\xi \Rightarrow \langle X, X\rangle = 1 \text{ iff } ds = d\xi\right]$

in which case we call X the 4-vel. of observer.

~~X spacelike~~;

8c

X spacelike: $ds(X)$ is the length in meters of a rod as measured by observer moving in a frame in which P_0 and P_1 occur at same time. E.g., in $\underset{\sim}{x}$-frame, $X^0 = 0 \Rightarrow ds(X) = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2}$ is pos def metric giving Euclidean lengths.

- Property of Flat Space: ① You can identify vectors in $T_p M$ with $T_q M$ by:

$$g_{ij} = \eta_{ij} \quad \text{in coords } \underset{\sim}{x}^i$$

$$\Leftrightarrow \quad X_p = a^i \frac{\partial}{\partial x^i}\Big|_p \longleftrightarrow X_q = a^i \frac{\partial}{\partial x^i}\Big|_q$$

"vectors with same components at different pts are said to be ∥-translations"

② You can identify $T_0 M$ with M by:

$$X = a^i \frac{\partial}{\partial x^i}\Big|_{\underset{\sim}{x}=0} \longleftrightarrow x^i = a^i \equiv \underset{\sim}{x} \text{ coords of a pt in } M$$

Define: $X \in T_p M$, $I(X) = q \in M : \underset{\sim}{x}^i(q) - \underset{\sim}{x}^i(p) = X^i$

Example : 1-d $\qquad g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

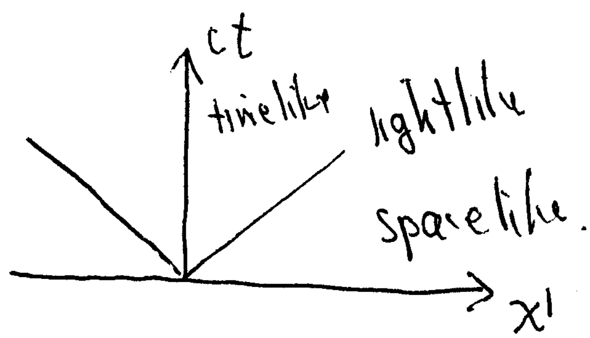
Check: $\left\{ \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1} \right\}$ form an o-N basis

$$\left\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0} \right\rangle = g_{ij}\, e_0^i\, e_0^j = -1$$

$$\left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1} \right\rangle = +1$$

$$\left\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1} \right\rangle = 0$$

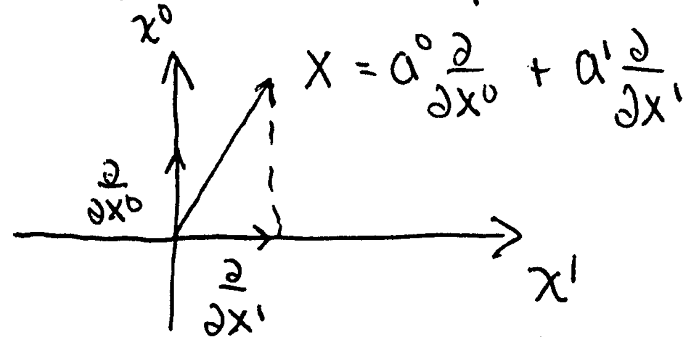
Check: $\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}$ lightlike $\quad [1, \pm 1] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \end{bmatrix} = 0$



Q: What are the other o-N frames, and how are they related to $\underline{x}$-coordinates?

<u>Ans</u>: Let X be vector $X = a^0 \frac{\partial}{\partial x^0} + a^1 \frac{\partial}{\partial x^1}$

Since metric everywhere the same, we can identify components with pts in the space:

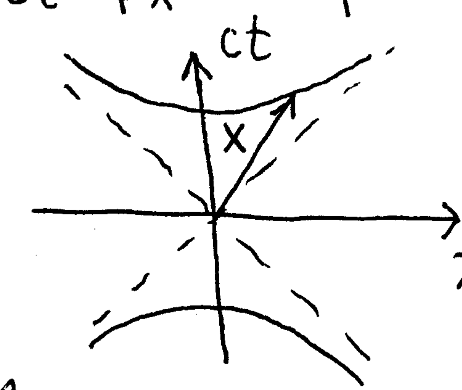


I.e. set $a^0 \equiv ct$, $a^1 \equiv x$, $X = ct \frac{\partial}{\partial x^0} + x \frac{\partial}{\partial x^1}$

Let X be timelike, unit length

$$\langle X, X \rangle = (ct, x) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} = -ct^2 + x^2 = -1$$

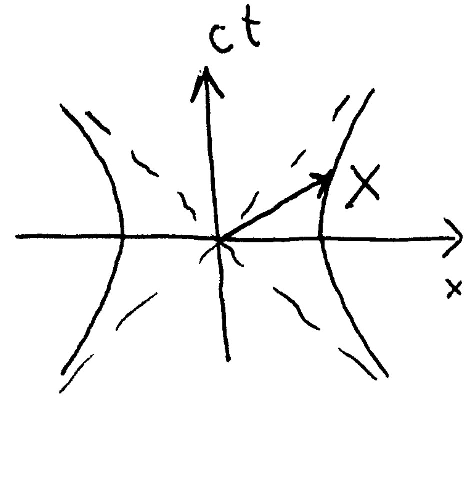
$\Rightarrow X$ lies on unit hyperbola$_1$



Let X be spacelike, unit length

$$\langle X, X \rangle = -ct^2 + x^2 = 1$$

$\Rightarrow X$ lies on unit hyperbola$_2$

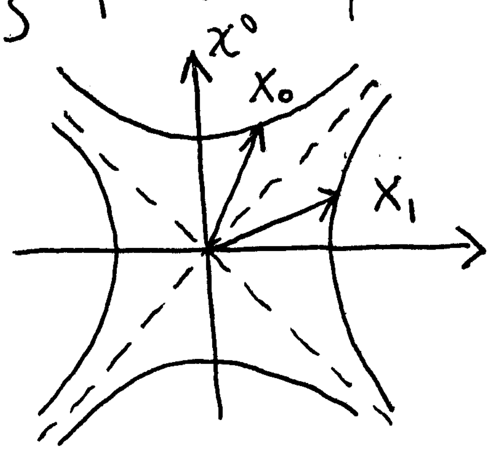


Thus: if $\langle X_0, X_1\rangle = 0$, $\langle X_0, X_0\rangle = -1$, $\langle X_1, X_1\rangle = 1$

$$0 = \langle X_0, X_1\rangle = (ct_0, x_0)\begin{bmatrix} -1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix} ct_1 \\ x_1\end{bmatrix} = (-ct_0, x_0)\cdot(ct_1, x_1)$$

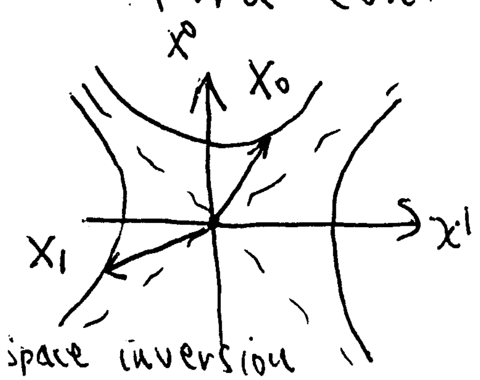
$\Rightarrow$ "$(ct_1, x_1) \parallel \pm(x_0, ct_0)$ $\Rightarrow$ X_1 is the reflection of X_0 in line $x = \pm ct$

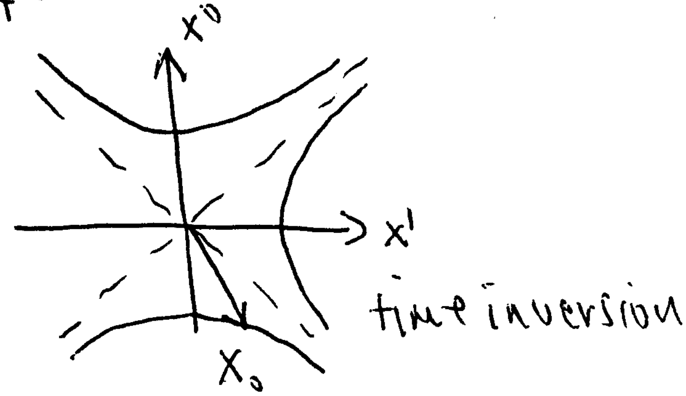
If we assume X_0 timelike, positive direction and $\{X_0, X_1\}$ positively oriented, then



<u>Note</u> ① cannot get to neg oriented frame thru cont transformations

② Cannot get to time inverted or space inverted thru cont sequ. of tran's.

