☐ <u>Special Relativity</u> in $\mathbb{R}^4$: Assume $g_{ij} = \eta_{ij} = \mathrm{diag}(-1,1,1,1)$ in $\underset{\sim}{x}$-coords $\Leftrightarrow$ $\underset{\sim}{x}$ a Lorentz frame.

Defn: A Lorentz transformation is a map between two coord. systems in which $g = \eta$.

Condition: $$\eta_{\alpha\beta} = \eta_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\alpha}$$

$\Leftrightarrow$ Matrix notation: $A = \dfrac{\partial x^{i} \leftarrow \text{row}}{\partial y^{\alpha} \leftarrow \text{colm}}$

$$\eta = A^t \eta A \qquad (L)$$

<u>Notation</u>: $x \equiv (x^0, \dots, x^3)$ $\quad \underset{\sim}{x} \equiv (x^1, x^2, x^3) \in \mathbb{R}^3$

<u>Theorem</u>: If A satisfies (L) at each point in a neighborhood of p, then $x \circ y^{-1}$ is a linear map; i.e. $\exists$ a constant matrix A such that

$$(x \circ y^{-1})(y) = A y + a$$

constant 4x4 matrix. constant vector

or

$$x^i = A^i_{\alpha} y^{\alpha} + a^i$$

$\leftarrow$ component form.

From Here on out, change notation:

$x \equiv (x^0, \dots, x^3)$ $\quad \underset{\sim}{x} \equiv (x^1, x^2, x^3)$

(Use in 1-d spec. rel)

Proof: assume that

$$(L) \qquad \eta_{\beta\gamma} = \eta_{ij} \frac{\partial x^i}{\partial y^\beta} \frac{\partial x^j}{\partial y^\gamma}$$

at each point in a nbhd of p. Differentiate both sides of (L) wrt y^α and obtain

$$(*) \qquad 0 = \eta_{ij} \frac{\partial^2 x^i}{\partial y^\beta \partial y^\alpha} \frac{\partial x^j}{\partial y^\gamma} + \eta_{ij} \frac{\partial x^i}{\partial y^\beta} \frac{\partial^2 x^j}{\partial y^\gamma \partial y^\alpha}$$

Since (*) holds for all α, β, γ, we can add various combinations of these derivatives to solve for the 2ND derivatives: Define

$$F(\alpha, \beta, \gamma) = \eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\gamma} \equiv F(\beta, \alpha, \gamma)$$

Then (*) reads

$$0 = F(\alpha, \beta, \gamma) + F(\gamma, \alpha, \beta).$$

Now add & subtract cyclic combination of (*):

$$0 = \cancel{F(\alpha,\beta,\gamma)} + F(\gamma,\alpha,\beta)$$
$$+ F(\gamma,\alpha,\beta) + \cancel{F(\beta,\gamma,\alpha)}$$
$$- \cancel{F(\beta,\gamma,\alpha)} - \cancel{F(\alpha,\beta,\gamma)}$$
$$= 2F(\alpha,\gamma,\beta) \qquad \forall\ \alpha.\gamma.\beta.$$

$$\therefore \quad 0 = \eta_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^j}{\partial y^\gamma}$$

nonsingular matrices

$$\therefore \quad \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} = 0$$

$\Rightarrow$ $x \circ y^{-1}$ is a linear function ✓

• Let

$$x^i = A^i_\alpha \, y^\alpha + a^i \qquad (L\text{-}T)_1$$

denote an arbitrary transformation that preserves η,

$$\eta_{ij} A^i_\alpha A^j_\beta = \eta_{\alpha\beta}, \qquad (L\text{-}T)_2$$

$A \equiv$ const. It is easy to show this is a group under composition. EIP

<u>Defn</u>: The set of all linear transformations (L-T) is called the <u>inhomogeneous</u> <u>Lorentz</u> <u>group</u> or the <u>Poincaré</u> group. The subgroup with $a=0$ is the <u>homogeneous</u> <u>Lorentz</u> <u>group</u>.

