Structure of the Lorentz Group 3+1 Dimensions

- **Defn**: A Lorentz/Minkowski coord. system $\underset{\sim}{x} = (x^0, x^1, x^2, x^3)$ is one that globally diagonalizes g at every point

$$g_{ij}(\underset{\sim}{x}) = \eta_{ij} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

- **Defn**: If such a coordinate system exists we say spacetime is flat $\Leftrightarrow$ no curvature $\Leftrightarrow$ special relativity

Thm: we can always achieve $g_{ij} = \eta_{ij}$ at P, with $g_{ij,k} = 0$ at P, but cannot get $g_{ij,kl} = 0$ at P when curvature is present

• Given one Lorentz coord system, all others are obtained by taking a Lorentz transformation of these:

$$\underset{\sim}{y} = \Lambda \underset{\wedge}{x}$$

where the constant 4x4 matrix Λ is characterized by the condition

$$\frac{\partial x^i}{\partial y^\alpha} g_{ij} \frac{\partial x^j}{\partial y^\beta} = \eta_{\alpha\beta}$$

or since $g_{ij} = \eta_{ij}$,

$$\boxed{\Lambda^T \eta \Lambda = \eta} \qquad (1)$$

- **Thm (1):** (A) The set of all 4x4 matrices Λ that satisfy (1) form a group called the (homogeneous) Lorentz group G

(i) $\Lambda \in G \Rightarrow \Lambda^{-1} \in G$

(ii) $\Lambda_1, \Lambda_2 \in G \Rightarrow \Lambda_1 \cdot \Lambda_2 \in G$

matrix mult

(iii) $I \in G$

(iv) $(AB)C = A(BC)$

(B) The proper homogeneous Lorentz transformations ($\Lambda^0{}_0 \geq 1$, $\det \Lambda > 0$) form a subgroup containing I.

4 Equivalence of the Lorentz Transformations with the purely imaginary rotations.

- Assume a flat Lorentzian spacetime:

$\underset{\sim}{x} = (x^0, x^1, x^2, x^3)$ & in $\underset{\sim}{x}$-coords

$$g_{ij} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \equiv \eta_{ij}$$

In terms of any Lorentzian coord system $\underset{\sim}{x}$, define Complex Lorentz coords

$$\underset{\sim}{\bar{x}} = (i x^0, x^1, x^2, x^3) \equiv (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$$

Thus:
$$\frac{\partial \bar{x}^{\bar{\alpha}}}{\partial x^\alpha} = \begin{bmatrix} i & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}, \quad \frac{\partial x^\alpha}{\partial \bar{x}^{\bar{\alpha}}} = \begin{bmatrix} -i & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$= B \qquad\qquad = B^{-1}$$

& $B^T = B$, $\quad B^T B = B^2 = \eta$, $\quad B^{-T} \eta B^{-1} = id$

Thm (2) (FIP): The set of Lorentz transformations consists of the set of all matrices

$$\Lambda = B^{-1} \bar{\Lambda} B$$

such that

(1) $\bar{\Lambda}$ complex & satisfies $\bar{\Lambda}^T \bar{\Lambda} = I$

(2) $\Lambda = B^{-1} \bar{\Lambda} B$ is real

Pf. Multiply (1) on left by B^{-T} & on right by B^{-1} & use $B^T B = \eta$ to see that (2) is equivalent to

$$B^{-T} \Lambda^T B^T B \Lambda B^{-1} = I$$

$$(B \Lambda B^{-1})^T (B \Lambda B^{-1}) = I$$

$$\bar{\Lambda}^T \bar{\Lambda} = I \quad \checkmark$$

- <u>Thm</u> (3): In 1+1 dimensions, a positive oriented Lorentz transformation is the boost

$$\Lambda(\theta) = \begin{bmatrix} \cosh\theta & \pm\sinh\theta \\ \pm\sinh\theta & \cosh\theta \end{bmatrix}$$

iff $\bar{\Lambda}(\theta)$ is a rotation thru complex angle $\beta = i\theta$

$$\bar{\Lambda}(\beta) = \begin{bmatrix} \cos(i\theta) & \pm\sin(i\theta) \\ \mp\sin(i\theta) & \cos(i\theta) \end{bmatrix}$$

Proof: $\sin\alpha = \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)$, $\cos\alpha = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right)$

$\Rightarrow \cos^2\alpha + \sin^2\alpha = 1$ even when α is imag.

