VECTOR FIELDS

- Recall: M^4 4-d spacetime manifold of events.
- $T_p M^4$ tangent space to M^4 at $p \in M^4$

$$X_p \in T_p M^4 \qquad X_p \leftrightarrow \frac{dc}{d\xi}\bigg|_p \leftrightarrow X^i \frac{\partial}{\partial x^i}\bigg|_p$$

"identify tangent vectors with 1st order operators in each coord. system"

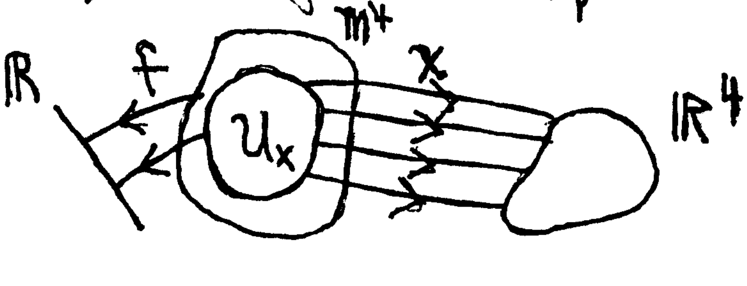
Vector Field: "smooth" assignment of a vector at each $p \in M^4$.

eg. $X = X^i(p) \dfrac{\partial}{\partial x^i}$ gives X in coord system $x : U \subset M^4 \to \mathbb{R}^4$

- Think of X_p as a linear operator on scalar functions:

$$X_p(f) = X^i(p) \frac{\partial}{\partial x^i}(f \circ x^{-1}) \equiv \text{"rate at which}$$

f changes in X_p direction at $p \in M^4$"



$f : M^4 \to \mathbb{R}$

- For fixed f, ask that the representation of X in each coordinate system give same derivative: $\Rightarrow$

$$X = X^i \frac{\partial}{\partial x^i} = \bar{X}^\alpha \frac{\partial}{\partial \bar{x}^\alpha}$$

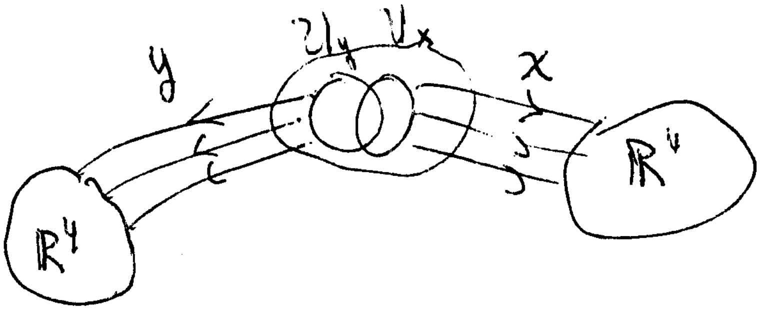
iff

$$X(f) = X^i \frac{\partial}{\partial x^i} f = \bar{X}^\alpha \frac{\partial}{\partial \bar{x}^\alpha} f$$

iff

$$X^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} \bar{X}^\alpha \qquad \frac{\partial}{\partial x^i} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial}{\partial \bar{x}^\alpha}$$

I.e.,



$x \circ \bar{x}^{-1} : \mathbb{R}^4 \to \mathbb{R}^4$ $\quad \dfrac{\partial x^i}{\partial \bar{x}^\alpha}$ Jacobian derivative

$(x \circ \bar{x}^{-1})^{-1} = \bar{x} \circ x^{-1} : \mathbb{R}^4 \to \mathbb{R}^4$ $\quad \left(\dfrac{\partial \bar{x}^\alpha}{\partial x^i}\right)^{-1} = \dfrac{\partial x^i}{\partial \bar{x}^\alpha}$ $\quad$ FIP

Conclude: The Einstein summation convention keeps track of the use of J, J^{-1}, J^t, J^{-t} in trans. laws.

• <u>Notation</u>: $X = x^i \frac{\partial}{\partial x^i}$ x-coords

$X = x^\alpha \frac{\partial}{\partial \bar{x}^\alpha}$ $\bar{x}$-coords

(let new set of indices keep track of "bar")

Indices on basis vector <u>down</u> $\frac{\partial}{\partial x^i} \leftrightarrow \underset{\sim}{e}_i$

Indices on vector coordinates <u>up</u> $x^i \frac{\partial}{\partial x^i}$

~~Convention: Mathematicians call up indices covariant~~

* Convention: <u>up</u> indices <u>contravariant</u>

<u>down</u> indices <u>covariant</u>

<u>Note</u>: no general agreement – * consistent with Dirac / Adler Bazin Schiffer / Misner Thorne Wh /

Mathematicians sometimes <u>reverse</u> this

Integral Curves of Vector Fields:

- A parameterized curve $c(\xi): \mathbb{R} \to M^n$ defines a vector at each point

$$\frac{dc}{d\xi} = X \leftrightarrow X^i \frac{\partial}{\partial x^i}$$

Here: $\frac{d}{d\xi} x^i \circ c(\xi) \equiv \frac{dx^i}{d\xi} = X^i$

- We can reverse this procedure: to every smooth vector field X there corresponds (locally) through each point a unique curve with a specified parameterization $\rightsquigarrow$ Flow on the manifold.