• Since time has a preferred direction, and space inversion is "singular", we expect there to be subgroups of the Lorentz group that do not invert time or space. These are called <u>proper</u> L-transformations. We now show that this is a subgroup separated from the rest of the L-G.

<u>Theorem</u>: The proper (L-T)'s are characterized by the conditions

$$A^0_0 \geq 1$$

and

$$\det A = 1.$$

Moreover, the are separated from all other L-T's in the sense that $\nexists$ a continuous 1-parameter family $A(s)$ of L-T's st $A(0)$ is proper and $A(1)$ is not.

<u>Proof</u>: From $\eta = A^t \eta A$ we obtain

$$(\det A)^2 = 1.$$

Moreover, from $\eta_{00} = \eta_{ij} A^i_0 A^j_0$ we have

$$-1 = -(A^0_0)^2 + \sum_{i=1}^{3} (A^i_0)^2$$

$$\Rightarrow \quad (A^0_0)^2 = 1 + \sum_{i=1}^{3} (A^i_0)^2 \geq 1.$$

Thus the L-T's with $A^0_0 \geq 1$ are separated from those with $A^0_0 \leq -1$, and those with $\det A = 1$ are separated from those with $\det A = -1$. The Thm follows by continuity from $A = id$. Since $\det A = 1$, we have

<u>Cor</u>: A proper L-T preserves the volume form

$$dx^0 \cdots dx^3 \longleftrightarrow dy^0 \cdots dy^3.$$

$$\int_V dx^0 \cdots dx^3 = \int_{L(V)} dy^0 \cdots dy^3.$$

<u>Defn</u>: a boost is a PHL that changes the velocity.

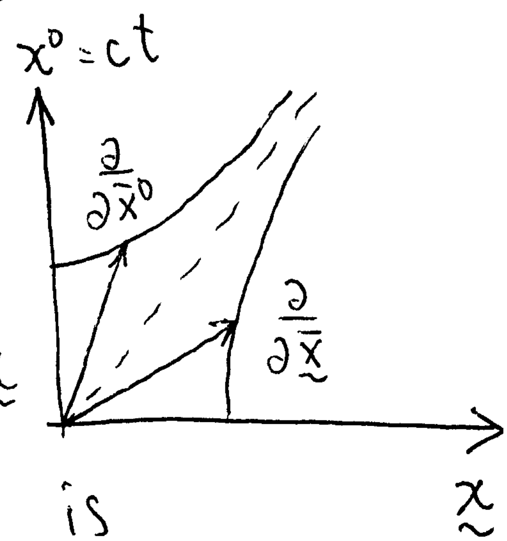
Eg, assume $x = A\bar{x}$ defines a boost.

Then
$$\frac{\partial}{\partial \bar{x}^0} = \frac{\partial x^i}{\partial \bar{x}^0}\frac{\partial}{\partial x^i} = A^i{}_0 \frac{\partial}{\partial x^i}$$

$$= A^0{}_0 \frac{\partial}{\partial x^0} + \underset{\sim}{A}{}_0 \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$

But the observor fixed in the barred frame follows the path

$$x(s) = s\frac{\partial}{\partial \bar{x}^0} = sA^0{}_0\frac{\partial}{\partial x^0} + s\underset{\sim}{A}{}_0 \cdot \frac{\partial}{\partial \underset{\sim}{x}}$$



Thus, the velocity of the observor is

$$\frac{1}{c}\frac{d\underset{\sim}{x}}{dt} = \frac{d\underset{\sim}{x}}{dx^0} = \frac{\underset{\sim}{A}{}_0}{A^0{}_0} \Rightarrow \boxed{v^i = c\,\frac{A^i{}_0}{A^0{}_0}}$$

$\uparrow$

$\underset{\sim}{v}$

☐ Characterization of the proper, homo., Lorentz Transformations in 4-d.

<u>Defn</u>: We call R a rotation if

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R} & \\ 0 & & & \end{bmatrix},$$

where $\underset{\sim}{R}$ is a 3-d rotation, $\underset{\sim}{R}^t \cdot \underset{\sim}{R} = id_3$.

<u>Lemma</u>: If $\det \underset{\sim}{R} > 0$, then R is a proper homo, L-transformation.