Now let $\beta = i\theta$, θ real $\Rightarrow$

$$\sin\beta = \frac{1}{2i}\left(e^{-\theta} - e^{\theta}\right) = i\sinh\theta$$

$$\cos\beta = \frac{1}{2}\left(e^{-\theta} + e^{\theta}\right) = \cosh\theta$$

Thus

$$\Omega = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{bmatrix} = \begin{bmatrix} \cos(i\theta) & \sin(i\theta) \\ -\sin(i\theta) & \cos(i\theta) \end{bmatrix}$$

$\checkmark$

- **Thm (4)**: The proper homogeneous Lorentz group consists of a 6-parameter Lie Group of matrices

Proof:

- **Lemma (1)**: Any rotation in space is a L-transformation

P.f. $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$, $\Lambda^T \eta \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & R^T R \end{bmatrix} = \eta$

since rotation $\Leftrightarrow R^T R = id$ ✓

Since L-transformations are complex rotations, we first characterize the 3x3 real rotations.

- The 3×3 rotations R with $\det R > 0$ form a manifold in $\mathbb{R}^9$ with a group structure on it $\Leftrightarrow$ Lie Group. The proper rotations are continuously connected to I. Call this G_R

- $R_0 \in G_R \Rightarrow T_{R_0} G_R = \{R_0 E : E \in T_I G_R\}$

I.e., $R(t)$ a smooth curve in G_R with $R(0) = R_0$, then $T_{R_0} G_R$ is the collection of all

$$\left.\frac{d}{dt} R(t)\right|_{t=0} \in \mathbb{R}^9 \equiv 3\times3 \text{ matrices}$$

Let $\bar{R}(t) = R_0^{-1} R(t) \Rightarrow \bar{R}(0) = I$

Fact: $A \in T_I G_R \Leftrightarrow A = \frac{d}{dt}\bar{R}(t)$ some $\bar{R}(t)$

$$\Rightarrow \frac{d}{dt}R(t) = R_0 \frac{d}{dt}\left(R_0^{-1}R(t)\right) = R_0 \frac{d}{dt}\bar{R}(t) = R_0 E \in T_{R_0} G_R$$

- <u>Defn</u>: If G is a Lie Group, then we call $T_I G = \mathfrak{g}$ the Lie Algebra of G.

<u>Then</u>: $T_{R_0} G = R_0 \mathfrak{g}$

- <u>Lemma(2)</u>: $A \in \mathfrak{g}_R$ iff $A^T = -A$

pf $A \in \mathfrak{g}_R \Leftrightarrow A = \frac{d}{dt} R(t)\Big|_{t=0}$ $R(t) \in G_R$, $R(0) = I$

But

$$R(t)^T R(t) = I \quad R(0) = I$$

So

$$0 = \frac{d}{dt}\left[R(t)^T R(t)\right] = R'(t)^T + R'(t)$$

$$\Leftrightarrow A^T = -A \quad \checkmark$$

- <u>Lemma(3)</u>: $A \in \mathfrak{g}_R \Rightarrow e^{At} \in G_R$

Pf $A^T = -A \Rightarrow \left(e^{At}\right)^T e^{At} = e^{-At} e^{At} = I \Rightarrow e^{At} \in G_R$

$\checkmark$

<u>Theorem</u> (Lie Groups) The component of a Lie Group G connected to I is the set of all e^{At} st $A \in \mathfrak{g}$.

<u>Conclude</u>: The proper rotations on $\mathbb{R}^3$ consist of the set of all e^A st $A \in \mathfrak{g}_R$ $\Leftrightarrow$ A is antisymmetric $A^T = -A$.

A is called an <u>infinitessimal</u> <u>rotation</u>.