- ~~I.e., In each coord system x, X defines (locally) a system of ODE's on M^n by:~~

- ~~I.e., In each coord system x, X defines (locally) an autonomous system of ode's~~

• I.e., in each coordinate system x, a vector field $X = X^i \frac{\partial}{\partial x^i}$ defines (locally) an <u>autonomous</u> system of ODE's for the solution curves $x(\xi)$:

$$\frac{dx^i(\xi)}{d\xi} = X^i(x(\xi)). \qquad (1)$$

We can specify initial conditions

$$x(\xi_0) = x_0. \qquad (2)$$

<u>Note</u>: (1) is of form $\frac{dx}{d\xi} = f(x)$, and is autonomous because f has no ξ-dependence. Thus, in M^4, (1) defines a system of 4 <u>coupled</u>, <u>nonlinear</u>, ODE's.

<u>Theorem</u> (1) (Local Existence Thm) If the function $X^i(x)$ is Lipschtz continuous,

$$|X^i(x) - X^i(y)| \leq K|x-y|,$$

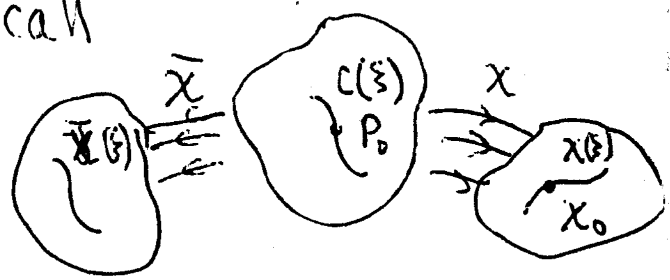
then (1), (2) has a <u>unique</u> maximal solution $x(\xi)$ defined in a nbhd of $\xi = \xi_0$. Soln's depend cont. on initial values

- **Defn**: The image of $x(\xi)$ is the integral curve of the vector field X in x-coordinates.

Since (1) is autonomous, $x(\xi)$ is an integral curve iff $x(\xi+\bar{\xi})$ is for any constant $\bar{\xi}$. This together with the uniqueness of solution curves implies $\Rightarrow$

Corollary: two different integral curves do not intersect. (FIP)

- Given $x(\xi)$, $x^{-1}\circ x(\xi) = C(\xi)$ defines a curve in the manifold, which we call the integral curve of X thru P_0, $x(P_0) = x_0$.



Lemma: The integral curve $\bar{x}(\xi)$ of X in the $\bar{x}$ coords determines the same integral curve $C(\xi)$ through P_0. (FIP)

• The integral curves of X defined in a nbhd $\mathcal{U} \subseteq M^4$ of p_0 define a <u>flow</u> Ψ on $\mathcal{U}$ as follows:

$$\Psi : I \times \mathcal{U} \longrightarrow M^4 , \quad I = (-\varepsilon, \varepsilon)$$

$$\Psi(\xi, p) \equiv \Psi_p(\xi)$$

where $\Psi_p(\xi)$ is the unique integral curve $c(\xi)$ satisfying

$$\frac{dc}{ds} = X , \quad c(0) = p .$$

~~If $X \neq 0$ (no rest points) then~~ For each $\xi \in I$,

$$\Psi_\xi(p) \equiv \Psi(\xi, p)$$

defines a map $\Psi_\xi : \mathcal{U} \to M^4$ which is 1-1 and regular, and $\Psi_{\xi_1+\xi_2}(p) = \Psi_{\xi_1} \circ \Psi_{\xi_2}(p)$

• This can be globalized to compact subsets of M^4 by using a finite cover of coord. charts (FIP).

- <u>Conclude</u>: each vector field X defines a natural length ξ on $\mathcal{M}^4$ via the parameter on the integral curves. Eg, if $X = \frac{\partial}{\partial x^i}$, then $\xi = x^i$ is the coordinate length (FIP)

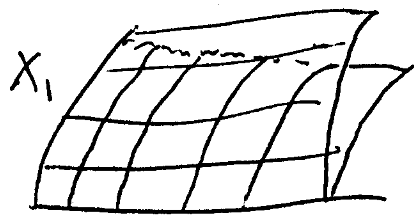
Q1: Given 4 indept vector fields $X_1, \dots, X_4$, one can construct coordinate systems based on these natural lengths. (This requires choosing the 3-surface where $\xi = 0$ for each vector field)

Q: When can a coord system based on these natural lengths be constructed so that the coord system has $X_1, \dots, X_4$ as coordinate basis at each point?