<u>Proof</u>: $R^0_0 = 1$, $\det R = \det \underset{\sim}{R} = 1$ ✓

But $$(A^0_0)^2 = 1 + \sum_{i=1}^{3} (A^i_0)^2$$

$$= 1 + \sum_{i=1}^{3} \left(\frac{v^i}{c} A^0_0\right)^2$$

$$\Rightarrow \quad A^0_0 = \frac{1}{\sqrt{1-\left(\frac{|v|}{c}\right)^2}} = \gamma \quad (= \cosh\theta \text{ later})$$

$$\Rightarrow \quad A^i_0 = A^0_0 \frac{v^i}{c} = \frac{v^i/c}{\sqrt{1-\frac{|v|^2}{c^2}}} = \frac{v^i}{c}\gamma \quad (= \sinh\theta)$$

<u>Lemma ②</u>: Let A and $\bar{A}$ be two boosts with same velocity $\underset{\sim}{v}$. Then

$$A = \bar{A} R$$

for some proper rotation R.

(A and $\bar{A}$ are boosts with <u>same</u> velocity if
$A^i_0 = \bar{A}^i_0 \quad i = 0,1,2,3$)

Proof: Let $B = A\bar{A}^{-1}$, itself a P-H L-T. We show $B^0_0 = 1$, $B^0_\sigma = B^\sigma_0 = 0$ $\sigma \neq 0$. Note first that

$$B\,\eta\,B^t = \eta$$

$$\Rightarrow \qquad B(\eta B^t \eta) = \mathrm{id} \qquad (\eta^2 = \mathrm{id})$$

$$\Rightarrow \qquad \eta B^t \eta = B^{-1}. \qquad (*)$$

$$B^{-1} = (\eta B^t \eta)^i_j = \eta_{jj} (B^t)^j_i\, \eta^{ii}, \qquad (\eta^{ij}) = (\eta_{ij})^{-1} = \eta_{ij}$$

$\Rightarrow$ B^{-1} equals B^t with factors of (-1) on the 0-row & 0-colm. First, it suffices to verify that $B^0_0 = 1$. To see this, use the identity

$$(B^0_0)^2 - \sum_{i=1}^{3} (B^i_0)^2 = 1,$$

to conclude $B^i_0 = 0$; and the same identity applied to $B^{-1} \approx B^t$ gives $B^0_i = 0$ thru

$$(B^0_0)^2 - \sum_{i=1}^{3} (B^0_i)^2 = 1. \quad \checkmark$$

To show that $B^0_0 = 1$, we use the condition

$$\frac{\partial}{\partial \bar{x}^0} = A^i_0 \frac{\partial}{\partial x^i} = \bar{A}^i_0 \frac{\partial}{\partial x^i}$$

which is equivalent to the condition that both A & $\bar{A}$ move at "same velocity" (ie have same timelike coord vector, since this gives world line of observer fixed in the frame).

This implies $A^i_0 = \bar{A}^i_0$, and so

$$B^0_0 = (A\bar{A}^{-1})^0_0 = A^\sigma_0 \eta_{\sigma\sigma} \bar{A}^\sigma_0 \eta^{00} = A^\sigma_0 \eta_{\sigma\sigma} A^\sigma_0 \eta^{00}$$

$$= (A^\sigma_0)^2 \eta_{\sigma\sigma} \eta^{00} = -(A^\sigma_0)^2 \eta_{\sigma\sigma}$$

$$= (A^0_0)^2 - \sum_{i=1}^{3} (A^i_0)^2 = 1$$

As claimed.

Thus,

$$B = A\bar{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \underset{\sim}{R} & \\ 0 & & & \end{bmatrix}$$

for some 3x3 matrix $\underset{\sim}{R}$, $\det \underset{\sim}{R} > 0$.

But (*) then gives $R^t R = D \Rightarrow R$ is a proper rotation, & Thm is proved.