- **Lemma (4):** The set of all proper rotations of (x^1, x^2, x^3) within spacetime (x^0, x^1, x^2, x^3) are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} = e^{\alpha^1 L_1 + \alpha^2 L_2 + \alpha^3 L_3} = e^{\alpha^i L_i}$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moreover:

(1) $\vec{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$ is the axis of rotation

(2) $\|\vec{\alpha}\|$ = magnitude of rotation in radians

(3) Right Hand Rule (RHR) gives direction of rotation.

Proof: for pf ignore time comp. & let $L_i \in$ 3x3 matrices. Then

$$A = \alpha^i L_i \;,\; R = e^A$$

• Use formula $e^A = \lim_{n \to \infty} \left(I + \frac{1}{n} A\right)^n$

$$e^A \approx \underbrace{\left(I + \tfrac{1}{n}A\right) \cdots \left(I + \tfrac{1}{n}A\right)}_{n\text{-times}}$$

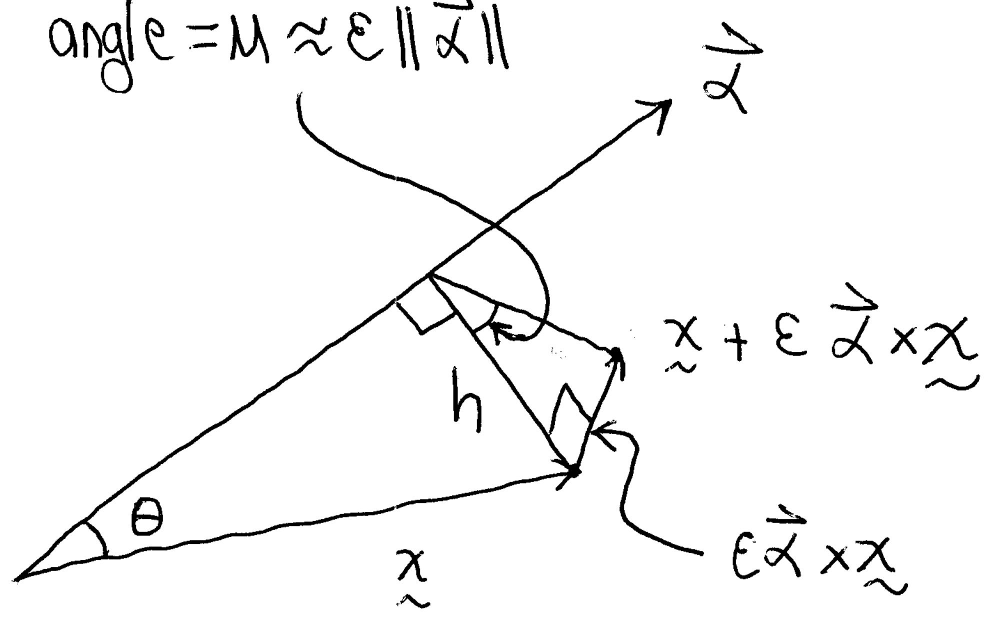
(a) $\underset{\sim}{x} = (x^1, x^2, x^3)$, $\varepsilon = \frac{1}{n}$

• Consider: $(I + \varepsilon A)\underset{\sim}{x} = \underset{\sim}{x} + \varepsilon \vec{\alpha} \times \underset{\sim}{x}$ (b)

[check: $A\underset{\sim}{x} = \vec{\alpha} \times \underset{\sim}{x}$ FIP]

Claim: $\underset{\sim}{x} \longmapsto \underset{\sim}{x} + \varepsilon \vec{\alpha} \times \underset{\sim}{x}$ is "a rotation about $\vec{\alpha}$ thru angle $\varepsilon \|\vec{\alpha}\|$ by RHR $+ O(\varepsilon^2)$"

I.e.,



- $\varepsilon \vec{\alpha} \times \underset{\sim}{x}$ points normal to $\underset{\sim}{x}$ & $\vec{\alpha}$ with magnitude $\|\varepsilon \vec{\alpha} \times \underset{\sim}{x}\| = \varepsilon \|\vec{\alpha}\| \|\underset{\sim}{x}\| \sin\theta$