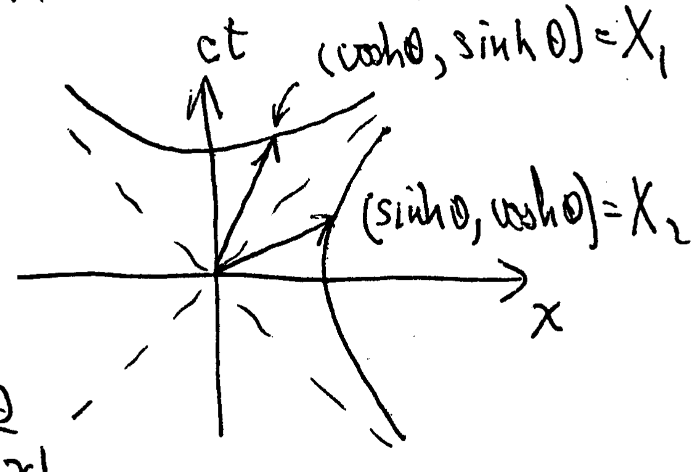




Note: $\cosh^2\theta - \sinh^2\theta = 1$

$\Rightarrow ct = \cosh\theta$, $x = \sinh\theta$ lies on unit hyperbola. Our notation

(a, b) $\quad a \equiv x^0$-coord
$\qquad\qquad b \equiv x^1$-coord



$$\Rightarrow X_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

gives all pos oriented, time oriented, o.n. frames, $-\infty < \theta < \infty$.

Note: ① $\forall$ X_0 with $|X_0| = 1$, you can complete it to an O.N. frame. If further X_0 is pos-time directed, then you can complete it uniquely to frame (X_0, X_1) with X_1 pos. space directed.

② All vectors can be completed to o.n. frame except lightlike vectors, which are $\perp$ to themselves.

3 For any X, $|X| \neq 0$, you can define the orthogonal projection onto X:

$$\mathrm{Proj}_X Y = \frac{\langle X, Y\rangle}{\langle X, X\rangle} X \qquad \text{(FIP)}$$

Keep $\pm$ sign from metric

Using this, you can define the Gramm-Schmidt process to construct an O-N basis in $\mathbb{R}^4$ from any 4 linearly indept non-light like vectors.

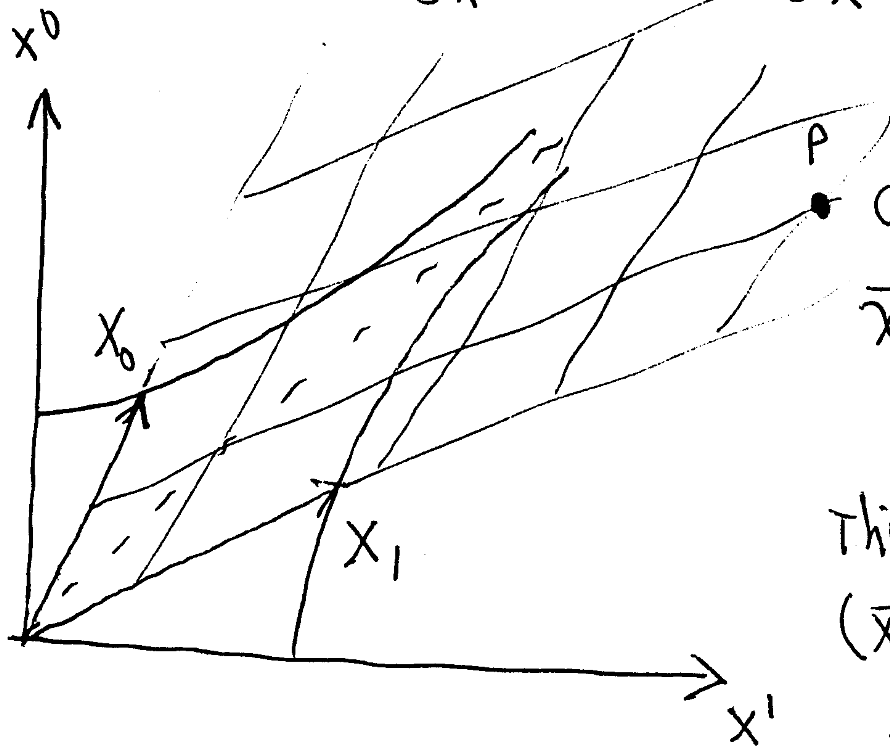
- <u>Lorentz Transformations</u>: Given $g_{ij} = \eta_{ij}$ in $\underset{\sim}{x} \equiv (x^0, x^1)$ coordinates. Construct another O-N frame for each

$$X_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

I.e., translate these vectors to each point of spacetime ("∥-translation in a flat spacetime") and choose $\{X_0, X_1\}$ to be the coord. basis vectors for a new coord system $(\bar{x}^0, \bar{x}^1)$ on spacetime as follows: We need

$$X_0 = \frac{\partial}{\partial \bar{x}^0}, \quad X_1 = \frac{\partial}{\partial \bar{x}^1}$$



$aX_0 + bX_1$

$\bar{x}^0(p) = a, \ \bar{x}^1(p) = b$

This defines the $(\bar{x}^0, \bar{x}^1)$ coord system.

Clearly, $\frac{\partial}{\partial \bar{x}^0} = X_0$, $\frac{\partial}{\partial \bar{x}^1} = X_1$ FIP, and

thus $\bar{g}_{ij} = \eta_{ij}$ because $\{X_0, X_1\}$ is an o-n basis at each point of spacetime.

Q: How is the (x^0, x^1) coord. system related to the $(\bar{x}^0, \bar{x}^1)$ coord system?

Ans: $x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} = \bar{x}^0 \frac{\partial}{\partial \bar{x}^0} + \bar{x}^1 \frac{\partial}{\partial \bar{x}^1}$

$\Leftrightarrow$ "coord's name same point p"

But: $$\frac{\partial}{\partial \bar{x}^0} = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}$$

$$\frac{\partial}{\partial \bar{x}^1} = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \frac{\partial}{\partial x^1}$$

$$\Rightarrow x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} = (\bar{x}^0 \cosh\theta + \bar{x}^1 \sinh\theta) \frac{\partial}{\partial x^0} + (\bar{x}^0 \sinh\theta + \bar{x}^1 \cosh\theta) \frac{\partial}{\partial x^1}$$

$$\Leftrightarrow \begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

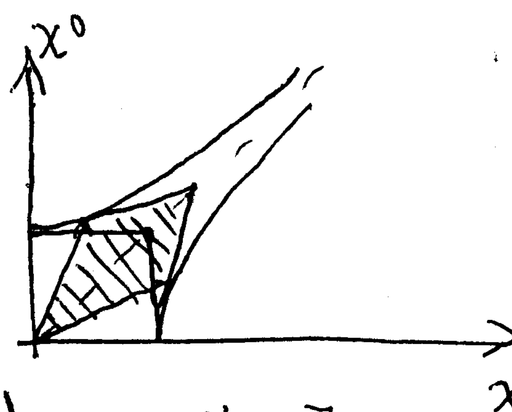
<u>Theorem</u>: The positively oriented, time oriented, homogeneous Lorentz transformations are given by $\underset{\sim}{x} = L(\theta)\underset{\sim}{\bar{x}}$, where $L(\theta)$ is given by

$$\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = L(\theta)\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}\begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

for $-\infty < \theta < \infty$.

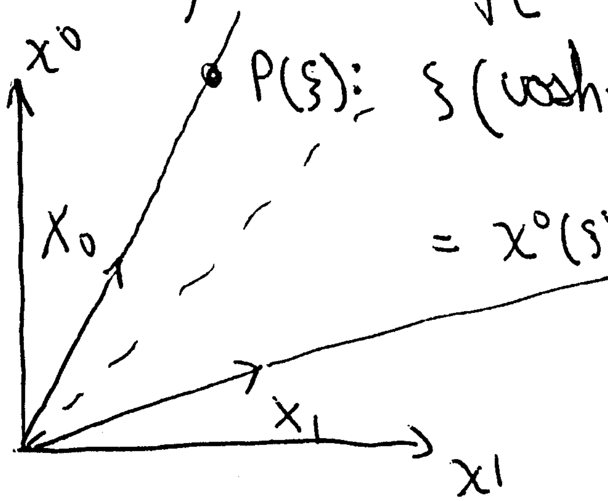
<u>Note ①</u>: $\det L(\theta) = \cosh^2\theta - \sinh^2\theta = 1 \Rightarrow$ Lorentz transformations preserve the coordinate volume;

I.e., Vol of  = 1.