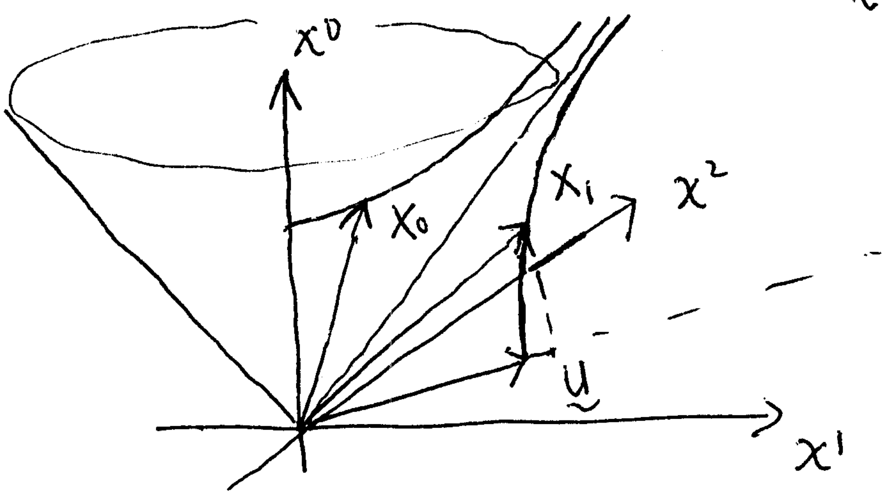
<u>Conclude</u>: All Rh-L-T's are given by

$$A = L(\underset{\sim}{v})\, R,$$

if we can construct a boost $L(\underset{\sim}{v})$ with velocity $\underset{\sim}{v}$.

② Construction of P-H-L-T (Pure Boost)

st barred frame moves w. vel. $\underset{\sim}{V} = \frac{d\underset{\sim}{x}}{dt}$ rel to unbarred frame. Set $\underset{\sim}{u} = \frac{\underset{\sim}{V}}{|\underset{\sim}{V}|}$



$X_0 \equiv$ "timelike unit vector in direction $\underset{\sim}{u}$" $\Rightarrow$

$$\boxed{X_0 = a \frac{\partial}{\partial x^0} + b\,\underset{\sim}{u}} \qquad (A)$$

$$\langle X_0, X_0 \rangle = -1 \Rightarrow -a^2 + b^2 |\underset{\sim}{u}|^2 = -1$$

$$\Rightarrow a^2 - b^2 = 1$$

$$\Rightarrow \quad X_0 = \cosh\theta \frac{\partial}{\partial x^0} + \sinh\theta\, \underset{\sim}{u}$$

skip

Let $X_1 \equiv$ "spacelike unit vector in direction $\underset{\sim}{u}$ orthogonal to X_0"

$$X_1 = \tilde{a} \frac{\partial}{\partial x^0} + \tilde{b} \underset{\sim}{u}$$

$$\langle X_1, X_1 \rangle = 1 \Rightarrow -\tilde{a}^2 + \tilde{b}^2 = 1$$

$$\Rightarrow \quad X_1 = \sinh\tilde{\theta} \frac{\partial}{\partial x^0} + \cosh\tilde{\theta} \underset{\sim}{u}$$

$$\langle X_0, X_1 \rangle = 0 \Rightarrow \tilde{\theta} = \theta$$

$$\boxed{X_1 = \sinh\theta \frac{\partial}{\partial x^0} + \cosh\theta \underset{\sim}{u} .} \qquad (B)$$

- The barred observer follows path

$$x(\xi) = \xi X_0, \quad (\equiv \xi \, \Phi(X_1) \text{ under } x \Leftrightarrow X)$$

$$\Rightarrow \quad \frac{dx}{d\xi} = X_0,$$

and so

$$\frac{1}{c} \underset{\sim}{V} = \frac{d\underset{\sim}{x}}{dx^0} = \frac{d\underset{\sim}{x}/d\xi}{dx^0/d\xi} = \tanh\theta \, \underset{\sim}{u}$$

$$\boxed{\underset{\sim}{V} = c\tanh\theta \, \underset{\sim}{u},}$$

$$\boxed{|\underset{\sim}{V}| = |c\tanh\theta|}$$

and thus

$$\cosh\theta = \frac{1}{\sqrt{1 - \frac{|\underset{\sim}{V}|^2}{c^2}}}$$

$$\sinh\theta = \frac{|V|/c}{\sqrt{1 - \frac{|\underset{\sim}{V}|^2}{c^2}}}$$

(note: $\sinh\theta > 0$ because $\underset{\sim}{V}$ in direction $\underset{\sim}{u} \Rightarrow \theta > 0$)