- $\tan \mu = \dfrac{\|\varepsilon \vec{\alpha} \times \underset{\sim}{x}\|}{h}$, $h = \|\underset{\sim}{x}\| \sin\theta = \dfrac{\|\varepsilon \vec{\alpha} \times \underset{\sim}{x}\|}{\varepsilon \|\vec{\alpha}\|}$

$$= \frac{\|\varepsilon \vec{\alpha} \times \underset{\sim}{x}\|}{\left(\dfrac{\varepsilon \vec{\alpha} \times \underset{\sim}{x}}{\varepsilon \|\vec{\alpha}\|}\right)} = \varepsilon \|\vec{\alpha}\|$$

$\Rightarrow \mu = \varepsilon \|\vec{\alpha}\| + O(\varepsilon^2)$

- Conclude: neglecting $O(\varepsilon^2)$ errors in (a), e^A is n infinitessimal rotations of mag $\frac{1}{n}\|\vec{\alpha}\|$ about axis $\vec{\alpha}$ by RHR ✓

◈ Consider now the general Lorentz Trans's.

- Let G denote the group of Lorentz trans. $\bar{\Lambda}$ such that $\Lambda = B^{-1}\bar{\Lambda}B$ is real &

$$\bar{\Lambda}^T\bar{\Lambda} = I_{4\times 4}.$$

- $\bar{\Lambda} = e^A$ some $A \in \mathfrak{g} = T_I G \equiv$ Lie Alg of G
- To get $\mathfrak{g}$: $\bar{\Lambda}(t) \in G$ a curve in G s.t. $\bar{\Lambda}(0) = I$, $\frac{d}{dt}\bar{\Lambda}(t)\Big|_{t=0} = A$. Then

$$0 = \frac{d}{dt}\bar{\Lambda}(t)^T\bar{\Lambda}(t)\Big|_{t=0} = A^T + A$$

$\Rightarrow$ $\bar{\Lambda} = e^A$ for some complex anti-symmetric A

$$\mathfrak{g} = \{A_{4\times 4} : A^T = -A \ \& \ B^{-1}AB \text{ real}\}$$

I.e., $\Lambda = B^{-1}\bar{\Lambda}B = e^{B^{-1}AB}$ real $\Leftrightarrow$ $B^{-1}AB$ real ✓

• <u>Conclude</u>: $A^T = -A$ & $B^{-1}AB$ real $\Rightarrow$

$$B^{-1}AB = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & 1 & & 0 \\ 0 & & 1 & \\ 0 & 0 & & 1 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} i & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A_{00} & -i A_{0j} \\ i A_{i0} & A_{ij} \end{bmatrix} \quad \text{real}$$

$\Rightarrow A_{00}, A_{ij}$ real $i,j = 1,2,3$

and A_{i0}, A_{0i} pure imaginary

Since $A^T_{ij} = -A_{ij}$ real it must be 3x3 infinitessimal rotation

$A \in \mathfrak{g}$

$$\Leftrightarrow \quad A = \begin{bmatrix} 0 & +i\Theta^T \\ -i\Theta & A_{ij} \end{bmatrix} \quad \text{some real } \Theta = (\Theta_1, \Theta_2, \Theta_3),$$

- Conclude: $\mathcal{L}$ is spanned by the 3 infinitessimal rotations L_1, L_2, L_3 and the 3 infinitessimal boosts M_1, M_2, M_3 where

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Defn</u>: a (pure) boost is a Lorentz transformation

$$\bar{\Lambda} = e^{B_i M_i}, \quad B_i = i\theta_i, \quad B = i\theta$$

(sum repeated i from 1-3)

- We now obtain a formula for pure boosts & find reln betw B & $\vec{V}$.

☒ Reln between $\beta = i\theta$ & $\vec{v}$

- Case 1+1 : $M_3 \equiv M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

<u>Thm</u>: $e^{\beta M} = \begin{bmatrix} \cos(\beta) & +\sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{bmatrix}$

$i^{2k+1} = i \cdot i^{2k}$
$= (-1)^k i$

Pf. $e^{i\theta M} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\theta)^n M^n$

$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} (i\theta)^{2k} M^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (i\theta)^{2k+1} M^{2k+1}$

$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $M^2 = \begin{bmatrix} 0 & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