<u>Note ②</u>: $L(\theta)^{-1} = L(-\theta) = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$

Note 2 we can rewrite in terms of the velocity of the $\bar{x}$-frame as observed in the x-frame by writing $\begin{Bmatrix}\cosh\theta \\ \sinh\theta\end{Bmatrix}$ as a fn of v:



$$P(\xi) : \xi\left(\cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1}\right)$$

$$= x^0(\xi)\frac{\partial}{\partial x^0} + x^1(\xi)\frac{\partial}{\partial x^1}$$

Conclude: $P(\xi)$ parameterizes the $\bar{x}^0$-axis

$$\Leftrightarrow \quad \frac{dx^1}{dx^0} = \frac{dx^1/d\xi}{dx^0/d\xi} = \frac{\sinh\theta}{\cosh\theta} \qquad \begin{cases} x^0(\xi) = \xi\cosh\theta \\ x^1(\xi) = \xi\sinh\theta \end{cases}$$

$$\therefore \quad \frac{dx^1}{dx^0} = \frac{1}{c}\frac{dx^1}{dt} = \frac{1}{c}v = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta$$

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta = \frac{1}{\cosh^2\theta} \Rightarrow \cosh^2\theta = \frac{1}{1-\left(\frac{v}{c}\right)^2}$$

$$-1 + \tanh^{-2}\theta = \operatorname{csch}^2\theta = \frac{1}{\sinh^2\theta} \Rightarrow \sinh^2\theta = \frac{\left(\frac{v}{c}\right)^2}{1-\left(\frac{v}{c}\right)^2}$$

$\Rightarrow$ Lorentz Transformation

$$\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & \frac{v/c}{\sqrt{1-(v/c)^2}} \\ \frac{v/c}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} \end{bmatrix} \begin{bmatrix} \bar{x}^0 \\ \bar{x}^1 \end{bmatrix}$$

gives L. trans, where the bar frame moves with vel. v rel to unbarred frame.

Here, $\sqrt{1-\left(\frac{v}{c}\right)^2} = 1 - \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$

$$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{1+\{\ \}} = 1 - \{\ \} + O(\{\ \})^2$$
$$= 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$$

$$\Rightarrow \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4$$

$\Rightarrow$ $L(\theta) =$

$\Rightarrow$ neglecting $O\left(\frac{v}{c}\right)^2$

$$x^0 = \bar{x}^0 + \frac{v}{c}\bar{x}^1 \Leftrightarrow ct = c\bar{t} + \frac{v}{c}\bar{x}^1 \Leftrightarrow \left\{ t = \bar{t} + O\left(\frac{1}{c^2}\right) \quad \begin{matrix} v = O(1) \\ \bar{x}^1 = O(1) \end{matrix} \right.$$

$$x^1 = \frac{v}{c}\bar{x}^0 + \bar{x}^1 \Leftrightarrow x^1 = v\bar{t} + \bar{x}^1 \Leftrightarrow \left\{ x^1 = v\bar{t} + \bar{x}^1 \right.$$

$\approx$ Galilean Trans

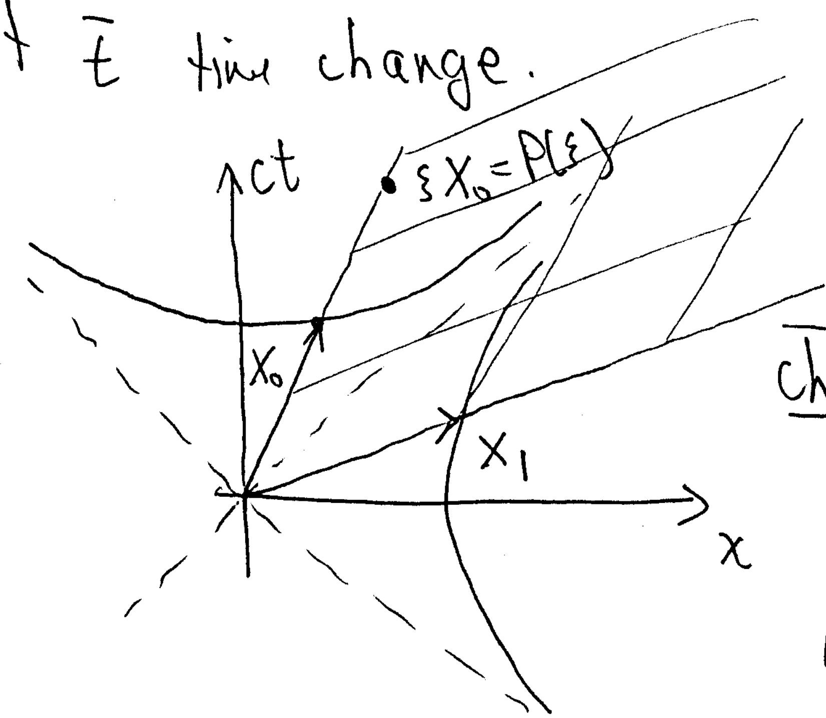
- Time Dialation: An observer fixed in the unbarred frame (say at origin) moves along a curve $x^1 = 0$, $x^0 = \xi$. His "aging time" betw t_1 and t_2 is given by

$$c\Delta\tau = \int_{ct_1}^{ct_2} \sqrt{|-\eta_{ij}\dot{x}^i \dot{x}^j|}\, d\xi = \int_{ct_1}^{ct_2} d\xi = c\Delta t = \Delta x^0$$

Conclude: Proper time & coordinate time agree for an observer fixed in L-frame.

By symmetry, an observer fixed on $\bar{x}^0$-coord axis ages according to the change in his $\bar{t}$-coordinate.

<u>But</u>: starting clocks at $t=\bar{t}=0$, the time change for observer fixed in $\bar{\underset{\sim}{x}}$-coordinate between O and $P(\xi)=\xi X_0$ is $\xi=\bar{t}c$ because X_0 represents a unit $\bar{t}$ time change.



Really $P'(\xi) = X_0$

$P(\xi) = \Phi(\xi X_0)$

<u>Check</u>: $c\bar{t} = \int_0^{\xi_0} \|X_0\|\, d\xi = \int_0^{\xi_0} d\xi = \xi_0$

But in $\underset{\sim}{x}$-coords, $\xi X_0 = (\xi\cosh\theta, \xi\sinh\theta) = (ct, x)$

$$\xi = c\bar{t}$$

$\Rightarrow ct = \xi\cosh\theta = c\bar{t}\cosh\theta$

$$\therefore\quad \bar{t} = \frac{1}{\cosh\theta}\, t = \sqrt{1-\tanh^2\theta}\; t = \sqrt{1-\left(\frac{v}{c}\right)^2}\; t$$

$1-\left(\frac{v}{c}\right)^2 = 1-\tanh^2\theta = \operatorname{sech}^2\theta$

OR: $\dfrac{t}{\bar{t}} = \cosh\theta = \dfrac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} > 1$

"Moving clocks appear to run slowly"

Conclude: $\bar{t} = \sqrt{1-\left(\frac{v}{c}\right)^2}\, t$

thus $\Delta\bar{t} = \sqrt{1-\left(\frac{v}{c}\right)^2}\, \Delta t < \Delta t$

"Moving observers carry clocks that appear to move slow rel to coord clocks"

<u>Twin Paradox</u>: Any observer that moves & returns will age less than fixed observer.

Q: Explain lack of symmetry?