- Now using the formula for projections, we can project an arbitrary vector Y onto the span of the 2-d spacelike plane orthog to Span$\{X_0, X_1\}$: $S = \text{Span}\{X_0, X_1\}^{\perp}$, $\Rightarrow$

$$\text{Proj}_S Y = Y - \frac{\langle Y, X_0\rangle}{\langle X_0, X_0\rangle} X_0 - \frac{\langle Y, X_1\rangle}{\langle X_1, X_1\rangle} X_1 \qquad (c)$$

Skip

or

$$\mathrm{Proj}_S Y = Y + \langle Y, X_0\rangle X_0 - \langle Y, X_1\rangle X_1 .$$

We now define the bar coordinate system by defining $\bar{x}(Y) \equiv \bar{x}(x(Y)) \equiv \bar{x}(Y^0, \ldots, Y^3)$ for each vector Y. We define $\bar{x}$ by asking that it satisfy the following 3 conditions:

① $\bar{x}^i(X_0) = \delta^i_0$ ("X_0 is the vector $\frac{\partial}{\partial \bar{x}^0}$")

② $\bar{x}^i(X_1) = u^i$ ("X_1 lies in the same direction relative to $\bar{x}$-coord system that $\underset{\sim}{u}$ lies in rel to x-coord system $\approx$ no rotation") (Set $u^0 = 0$)

③ $\bar{x}^i(Y) = Y^i$ for all $Y \in S$ ("the coords of vectors orthogonal to the direction of motion are unchanged $\approx$ no rotation")

④ $\bar{x}(Y)$ is linear in Y.

Theorem: $\exists$ a unique Lorentz transformation satisfying ①-④, and under the identification $x^i \longleftrightarrow y^i$, this is given by

$$\bar{x}^i(x) = L(\theta, \underset{\sim}{u})^i_j \, x^j$$

where

row

$$L(\theta,\underset{\sim}{u})^i_j = \begin{array}{c} \\ \left[\begin{array}{cc} \cosh\theta & -\sinh\theta \, \underset{\sim}{u}^t \\ -\sinh\theta \, \underset{\sim}{u} & \delta^i_j + (\cosh\theta - 1)\, u_j u^i \end{array}\right]^{i} \\ \end{array}_{j}$$

colm

Here, $\underset{\sim}{u} = (u^1, u^2, u^3)^t$, $u^i = u_j$ $\; i=1,\dots,3$, and

$\delta^i_j + (\cosh\theta - 1)\, u_j u^i$ is the 3×3 matrix, $i j = 1,2,3$.

Note: $\underset{\sim}{u} = (1,0,0) = \underset{\sim}{e}_1 \Rightarrow$ reduction to previous formula.

Note: $L(\theta, \underset{\sim}{u})^{-1} = L(-\theta, \underset{\sim}{u}) = \begin{bmatrix} \cosh\theta & \sinh\theta \, \underline{u}^t \\ \sinh\theta \, \underline{u}^t & \delta^i_j + (\cosh\theta - 1) u_j u^i \end{bmatrix}$

$$L(\theta_1 + \theta_2, \underset{\sim}{u}) = L(\theta_1, \underset{\sim}{u}) L(\theta_2, \underset{\sim}{u})$$

(FIP) checked 1-92

Proof: Using the orthog proj lemma we can $\boxed{\text{SKIP}}$ write

$$Y = -\langle Y, X_0\rangle X_0 + \langle Y, X_1\rangle X_1 + \mathrm{Proj}_S Y$$

$$\begin{aligned} Y = {} & -\langle Y, X_0\rangle X_0 + \langle Y, X_1\rangle X_1 \\ & + Y + \langle Y, X_0\rangle X_0 - \langle Y, X_1\rangle X_1 . \end{aligned}$$