$M^3 = -M$, $M^4 = I \Rightarrow$

$\boxed{M^{2k} = (-1)^k I}$, $\boxed{M^{2k+1} = (-1)^k M}$

Thus:

$$e^{i\theta M} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (i\theta)^{2k} (-1)^k I + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (i\theta)^{2k+1} M$$

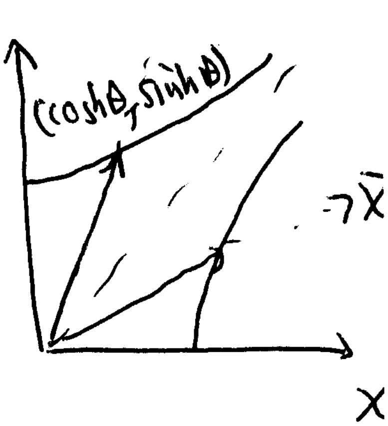
$$= \begin{bmatrix} \cos(i\theta) & 0 \\ 0 & \cos(i\theta) \end{bmatrix} + \begin{bmatrix} 0 & +\sin(i\theta) \\ -\sin(i\theta) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos i\theta & +\sin i\theta \\ -\sin i\theta & \cos i\theta \end{bmatrix} = B \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} B^{-1}$$

$\Rightarrow \theta$ is our old θ, $B = i\theta r$

• Thus Eg $B e^{i\theta M} B^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cosh\theta\\ \sinh\theta\end{bmatrix}$

gives the timelike vector fixed in the moving frame as seen in the fixed frame:



$\cosh^2\theta - \sinh^2\theta = 1$

$$v = \text{"}\frac{\Delta x^1}{\Delta x^0}\text{"} = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta$$

$$1 - \tanh^2\theta = \frac{1}{\cosh^2\theta} \Rightarrow \cosh^2\theta = \frac{1}{1-v^2}$$

$$\sinh^2\theta = \frac{v^2}{1-v^2}$$

So: $\Lambda = B e^{i\theta M} B^{-1}$ takes $\begin{pmatrix}0\\1\end{pmatrix}$ to $\begin{pmatrix}\cosh\theta\\ \sinh\theta\end{pmatrix}$

$$\boxed{\begin{pmatrix}\cosh\theta\\ \sinh\theta\end{pmatrix} = \frac{1}{\sqrt{1-v^2}}\begin{pmatrix}1\\ v\end{pmatrix}}$$

$\boxed{v = \tanh\theta}$ gives v in terms of θ ✓

$|B| \equiv i|\theta|$

General Reln betw $\vec{V}$, θ, $\beta = i\theta$, $\theta = (\theta_1, \theta_2, \theta_3)$

- $$A = \beta_k M_k = \begin{bmatrix} 0 & +\beta_1 & +\beta_2 & +\beta_3 \\ -\beta_1 & & & \\ -\beta_2 & & \underset{\sim}{0} & \\ -\beta_3 & & & \end{bmatrix} = \begin{bmatrix} 0 & +\beta^T \\ -\beta & \underset{\sim}{0} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & +\beta_1 & +\beta_2 & +\beta_3 \\ -\beta_1 & & & \\ -\beta_2 & & \underset{\sim}{0} & \\ -\beta_3 & & & \end{bmatrix} \begin{bmatrix} 0 & +\beta_1 & +\beta_2 & +\beta_3 \\ -\beta_1 & & & \\ -\beta_2 & & \underset{\sim}{0} & \\ -\beta_3 & & & \end{bmatrix} = \begin{bmatrix} -|B|^2 & 0 & 0 & 0 \\ 0 & -\beta_1^2 & -\beta_1\beta_2 & -\beta_1\beta_3 \\ 0 & -\beta_1\beta_2 & -\beta_2^2 & -\beta_2\beta_3 \\ 0 & -\beta_1\beta_3 & -\beta_2\beta_3 & -\beta_3^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -|B|^2 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -\beta_i\beta_j & \\ 0 & & & \end{bmatrix} \begin{bmatrix} 0 & +\beta_1 & +\beta_2 & +\beta_3 \\ -\beta_1 & & & \\ -\beta_2 & & \underset{\sim}{0} & \\ -\beta_3 & & & \end{bmatrix} = \begin{bmatrix} 0 & -|B|^2\beta^T \\ +|B|^2\beta & \underset{\sim}{0} \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & -|B|^2\beta^T \\ +|B|^2\beta & \underset{\sim}{0} \end{bmatrix} \begin{bmatrix} 0 & +\beta_1 & +\beta_2 & +\beta_3 \\ -\beta_1 & & & \\ -\beta_2 & & \underset{\sim}{0} & \\ -\beta_3 & & & \end{bmatrix} = \begin{bmatrix} |B|^4 & 0 & 0 & 0 \\ 0 & |B|^2\beta_1^2, & |B|^2\beta_1\beta_2, & |B|^2\beta_1\beta_3 \\ 0 & |B|^2\beta_1\beta_2, & |B|^2\beta_2^2, & |B|^2\beta_2\beta_3 \\ 0 & |B|^2\beta_1\beta_3, & |B|^2\beta_2\beta_3, & |B|^2\beta_3^2 \end{bmatrix}$$