<u>Homework</u>: Show $L(\theta)L(\bar{\theta}) = L(\theta+\bar{\theta})$, and use this to show that if the $\underset{\sim}{\bar{x}}$-frame moves with vel v relative to $\underset{\sim}{x}$-frame, and the $\underset{\sim}{\bar{\bar{x}}}$-frame moves with velocity $\bar{v}$ rel to the barred frame, then the following velocity transformation law holds:

$$\bar{\bar{v}} = \frac{v+\bar{v}}{1+\frac{v\bar{v}}{c^2}}$$

Time Dilation & Lorentz Contr by Lorentz Projection –

- $\underline{X}_0, \underline{X}_1, \underline{X}_2, \underline{X}_3$ ON basis ($\Rightarrow$ non-lightlike)

$\underline{Y} = a^i \underline{X}_i \qquad a^i \equiv$ comps of $\underline{Y}$ wrt $\underline{X}_i$

$$\langle \underline{Y}, \underline{X}_i \rangle = a^i \langle \underline{X}_i, \underline{X}_i \rangle = \delta_i a^i, \quad \delta_i = \begin{cases} -1 & i=0 \\ +1 & i \neq 0 \end{cases}$$

$$\mathrm{Proj}_{\underline{X}_i} \underline{Y} = a^i \underline{X}_i = \underbrace{\frac{\langle \underline{Y}, \underline{X}_i \rangle}{\langle \underline{X}_i, \underline{X}_i \rangle}}_{} \underline{X}_i$$

"orth comp of $\underline{Y}$ in direction $\underline{X}_i$"

More generally – $$\mathrm{Proj}_{Z} \underline{Y} = \frac{\langle \underline{Y}, Z \rangle}{\langle Z, Z \rangle} Z$$

i.e. $Z = |\langle Z, Z \rangle|^{1/2} \underset{\text{unit}}{\underline{X}} \Rightarrow \mathrm{Proj}_Z \underline{Y} = \mathrm{Proj}_{\underline{X}} \underline{Y}$

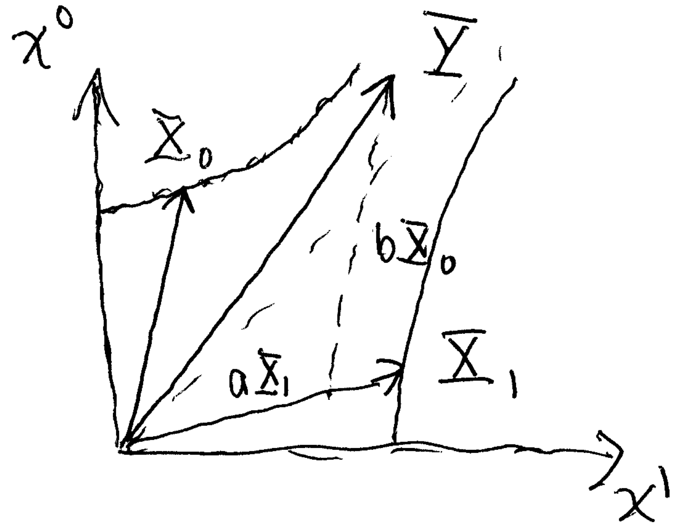
$$= \frac{\langle \underline{Y}, \underline{X} \rangle}{\langle \underline{X}, \underline{X} \rangle} \underline{X} = \frac{\langle \underline{Y}, Z \rangle}{\langle \underline{X}, \underline{X} \rangle |\langle Z, Z \rangle|} Z$$

• In $n=2$:

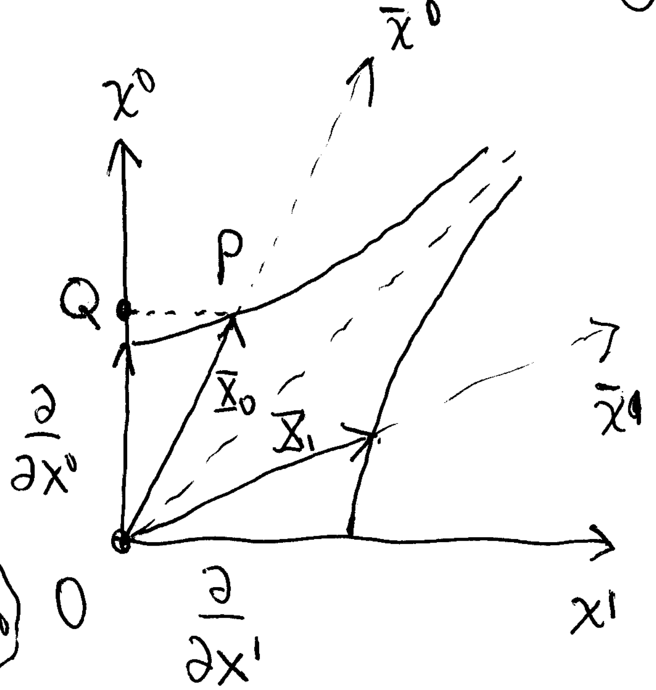
$$\bar{Y} = a\bar{X}_1 + b\bar{X}_0$$

$$a = \langle \bar{Y}, \bar{X}_1 \rangle$$

$$b = -\langle \bar{Y}, \bar{X}_0 \rangle > 0$$ ✓



☑ Time Dilation:



- when observer passing thru O reaches $\bar{x}^0 = 1$, he is at $P = \bar{X}_0$. But this is time $x^0 = a$ where

$\overline{OQ} = a\frac{\partial}{\partial x^0}$

& $a\frac{\partial}{\partial x^0} + b\frac{\partial}{\partial x^1} = \bar{X}_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta \frac{\partial}{\partial x^1} \Rightarrow a = \cosh\theta$

$$\left\langle a\frac{\partial}{\partial x^0} + b\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^0} \right\rangle = \left\langle \bar{X}_0, \frac{\partial}{\partial x^0} \right\rangle = -\cosh\theta$$

$$-a = -\cosh\theta$$

$$a = \cosh\theta = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma = \left|\left\langle \frac{\partial}{\partial x^0}, \bar{X}_0 \right\rangle\right|$$

$$a = \left\| \mathrm{Proj}_{\frac{\partial}{\partial x^0}} \bar{X}_0 \right\| = \left| \frac{\left\langle \frac{\partial}{\partial x^0}, \bar{X}_0 \right\rangle}{\left\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0} \right\rangle} \right| = \gamma = \frac{\Delta \bar{t}}{\Delta t}$$

$\|Y\| = \sqrt{|\langle Y, Y\rangle|}$

<u>Note</u>: $\langle \frac{\partial}{\partial x^0}, \bar X_0 \rangle = \langle \bar X_0, \frac{\partial}{\partial x^0} \rangle$ and since they are both <u>unit vectors</u>

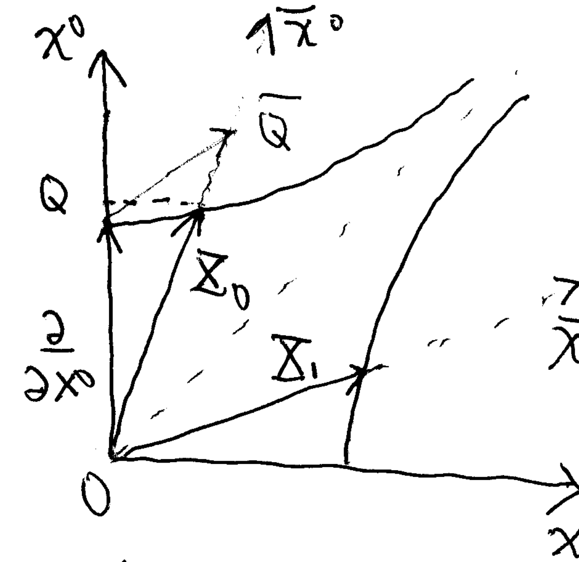
$$\mathrm{Proj}_{\frac{\partial}{\partial x^0}} \bar X_0 = \langle \bar X_0, \frac{\partial}{\partial x^0} \rangle = \langle \frac{\partial}{\partial x^0}, \bar X_0 \rangle = \mathrm{Proj}_{\bar X_0} \frac{\partial}{\partial x^0}$$

I.e.,

$$\underbrace{\overset{(\cosh\theta)}{\frac{\langle \bar X_0, \frac{\partial}{\partial x^0} \rangle}{\langle \frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0} \rangle}} \frac{\partial}{\partial x^0}}_{\mathrm{Proj}_{\frac{\partial}{\partial x^0}} \bar X_0 = \vec{OQ}} + \overset{(\sinh\theta)}{\langle \bar X_0, \frac{\partial}{\partial x^1} \rangle} \frac{\partial}{\partial x^1} = \bar X_0$$