Now using the assumed linearity of $\bar{x}$, together with ①-③ we obtain

$$\begin{aligned} \bar{x}^i(Y) = {} & -\langle Y, X_0\rangle \delta^i_0 + \langle Y, X_1\rangle u^i \\ & + Y^i + \langle Y, X_0\rangle X^i_0 - \langle Y, X_1\rangle X^i_1 \end{aligned}$$

$\boxed{i=0}$
$$\begin{aligned} \bar{x}^0(Y) = {} & -\langle Y, X_0\rangle + Y^0 + \langle Y, X_0\rangle \cosh\theta \\ & - \langle Y, X_1\rangle \sinh\theta \end{aligned}$$

$$\cancel{= -(-\cosh\theta, \sinh\theta\, \underset{\sim}{u})_0 Y^0 + Y^0}$$

$$\cancel{+ \{(-\cosh\theta, \sinh\theta\, \underset{\sim}{u})_0 \cosh\theta - (-\sinh\theta, \cosh\theta\, \underset{\sim}{u})_0 \sinh}$$

$\boxed{\text{Skip}}$

Substituting (A) & (B) gives

$$\bar{x}^0(Y) = -(-\cosh\theta, \sinh\theta\, \underset{\sim}{u})_\sigma Y^\sigma + Y^0$$

$$+\{(-\cosh\theta, \sinh\theta\, \underset{\sim}{u})_\sigma \cosh\theta - (-\sinh\theta, \cosh\theta\, \underset{\sim}{u})_\sigma \sinh\theta\} Y^\sigma$$

$$= \cancel{Y^0} + (\cosh\theta, -\sinh\theta\, \underset{\sim}{u})_\sigma Y^\sigma$$

$$+ \cancel{(-\cosh^2\theta + \sinh^2\theta, \underset{\sim}{0})_\sigma Y^\sigma}$$

$$= (\cosh\theta, -\sinh\theta\, \underset{\sim}{u})_\sigma Y^\sigma$$

$\boxed{i \neq 0}$

$$\bar{x}^i(Y) = \langle Y, X_1 \rangle u^i + Y^i$$

$$+ \langle Y, X_0 \rangle \sinh\theta\, u^i - \langle Y, X_1 \rangle \cosh\theta\, u^i$$

$$= (-\sinh\theta, \cosh\theta\, \underset{\sim}{u})_\sigma Y^\sigma u^i + Y^i$$

$$+ (-\cosh\theta \sinh\theta, \sinh^2\theta\, \underset{\sim}{u})_\sigma Y^\sigma u^i$$

$$- (-\cosh\theta \sinh\theta, \cosh^2\theta\, \underset{\sim}{u})_\sigma Y^\sigma u^i$$

$\boxed{\text{skip}}$

$$= \left(-\sinh\theta, \{\cosh\theta + \sinh^2\theta - \cosh^2\theta\}\underset{\sim}{u}\right)_\sigma y^\sigma u^i + y^i$$

$$= \left(-\sinh\theta, (\cosh\theta - 1)\underset{\sim}{u}\right)_\sigma y^\sigma u^i + y^i \qquad u^0 = 0$$

$$= \delta^i_\sigma y^\sigma - \sinh\theta\, u^i y^0 + (\cosh\theta - 1) u_\sigma u^i y^\sigma$$

$$\bar{x}^i(y) = \begin{bmatrix} & \text{———} \\ | & \\ -\sinh\theta\, \underset{\sim}{u}^i & \delta^i_\sigma + (\cosh\theta - 1) u_\sigma u^i \\ | & \end{bmatrix} \begin{bmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{bmatrix}$$

which completes the proof.

Substituting $\cosh\theta = \dfrac{1}{\sqrt{1 - |\frac{v}{c}|^2}}$, $\sinh\theta = \dfrac{|v/c|}{\sqrt{1 - |\frac{v}{c}|^2}}$

$\underset{\sim}{u} = |v|^{-1}\underset{\sim}{v}$ gives

$$L(\underset{\sim}{v}) = L(\theta, \underset{\sim}{u}).$$