$$A^{2k} = (-1)^k |B|^{2k} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \dfrac{\beta_i\beta_j}{|B|^2} & \\ 0 & & & \end{bmatrix}, \quad A^{2k+1} = (-1)^k |B|^{2k+1} \begin{bmatrix} 0 & +\dfrac{\beta^T}{|B|} \\ -\dfrac{\beta}{|B|} & \underset{\sim}{0} \end{bmatrix}$$

Or in terms of θ use $\beta = i\theta$, $|\beta| = i|\theta|$

$$A^{2k} = (-1)^k (i|\theta|)^{2k} \begin{bmatrix} 1 & 0\;0\;0 \\ \begin{matrix}0\\0\\0\end{matrix} & \dfrac{\theta_i \theta_j}{|\theta|^2} \end{bmatrix}, \qquad A^{2k+1} = (-1)^k (i|\theta|)^{2k+1} \begin{bmatrix} 0 & +\dfrac{\theta}{|\theta|} \\ -\dfrac{\theta}{|\theta|} & \underset{\sim}{0} \end{bmatrix}$$

$(k \geq 1)$

Conclude:

$$\bar{\Lambda} = e^A = I + \sum_{k=0}^{\infty} \frac{(-1)^k (i|\theta|)^{2k}}{(2k)!} \begin{bmatrix} 1 & 0\;0\;0 \\ \begin{matrix}0\\0\\0\end{matrix} & \dfrac{\theta_i \theta_j}{|\theta|^2} \end{bmatrix} + \sum_{k=0}^{\infty} \frac{(-1)^k (i|\theta|)^{2k+1}}{(2k+1)!} \begin{bmatrix} 0 & +\dfrac{\theta}{|\theta|} \\ -\dfrac{\theta}{|\theta|} & \underset{\sim}{0} \end{bmatrix}$$

to get I in cos

$$\begin{bmatrix} 0 & 0\;0\;0 \\ \begin{matrix}0\\0\\0\end{matrix} & I - \dfrac{\theta_i \theta_j}{|\theta|^2} \end{bmatrix} + \cos(i|\theta|) \begin{bmatrix} 1 & 0\;0\;0 \\ \begin{matrix}0\\0\\0\end{matrix} & \dfrac{\theta_i \theta_j}{|\theta|^2} \end{bmatrix} + \sin(i|\theta|) \begin{bmatrix} 0 & +\dfrac{\theta}{|\theta|} \\ -\dfrac{\theta}{|\theta|} & \underset{\sim}{0} \end{bmatrix}$$

$$\Rightarrow \quad \bar{\Lambda} = \begin{bmatrix} \cos(i|\theta|) & +\dfrac{\theta}{|\theta|} \sin(i|\theta|) \\ -\dfrac{\theta}{|\theta|} \sin(i|\theta|) & I + \dfrac{\theta_i \theta_j}{|\theta|^2} \left(\cos(i|\theta|) - 1 \right) \end{bmatrix}$$