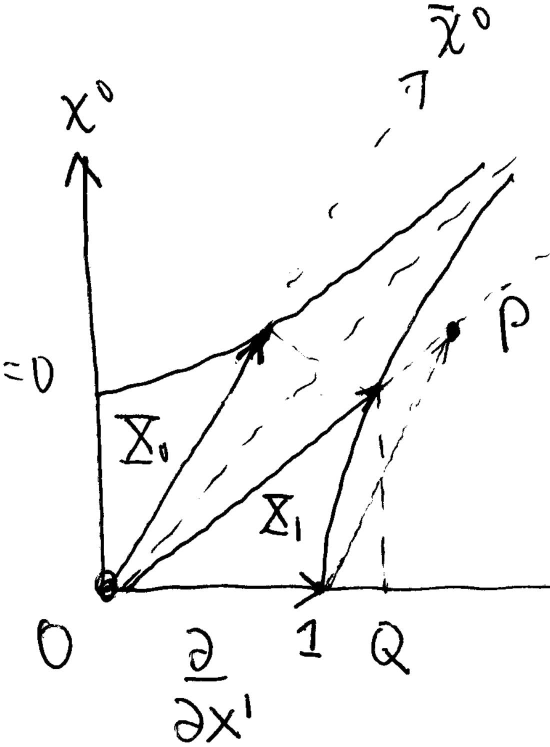
$$\underbrace{\overset{(\cosh\theta)}{\frac{\langle \frac{\partial}{\partial x^0}, \bar X_0 \rangle}{\langle \bar X_0, \bar X_0 \rangle}} \bar X_0}_{\mathrm{Proj}_{\frac{\partial}{\partial x^0}} \bar X_0 = \vec{O\bar Q}} + \overset{(-\sinh\theta)}{\langle \frac{\partial}{\partial x^0}, \bar X_1 \rangle} \bar X_1 = \frac{\partial}{\partial x^0}$$

$\vec{OQ} = \mathrm{Proj}_{\frac{\partial}{\partial x^0}} \bar X_0$; $\vec{O\bar Q} = \mathrm{Proj}_{\bar X_0} \frac{\partial}{\partial x^0}$



$$\|\vec{OQ}\| = \|\vec{O\bar Q}\| = \left|\langle \bar X_0, \frac{\partial}{\partial x^0} \rangle\right| = \gamma = \cosh\theta = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

Lorentz Contraction:



- Note: A rod at rest in bar frame at $\overline{OP}$ when $\bar{x}^0 = 0$ is observed to lie betw 0 & 1 at $x^0 = 0$; i.e., just trace the endpts to $x^0 = 0$.

- But:

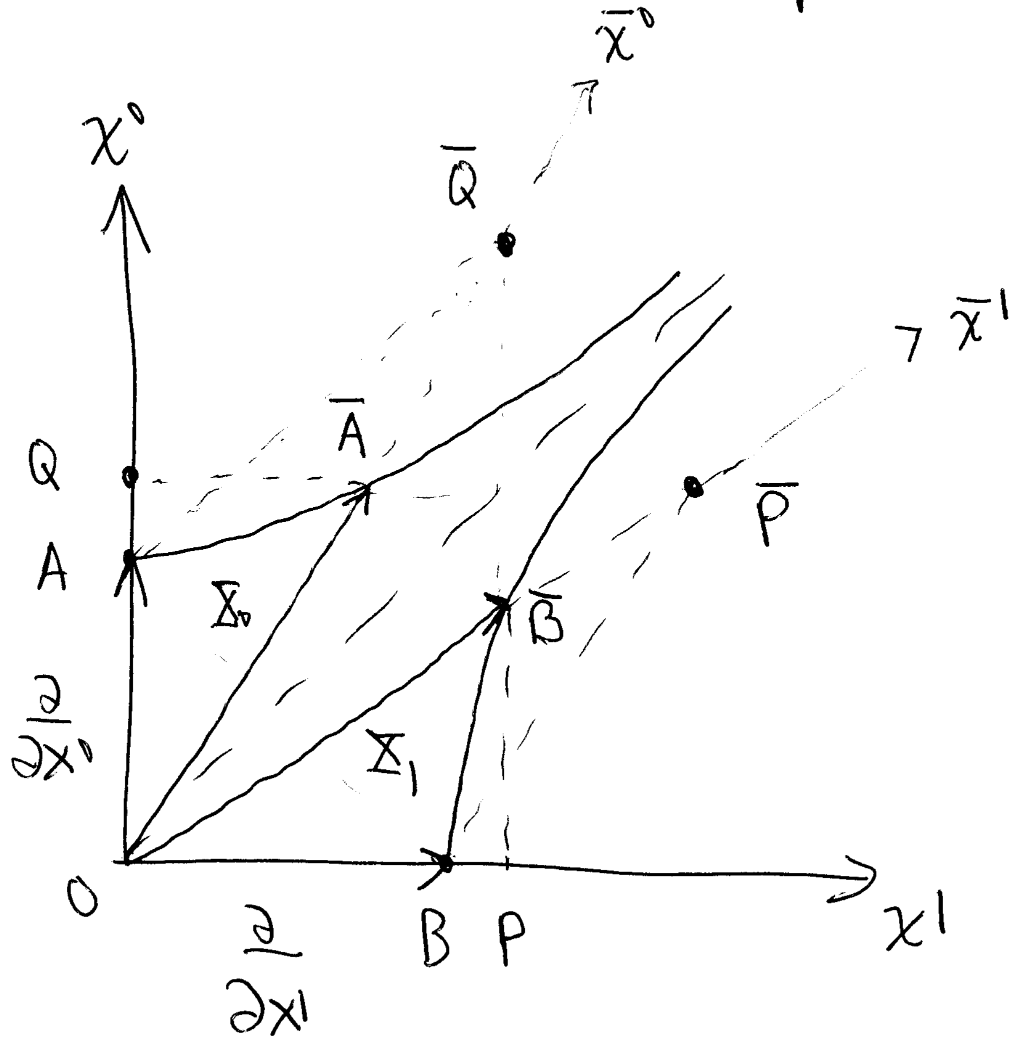
$$\vec{OP} = \bar{a}\,\underline{X}_1 \quad \text{with} \quad \underbrace{\bar{a}\,\underline{X}_1}_{\text{Proj}_{\underline{X}_1}\frac{\partial}{\partial x^1}} - \underline{X}_0 = \frac{\partial}{\partial x^1}$$

$\bar{a}$ = length of a bar in its rest frame when it measures one in moving frame

$$\vec{OQ} = a\frac{\partial}{\partial x^1} \quad \text{with} \quad \underbrace{a\frac{\partial}{\partial x^1}}_{\text{Proj}_{\frac{\partial}{\partial x^1}}\underline{X}_1} + b\frac{\partial}{\partial x^0} = \underline{X}_1$$

So $\bar{a} = \|\text{Proj}_{\underline{X}_1}\frac{\partial}{\partial x^1}\| = \langle \underline{X}_1, \frac{\partial}{\partial x^1}\rangle = \langle \frac{\partial}{\partial x^1}, \underline{X}_1\rangle = \overset{= a}{\|\text{Proj}_{\frac{\partial}{\partial x^1}}\underline{X}_1\|}$

Picture is symmetric but interpretation is not.



Time

- Moving clock at $\bar{A}$ reads $\bar{x}^0 = 1$ while $x^0 = Q = \cosh\theta > 1$
- Fixed observer at A reads $x^0 = 1$ while moving clock at $\bar{Q}$ reads $\bar{x}^0 = \bar{Q} = \cosh\theta > 1$

Space

- Moving rod at $\overline{OP}$ has length $\gamma > 1$ while fixed observer sees it at $x^0 = 0$ on $\overrightarrow{OB}$ of length one
- Fixed ~~rod~~ rod of length $\overrightarrow{OP} = \gamma$ is at $\bar{x}^0 = 0$ at $\overline{OB} = 1$.

☒ <u>Special Relativity</u> in (3+1): Assume $g_{ij} = \eta_{ij} = \mathrm{diag}(-1,1,1,1)$ in $\underset{\sim}{x}$-coords $\Leftrightarrow$ $\underset{\sim}{x}$ a Lorentz frame.

<u>Defn</u>: A Lorentz transformation is a map between two coord. systems in which $g = \eta$.

Condition: $$\eta_{\alpha\beta} = \eta_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\alpha}$$

$\Leftrightarrow$ Matrix notation: $A = \dfrac{\partial x^i}{\partial y^\alpha} \begin{matrix} \leftarrow \text{row} \\ \leftarrow \text{colm} \end{matrix}$

$$\eta = A^t \eta A \qquad (L)$$

<u>Notation</u>: $x \equiv (x^0, \dots, x^3)$ $\quad \underset{\sim}{x} \equiv (x^1, x^2, x^3) \in \mathbb{R}^3$

Theorem: If A satisfies (L) at each point in a neighborhood of p, then $\underset{y \to x}{x \circ y^{-1}}$ is a linear map; i.e. $\exists$ a constant matrix A such that

$$(x \circ y^{-1})(y) = Ay + a$$

constant 4x4 matrix. constant vector

or

$$x^i = A^i_\alpha y^\alpha + a^i$$ $\leftarrow$ component form.

From Here on out, change notation:

$x \equiv (x^0, \ldots, x^3)$ $\quad$ $\underset{\sim}{x} \equiv (x^1, x^2, x^3)$

(Use in 1-d spec. rel)

<u>Theorem</u>: If A satisfies (L) at each point in a neighborhood of p, then $x \circ y^{-1}$ is a linear map; i.e. $\exists$ a constant matrix A such that

$$\left(x \circ y^{-1}\right)(y) = \underset{\substack{\text{constant} \\ 4\times 4 \text{ matrix.}}}{A}\, y + \underset{\text{constant vector}}{a}$$

or

$$x^i = A^i_\alpha\, y^\alpha + a^i \quad \leftarrow \text{component form.}$$

From Here on out, change notation:

$$x \equiv (x^0, \dots, x^3) \qquad \underset{\sim}{x} \equiv (x^1, x^2, x^3)$$

(Use in 1-d spec. rel)

Proof: assume that

$$(L) \qquad \eta_{\beta\gamma} = \eta_{ij} \frac{\partial x^i}{\partial y^\beta} \frac{\partial x^j}{\partial y^\gamma}$$

at each point in a nbhd of P. Differentiate both sides of (L) wrt y^α and obtain

$$(*) \qquad 0 = \eta_{ij} \frac{\partial^2 x^i}{\partial y^\beta \partial y^\alpha} \frac{\partial x^j}{\partial y^\gamma} + \eta_{ij} \frac{\partial x^i}{\partial y^\beta} \frac{\partial^2 x^j}{\partial y^\gamma \partial y^\alpha}$$

Since (*) holds for all α, β, γ, we can add various combinations of these derivatives to solve for the 2ND derivatives: Define

$$F(\alpha, \beta, \gamma) = \eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\gamma} \equiv F(\beta, \alpha, \gamma)$$

Then (*) reads

$$0 = F(\alpha, \beta, \gamma) + F(\gamma, \alpha, \beta).$$

Now add & subtract cyclic combination of (*):

$$
\begin{aligned}
0 &= \cancel{F(\alpha,\beta,\gamma)} + F(\gamma,\alpha,\beta) \\
&\quad + F(\gamma,\alpha,\beta) + \cancel{F(\beta,\gamma,\alpha)} \\
&\quad - \cancel{F(\beta,\gamma,\alpha)} - \cancel{F(\alpha,\beta,\gamma)} \\
&= 2F(\alpha,\gamma,\beta) \qquad \forall\ \alpha.\gamma.\beta.
\end{aligned}
$$

$$
\therefore \quad 0 = \eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\gamma}
$$

nonsingular matrices

$$
\therefore \quad \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} = 0
$$

$\Rightarrow$ $x \circ y^{-1}$ is a linear function ✓

- Let

$$x^i = A^i_\alpha \, y^\alpha + a^i \qquad (L\text{-}T)_1$$

denote an arbitrary transformation that preserves η,

$$\eta_{ij} A^i_\alpha A^j_\beta = \eta_{\alpha\beta}, \qquad (L\text{-}T)_2$$

$A \equiv$ const. It is easy to show this is a group under composition. EFP

<u>Defn</u>: The set of all linear transformations (L-T) is called the <u>inhomogeneous Lorentz group</u> or the <u>Poincaré</u> group. The subgroup with $a=0$ is the <u>homogeneous Lorentz group</u>.

- Since time has a preferred direction, and space inversion is "singular", we expect there to be subgroups of the Lorentz group that do not invert time or space. These are called <u>proper</u> L-transformations. We now show that this is a subgroup separated from the rest of the L-G.

<u>Theorem</u>: The proper (L-T)'s are characterized by the conditions

$$A^0_{\ 0} \geqslant 1$$

and

$$\det A = 1 .$$

Moreover, the are separated from all other L-T's in the sense that $\nexists$ a continuous 1-parameter family $A(\xi)$ of L-T's st $A(0)$ is proper and $A(1)$ is not.

<u>Proof</u>: From $\eta = A^t \eta A$ we obtain

$$(\det A)^2 = 1 .$$

Moreover, from $\eta_{00} = \eta_{ij} A^i_{\ 0} A^j_{\ 0}$ we have

$$-1 = -(A^0_{\ 0})^2 + \sum_{i=1}^{3} (A^i_{\ 0})^2$$

$$\Rightarrow \qquad (A^0_{\ 0})^2 = 1 + \sum_{i=1}^{3} (A^i_{\ 0})^2 \geqslant 1 .$$

Thus the L-T's with $A^0_0 \geq 1$ are separated from those with $A^0_0 \leq -1$, and those with $\det A = 1$ are separated from those with $\det A = -1$. The Thm follows by continuity from $A = id$. Since $\det A = 1$, we have

<u>Cor</u>: A proper L-T preserves the volume form

$$dx^0 \cdots dx^3 \longleftrightarrow dy^0 \cdots dy^3.$$

$$\int_V dx^0 \cdots dx^3 = \int_{L(V)} dy^0 \cdots dy^3 .$$

☐ Characterization of the proper, homo., Lorentz Transformations in 4-d.

<u>Defn</u>: We call R a rotation if

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R} & \\ 0 & & & \end{bmatrix},$$

where $\underset{\sim}{R}$ is a 3-d rotation, $\underset{\sim}{R}^t \cdot \underset{\sim}{R} = id_3$.

<u>Lemma</u>: If $\det \underset{\sim}{R} > 0$, then R is a proper homo, L-transformation.

<u>Proof</u>: $R^0_0 = 1$, $\det R = \det \underset{\sim}{R} = 1$ ✓

<u>Defn</u>: a boost is a PHLT that changes the velocity.

Eg, assume $x = A\bar{x}$ defines a boost.

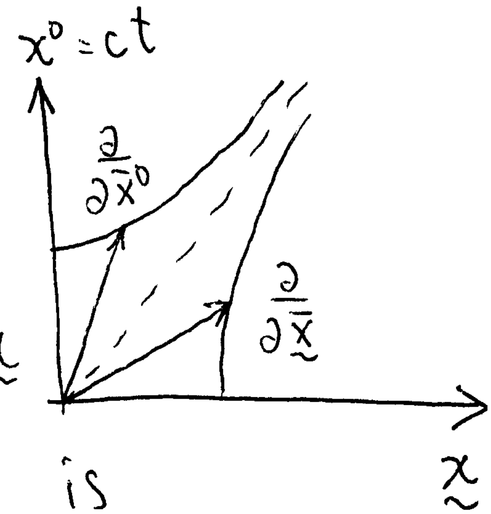
Then

$$\frac{\partial}{\partial \bar{x}^0} = \frac{\partial x^i}{\partial \bar{x}^0}\frac{\partial}{\partial x^i} = A^i{}_0 \frac{\partial}{\partial x^i}$$

$$= A^0{}_0 \frac{\partial}{\partial x^0} + \underset{\sim}{A}_0 \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$

But the observer fixed in the barred frame follows the path

$$x(s) = s\frac{\partial}{\partial \bar{x}^0} = sA^0{}_0\frac{\partial}{\partial x^0} + s\underset{\sim}{A}_0 \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$



Thus, the velocity of the observer is

$$\frac{1}{c}\frac{d\underset{\sim}{x}}{dt} = \frac{d\underset{\sim}{x}}{dx^0} = \frac{\underset{\sim}{A}_0}{A^0{}_0} \Rightarrow \boxed{v^i = c\,\frac{A^i{}_0}{A^0{}_0}}$$

$\uparrow$

$\underset{\sim}{v}$

But $\left(A^0_{\ 0}\right)^2 = 1 + \sum_{i=1}^{3} \left(A^i_{\ 0}\right)^2$

$$= 1 + \sum_{i=1}^{3} \left(\frac{v^i}{c} A^0_{\ 0}\right)^2$$

$$\Rightarrow \qquad A^0_{\ 0} = \frac{1}{\sqrt{1-\left(\frac{|v|}{c}\right)^2}} = \gamma \quad (= \cosh\theta \text{ later})$$

$$\Rightarrow \qquad A^i_{\ 0} = A^0_{\ 0}\,\frac{v^i}{c} = \frac{v^i/c}{\sqrt{1-\frac{|v|^2}{c^2}}} = \frac{v^i}{c}\gamma \quad (= \sinh\theta)$$

<u>Lemma ②</u>: Let A and $\bar{A}$ be two boosts with same velocity $\underset{\sim}{v}$. Then

$$A = \bar{A} R$$

for some proper rotation R.