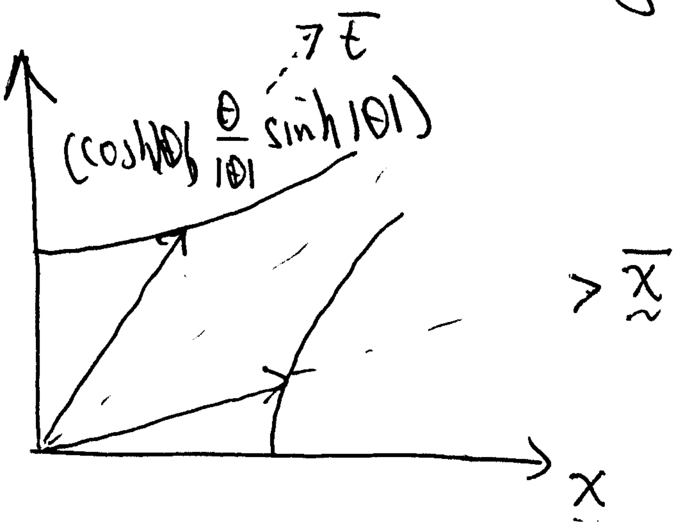
Thus we obtain a general formula for a pure boost —

$$\Lambda = B^{-1}\bar{\Lambda}B = \begin{bmatrix} \cosh|\theta| & \frac{\theta}{|\theta|}\sinh|\theta| \\ \frac{\theta}{|\theta|}\sinh|\theta| & I + \frac{\theta_i\theta_j}{|\theta|^2}(\cosh|\theta|-1) \end{bmatrix}$$

- Note: this generalizes 1+1 case where $\frac{\theta}{|\theta|} = 1$
- Note: $\Lambda \begin{bmatrix} 1 \\ \underset{\sim}{0} \end{bmatrix} = \begin{bmatrix} \cosh|\theta| \\ \frac{\theta}{|\theta|}\sinh|\theta| \end{bmatrix}$ gives the

unit timelike vector of the moving frame as expressed in the original frame:

I.e.



<u>Conclude</u>: the velocity $\vec{v}$ of the moving frame as expressed in the original frame is

$$\boxed{\vec{v} = \frac{\sinh|\theta|}{\cosh|\theta|}\frac{\theta}{|\theta|} = \frac{\theta}{|\theta|}\tanh|\theta|}$$

giving the relation betw $\vec{v}$ & θ that generalizes the 1+1 formula.

Also: $|\vec{v}| = \frac{\sinh\theta}{\cosh\theta}$ gives $\boxed{\frac{\vec{v}}{|\vec{v}|} = \frac{\theta}{|\theta|}}$

- Since $\cosh|\theta| = \frac{1}{1-|\vec{v}|^2}$, $\sinh|\theta| = \frac{|\vec{v}|}{1-|\vec{v}|^2}$

we obtain Λ in terms of $\vec{v}$: (Formula for PURE BOOST in direction $\vec{v}$!)

$$\Lambda = \begin{bmatrix} \frac{1}{\sqrt{1-|v|^2}} & \frac{\vec{v}^T}{\sqrt{1-|\vec{v}|^2}} \\ \frac{\vec{v}}{\sqrt{1-|\vec{v}|^2}} & I + \frac{v_i v_j}{|\vec{v}|^2}\left(\frac{1}{\sqrt{1-|\vec{v}|^2}} - 1\right) \end{bmatrix}$$

- **Note:** G_R is a compact group: i.e.,

$\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $0 \le |\alpha| \le 2\pi$ (since $|\alpha|$ is rotation in radians)

In contrast, G is non-compact since

$$\beta = i\theta, \quad \theta = (\theta_1, \theta_2, \theta_3)$$

$$-\infty < \theta_i < +\infty$$

"Thm: compact Lie Gps have unitary representations"
[see Veltmann & ref's therein...]

Structure of Lorentz Group:

• Q: if $A, B \in \mathfrak{g} \equiv$ Lie Alg of G [Proper Lorentz Gp],

then for what C is

$$e^A e^B = e^C$$

Eg: $e^{\alpha^i L_i + \beta^j M_j} \cdot e^{\bar{\alpha}^i L_i + \bar{\beta}^j M_j} \stackrel{?}{=} e^{\bar{\bar{\alpha}}^i L_i + \bar{\bar{\beta}}^j M_j}$

How are $\bar{\bar{\alpha}}$ $\bar{\bar{\beta}}$ related to $\alpha, \beta, \bar{\alpha}, \bar{\beta}$?