(A and $\bar{A}$ are boosts with <u>same</u> velocity if
$A^i_{\ 0} = \bar{A}^i_{\ 0} \quad i = 0,1,2,3$)

Proof: Let $B = A\bar{A}^{-1}$, itself a R-H L-T. We show $B^0_0 = 1$, $B^0_\sigma = B^\sigma_0 = 0$ $\sigma \neq 0$. Note first that

$$B\,\eta\,B^t = \eta$$

$$\Rightarrow \qquad B(\eta B^t \eta) = id \qquad (\eta^2 = id)$$

$$\Rightarrow \qquad \eta B^t \eta = B^{-1}. \qquad (*)$$

$$B^{-1} = (\eta B^t \eta)^i_j = \eta_{jj}(B^t)^j_i\,\eta^{ii}, \quad (\eta^{ij}) = (\eta_{ij})^{-1} = \eta_{ij}$$

$\Rightarrow$ B^{-1} equals B^t with factors of (-1) on the 0-row & 0-colm. First, it suffices to verify that $B^0_0 = 1$. To see this, use the identity

$$(B^0_0)^2 - \sum_{i=1}^{3}(B^i_0)^2 = 1,$$

to conclude $B^i_0 = 0$; and the same identity applied to $B^{-1} \approx B^t$ gives $B^0_i = 0$ thru

$$(B^0_0)^2 - \sum_{i=1}^{3}(B^0_i)^2 = 1. \quad \checkmark$$

To show that $B^0_0 = 1$, we use the condition

$$\frac{\partial}{\partial \bar{x}^0} = A^i_0 \frac{\partial}{\partial x^i} = \bar{A}^i_0 \frac{\partial}{\partial x^i}$$

which is equivalent to the condition that both A & $\bar{A}$ move at "same velocity" (ie have same timelike coord vector, since this gives world line of observer fixed in the frame).

This implies $A^i_0 = \bar{A}^i_0$, and so

$$B^0_0 = (A\bar{A}^{-1})^0_0 = A^\sigma_0 \eta_{\sigma\sigma} \bar{A}^\sigma_0 \eta^{00} = A^\sigma_0 \eta_{\sigma\sigma} A^\sigma_0 \eta^{00}$$

$$= (A^\sigma_0)^2 \eta_{\sigma\sigma} \eta^{00} = -(A^\sigma_0)^2 \eta_{\sigma\sigma}$$

$$= (A^0_0)^2 - \sum_{i=1}^{3} (A^i_0)^2 = 1$$

As claimed.

Thus,

$$B = A\bar{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R} & \\ 0 & & & \end{bmatrix}$$

for some 3x3 matrix $\underset{\sim}{R}$, $\det \underset{\sim}{R} > 0$.
But (*) then gives $R^t R = I \Rightarrow R$ is a proper rotation, & Thm is proved.

<u>Conclude</u>: All Rh-L-T's are given by

$$A = L(\underset{\sim}{v})R,$$

if we can construct a boost $L(\underset{\sim}{v})$ with velocity $\underset{\sim}{v}$.

Characterization of Proper Homogeneous Lorentz Transformations (PHLT) in 3+1 dimensions.

- The defining property of a Lorentz Trans. Λ is:

$$\eta_{ij}\Lambda^{i}{}_{\alpha}\Lambda^{j}{}_{\beta} = \eta_{\alpha\beta} \iff \Lambda^{T}\eta\Lambda = \eta$$

- $\eta = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \Rightarrow \eta_{ii}\Lambda^{i}{}_{\alpha}\Lambda^{i}{}_{\beta} = \eta_{\alpha\alpha} \quad \forall\ \alpha = 0,\dots,3.$

$$\Rightarrow \eta_{ii}\Lambda^{i}{}_{0}\Lambda^{i}{}_{0} = \eta_{00} \iff -(\Lambda^{0}{}_{0})^{2} + \sum_{i=1}^{3}(\Lambda^{i}{}_{0})^{2} = -1$$

$$\eta_{ii}\Lambda^{i}{}_{j}\Lambda^{i}{}_{j} = \eta_{jj} \iff -(\Lambda^{0}{}_{0})^{2} + \sum_{i=1}^{3}(\Lambda^{i}{}_{j})^{2} = 1$$

$j = 1,2,3$

- $\Lambda^{T}\eta\Lambda = \eta \iff \begin{bmatrix} \Lambda^{T}_{00} & — \\ | & (\Lambda^{T})^{i}{}_{j} \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & 0 & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \Lambda_{00} & — \\ | & \Lambda^{i}{}_{j} \end{bmatrix}$

$$\Rightarrow \tilde{\Lambda}^{T}\tilde{\Lambda} = I = \tilde{\Lambda}\tilde{\Lambda}^{T} \iff \tilde{\Lambda}^{T} = \tilde{\Lambda}^{-1} = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & 0 & 1 & \\ & & & 1 \end{bmatrix}$$

$\tilde{\Lambda}^{T}_{3\times3} = (\Lambda^{T})^{i}{}_{j}$, $\tilde{\Lambda}_{3\times3} = \Lambda^{i}{}_{j}$ $\quad$ $\therefore \tilde{\Lambda}$ ~ Rotation

(8B)

- **Defn**: We call $R_{4\times4}$ a rotation if

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R} & \\ 0 & & & \end{bmatrix} \qquad \underset{\sim}{R}^{T}_{3\times3} = \underset{\sim}{R}^{-1}_{3\times3}$$

Lemma: if $\det \underset{\sim}{R} > 0$, then R is a PHLT

Proof:

$$R^T \eta R \overset{?}{=} \eta$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R}^T & \\ 0 & & & \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R} & \\ 0 & & & \end{bmatrix} \overset{?}{=} \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R}^T \underset{\sim}{R} & \\ 0 & & & \end{bmatrix} \overset{?}{=} \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \checkmark$$

Since $R^0_0 = 1 \geq 1$, $\det R > 0$ iff $\det \underset{\sim}{R} > 0$

so $R^0_0 = 1 \Rightarrow R$ is PHLT $\checkmark$

<u>Defn</u>: a boost is a PHLT that changes the velocity.

Eg, assume $x = A\bar{x}$ is any PHLT

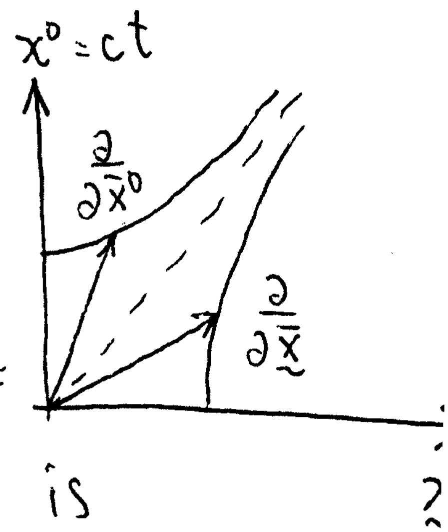
Then

$$\frac{\partial}{\partial \bar{x}^0} = \frac{\partial x^\sigma}{\partial \bar{x}^0}\frac{\partial}{\partial x^\sigma} = A^\sigma{}_0 \frac{\partial}{\partial x^\sigma}$$

$$= A^0_0 \frac{\partial}{\partial x^0} + \underset{\sim}{A}_0 \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$

But the observer fixed in the barred frame follows the path

$$x(s) = s\frac{\partial}{\partial \bar{x}^0} = sA^0_0 \frac{\partial}{\partial x^0} + s\underset{\sim}{A}_0 \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$



Thus, the velocity of the observer is

$$\underbrace{\frac{1}{c}\frac{d\underset{\sim}{x}}{dt}}_{\underset{\sim}{v}} = \frac{d\underset{\sim}{x}}{dx^0} = \frac{\underset{\sim}{A}_0}{A^0_0} \Rightarrow \boxed{v^i = c\,\frac{A^i_0}{A^0_0}}$$

But $$(A^0_0)^2 = 1 + \sum_{i=1}^{3} (A^i_0)^2$$
$$= 1 + \sum_{i=1}^{3} \left(\frac{v^i}{c} A^0_0\right)^2$$

$$A^i_0 = \frac{v^i}{c} A^0_0$$

So

$$A^0_0 = \frac{1}{\sqrt{1-\left(\frac{|v|}{c}\right)^2}} = \gamma \equiv \cosh\theta$$

$$A^i_0 = A^0_0 \frac{v^i}{c} = \frac{v^i/c}{\sqrt{1-\frac{|v|^2}{c^2}}} = \frac{v^i}{c}\gamma$$

Conclude:

$v = \tanh\theta$, $\operatorname{sech}^2\theta = 1 - \tanh^2\theta$

$$(1+1) \quad A = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-\frac{|v|^2}{c^2}}} & \frac{v}{\sqrt{1-\left|\frac{v}{c}\right|^2}} \\ \frac{v}{\sqrt{1-\left|\frac{v}{c}\right|^2}} & \frac{1}{\sqrt{1-\left|\frac{v}{c}\right|^2}} \end{bmatrix}$$

$$(3+1) \quad A = \begin{bmatrix} \cosh\theta & \text{---} \\ \underset{\sim}{v}^T \cosh\theta & \underset{\sim}{A}_{3\times 3} \end{bmatrix}$$

But $A^T \neq A$ & $RA \neq AR$ in general.