Hard Question at the Heart of gp theory

• Eg $e^A e^B \neq e^{A+B}$ unless $[A,B]=0$.

Ex: $$\left(\sum_{k=0}^{\infty} \frac{A^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{B^n}{n!}\right) \neq \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$$

$$\left(1+A+\tfrac{1}{2}A^2\right)\left(1+B+\tfrac{1}{2}B^2\right) = 1+A+B+AB+\tfrac{1}{2}A^2+\tfrac{1}{2}B^2$$

$$\left(1+(A+B)+\tfrac{1}{2}(A+B)^2\right) = I+A+B+\tfrac{1}{2}AB+\tfrac{1}{2}BA+\tfrac{1}{2}A^2+\tfrac{1}{2}B^2$$

$\Rightarrow$ don't agree 2nd order unless $[A,B]=0$

In fact: if $\boxed{C = A + B + \frac{1}{2}[A,B]}$ then

$$(1 + A + \tfrac{1}{2}A^2)(1 + B + \tfrac{1}{2}B^2) = 1 + C + \tfrac{1}{2}C^2$$

up to 2nd order.

<u>Thm</u>: Cambell-Baker-Hausdorff Formula

$\exists$ an algorithm for computing C up to any order, and each term can be computed using only A, B & $[A,B]$ and brackets of all orders

$$C = A + B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}[A,[A,B]]$$

$$+ \tfrac{1}{2}[[A,B],B] + \tfrac{1}{48}[A[[A,B],B]] + \cdots$$

The point: you only need to know formulas for brackets of all orders in order to compute C from A, B. $\Rightarrow$ structure of G.

• For example, $\mathcal{A}$ is closed under taking brackets

I.e., $A \in \mathcal{A}$ if $A^T = -A$

$[A,B] = AB - BA \qquad [A,B]^T = B^T A^T - A^T B^T = BA - AB$

$= -[A,B]$ ✓

$\therefore$ if $H_1, \ldots, H_6 = L_1, \ldots, L_3, M_1, \ldots, M_3$

& we know

$$[H_i, H_j] = C_{ij}^{k} H_k$$

Then CBH gives way to compute gp structure

$C_{ij}^{k} \equiv$ structure constants.

In fact: $[L_i, L_j] \neq 0$, $[L_i, M_j] \neq 0$

$[M_i, M_j] \neq 0$

- Lemma 1: $[L_1, L_2] = -L_3$

& cyclically permute gives structure constants for Lorentz group.

- Lemma 2: The following gives the structure constants for all of $\mathfrak{L}$:

$$[K_{ab}, K_{cd}] = C^{ij}_{abcd} K_{ij}$$

$$C^{ij}_{abcd} = \delta_{bc}\delta_{ai}\delta_{dj} - \delta_{bd}\delta_{ai}\delta_{cj}$$

$$- \delta_{ac}\delta_{bi}\delta_{dj} + \delta_{ad}\delta_{bi}\delta_{cj}$$

$\alpha_1 \Leftrightarrow \alpha_{23}$ $\quad B_1 \Leftrightarrow \alpha_{10}$ $\quad L_1 = K_{23}$ $\quad M_1 = +K_{10}$

$\alpha_2 \Leftrightarrow \alpha_{31}$ $\quad B_2 \Leftrightarrow \alpha_{20}$ $\quad L_2 = -K_{13}$ $\quad M_2 = +K_{20}$

$\alpha_3 \Leftrightarrow \alpha_{12}$ $\quad B_3 \Leftrightarrow \alpha_{30}$ $\quad L_3 = K_{12}$ $\quad M_3 = K_{30}$

$K_{\mu\nu}$ = matrix with 1 in row μ, colm ν & -1 in (ν, μ).

(Proof: see Veltmann)

Pf Tedious Calculation (See Veltmann)

- Note: since $[M_i, M_j] \neq 0$, two non-collinear boosts will not be a pure boost $\Rightarrow$ creates a rotation call Thomas Rotation

1st Calculated by Wigner ~1920's

- Note: if you go through a sequence of boosts that take you back to your original inertial frame, there will be a net rotation by same principle