Lemma ②: Let A & $\bar{A}$ be two boosts with the SAME velocity $\underset{\sim}{v}$. Then

$$A = \bar{A} R$$

for some proper rotation R.

Note ① In general, $\bar{A} R \neq R \bar{A}$

The RHL Group of transformations is complicated. ⇝ Description by Lie Groups / Lie Algebras

Proof: Let $B = A\bar{A}^{-1}$, itself a P-H L-T. We show $B^0_0 = 1$, $B^0_\sigma = B^\sigma_0 = 0$ $\sigma \neq 0$. Note first that

$$B \eta B^t = \eta$$

$$\Rightarrow \qquad B(\eta B^t \eta) = id \qquad (\eta^2 = id)$$

$$\Rightarrow \qquad \eta B^t \eta = B^{-1}. \qquad (*)$$

$$B^{-1} = (\eta B^t \eta)^i_j = \eta_{jj} (B^t)^j_i \eta^{ii}, \quad (\eta^{ij}) = (\eta_{ij})^{-1} = \eta_{ij}$$

$\Rightarrow$ B^{-1} equals B^t with factors of (-1) on the 0-row & 0-colm. Claim — it suffices to verify that $B^0_0 = 1$. To see this, use the identity

$$(B^0_0)^2 - \sum_{i=1}^{3} (B^i_0)^2 = 1,$$

to conclude $B^i_0 = 0$; and the same identity applied to $B^{-1} \approx \pm B^t$ gives $B^0_i = 0$ thru

$$(\bar{B}^0_0)^2 - \sum_{i=1}^{3} (\bar{B}^{-1\,0}_{\ \ i})^2 = 1. \quad \checkmark$$

To show that $B^0_0 = 1$, we use the condition (12)

$$\frac{\partial}{\partial \bar{x}^0} = A^i_0 \frac{\partial}{\partial x^i} = \bar{A}^i_0 \frac{\partial}{\partial x^i}$$

which is equivalent to the condition that both A & $\bar{A}$ move at "same velocity" (ie have same timelike coord vector, since this give world line of observer fixed in the frame).

This implies $A^i_0 = \bar{A}^i_0$, and so using $\bar{A}^{-1} = \eta \bar{A}^T \eta$

$$B^0_0 = (A\bar{A}^{-1})^0_0 = A^\sigma_0 \eta_{\sigma\sigma} \bar{A}^\sigma_0 \eta^{00} = A^\sigma_0 \eta_{\sigma\sigma} A^\sigma_0 \eta^{00}$$

$$= (A^\sigma_0)^2 \eta_{\sigma\sigma} \eta^{00} = -(A^\sigma_0)^2 \eta_{\sigma\sigma}$$

$$= (A^0_0)^2 - \sum_{i=1}^{3} (A^i_0)^2 = 1$$

As claimed.

Thus,

$$B = A\bar{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R} & \\ 0 & & & \end{bmatrix}$$

(13)

for some 3×3 matrix $\underset{\sim}{R}$, $\det \underset{\sim}{R} > 0$.

But (*) then gives $R^t R = I \Rightarrow R$ is a proper rotation, & Thm is proved.

<u>Conclude</u>: All Rh-L-T's are given by

$$A = L(\underset{\sim}{v})\, R,$$

if we can construct a boost $L(\underset{\sim}{v})$ with velocity $\underset{\sim}{v}$.

<u>Thm</u>: Simplest boost in direction $\underset{\sim}{v}$

$$A = \begin{bmatrix} \gamma & -\gamma\left(\frac{v^T}{c}\right) \\ -\gamma \frac{v}{c} & I + (\gamma-1)\frac{\underset{\sim}{v}\,\underset{\sim}{v}^T}{|\underset{\sim}{v}|^2} \end{bmatrix}$$

(Move on to Connection/Curvature)

Proof: In 1+1, $\bar{A}(\theta) = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$ is the map from $\bar{x}$ to x. That is,

$$\bar{A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cosh\theta \\ \sinh\theta \end{bmatrix} = \text{coords of } \frac{\partial}{\partial \bar{x}^0} \text{ in } (x^0, x^1)$$

Thus $A = \bar{A}^{-1} = \bar{A}(-\theta) = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$ (1)

maps from x to $\bar{x}$.

- To construct a boost with velocity $\vec{v} \in \mathbb{R}^3$, start with 1+1 form of (1)

$$A = \begin{bmatrix} \gamma & -\gamma\frac{v}{c} \\ -\gamma\frac{v}{c} & \gamma \end{bmatrix}.$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \end{pmatrix} = A\begin{bmatrix} x^0 \\ x^1 \end{bmatrix} = \begin{bmatrix} \gamma x^0 - \gamma\frac{vx^1}{c} \\ -\gamma\frac{vx^0}{c} + \gamma x^1 \end{bmatrix}; \quad \begin{aligned} \bar{x}^0 &= \gamma\left(x^0 - \frac{vx^1}{c}\right) \\ \bar{x}^1 &= \gamma\left(x^1 - \frac{vx^0}{c}\right) \end{aligned} \quad (2)$$

- Given $\vec{v}$, we can define the boost that is (1) in direction $\underset{\sim}{v}$, with no change in directions orthogonal to $\vec{v}$.

I.e., let $\underset{\sim}{x} = \underset{\sim}{x}_{\perp} + \underset{\sim}{x}_{\parallel} \in \mathbb{R}^3$

$$\underset{\sim}{x}_{\parallel} = \frac{\underset{\sim}{x} \cdot \vec{v}}{|\vec{v}|^2} \underset{\sim}{v}$$

$$\underset{\sim}{x}_{\perp} = \underset{\sim}{x} - \underset{\sim}{x}_{\parallel}$$

Thus (2) in direction $\vec{v}$ reads –

$$\bar{x}^0 = \gamma\left(x^0 - \frac{\underset{\sim}{v}}{c} \cdot \underset{\sim}{x}\right)$$

$$\bar{\underset{\sim}{x}} = \underbrace{\underset{\sim}{x} - \underset{\sim}{x}_{\parallel}}_{\substack{\text{no change} \\ \text{in } \underset{\sim}{x}^{\perp}}} + \underbrace{\gamma\left(\underset{\sim}{x}_{\parallel} - \frac{\vec{v} x^0}{c}\right)}_{\text{boost in direction } \underset{\sim}{x}_{\parallel}}$$

$$= \underset{\sim}{x} + (\gamma - 1)\underset{\sim}{x}_{\parallel} - \gamma \frac{\vec{v} x^0}{c}, \quad \underset{\sim}{x}_{\parallel} = \frac{\underset{\sim}{x} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

$$= \underset{\sim}{x} + \left(\frac{\gamma - 1}{v^2} \underset{\sim}{x} \cdot \vec{v} - \frac{\gamma x^0}{c}\right) \vec{v}$$

OR

$$\bar{x}^0 = \gamma\left(x^0 - \overbrace{\frac{\vec{v}}{c}}^{A^0_i}\cdot \underset{\sim}{x}\right)$$

$$\bar{x}^i = x^i + \left(\frac{\gamma-1}{v^2}\,\underset{\sim}{x}\cdot\vec{v} - \underbrace{\frac{\gamma x^0}{c}}_{A^i_0}\right)v^i$$

OR

$$\begin{pmatrix} \bar{x}^0 \\ \bar{\underset{\sim}{x}} \end{pmatrix} = \begin{bmatrix} \gamma & -\frac{\vec{v}^T}{c}\gamma \\ -\frac{\vec{v}}{c}\gamma & I + \frac{\gamma-1}{v^2}\frac{\vec{v}*\vec{v}^T}{v^2} \end{bmatrix} \begin{bmatrix} x^0 \\ \underset{\sim}{x} \end{bmatrix}$$

as claimed. (Real pf need to check it satisfies $\eta^T A \eta = \eta$!)