Ans: when $[X_i, X_j] = 0$, $i j = 1, \dots, 4$

Q2: Given $\{X_1, X_2\}$ indept, when do the integral curves of X_i form a 2-d surface?

Ans: when $[X_1, X_2] \in \text{Span}\{X_1, X_2\}$



Q3: Given $\{X_1, X_2, X_3\}$ indept,

when do the integral curves of X_i all lie in the same 3-d surfaces?

Ans: when $[X_i, X_j] \in \text{Span}\{X_1, X_2, X_3\}$ $\forall p$

<u>Defn</u>: $\{X_1, \ldots, X_k\} \equiv \Delta$ (indept) called a k-distribution.

The distribution is <u>integrable</u> if the integral curves form a k-d surface thru each point.

Precisely, M^k is a k-dim integral manifold for Δ passing thru p_0 if for every $p \in M^k$,

$T_p M^k = \Delta$.

<u>Theorem</u> (Frobenious) Δ is integrable iff

$[X_i, X_j] \in \Delta$ $\forall$ X_i, X_j in Δ.

Lie Bracket / Derivative

- Let X, Y be vector fields. We view these as 1st order linear operators acting on scalar functions:

$$X: \text{scalar} \longrightarrow \text{scalar}$$

$$f \longmapsto X(f)$$

in coordinates: $X = X^i \frac{\partial}{\partial x^i}$

$$X(f) = X^i \frac{\partial f}{\partial x^i}$$

$$(aX + bY)(f) = aX(f) + bY(f)$$

- But $X(f)$ is itself a scalar $\Rightarrow$

$$Y(X(f)) = (YX)(f)$$

is meaningful. However, YX is linear, but <u>not</u> 1st order on f $\Rightarrow$ not vector:

$$(YX)(f) = Y(X(f)) = Y^i \frac{\partial}{\partial x^i}(X^j f_{,j})$$

$$= Y^i X^j{}_{,i} \frac{\partial}{\partial x^j} f + Y^i X^j f_{,ij}$$

Notation: $_{,i} = \frac{\partial}{\partial x^i}$, a coord dependent expression.

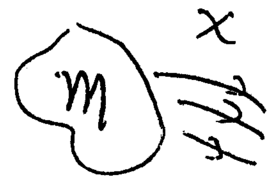
• The commutator is a vector: 1st order ☺ ↙ operator.

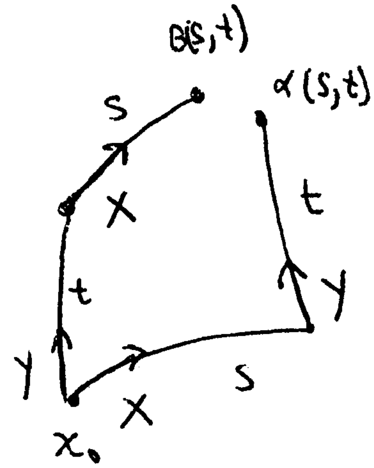
$$(XY - YX)(f) = \left(X^i Y^j_{,i} - Y^i X^j_{,i}\right) \frac{\partial f}{\partial x^j}$$

Conclude: $[X, Y] \equiv XY - YX$ is a vector field with components

$$[X, Y]^j = X^\sigma Y^j_{,\sigma} - Y^\sigma X^j_{,\sigma}.$$

□ Geometric Interpretation:





Let $\psi^X_s(x)$ be the pt s units along the int curve of X starting at x; $\psi^Y_t(x)$ the point t units along int curve of Y starting at x, in x-coords.

Let

$$\alpha(s,t) = \psi^Y_t \circ \psi^X_s(x_0) \in \mathbb{R}^4$$

$$\beta(s,t) = \psi^X_s \circ \psi^Y_t(x_0) \in \mathbb{R}^4.$$

Theorem:
$$[X, Y]^i_x = \lim_{\substack{s,t \to 0 \\ c^{-1}|t| \le |s| \le c|t|}} \frac{\alpha(s,t)^i - \beta(s,t)^i}{st} \qquad \text{(FIP)}$$

<u>Conclude</u>: "$[X, Y]$ measures the failure of the flows of X & Y to commute"

<u>Lie Derivative</u>:

- A vector field X defines a flow:

$$\varphi_s : M \longrightarrow M.$$

This induces a map on the tangent space of M:

$$(\varphi_s)_* : T_pM \longrightarrow T_{\varphi_s(p)}M \qquad \text{for each } p.$$

<u>2 ways</u>: ① If $c(s) \subset M$, then

$$\varphi_s : c(s) \longmapsto \varphi_s \circ c(s) = \bar{c}(s)$$

Then
$$\varphi_{s*} : \frac{dc}{ds} \longmapsto \frac{d}{ds}\bar{c}(s)$$

Alternatively: ② If $f : M \to \mathbb{R}$, then

$$(\varphi_* X_p)(f) = X_p(f \circ \varphi)$$

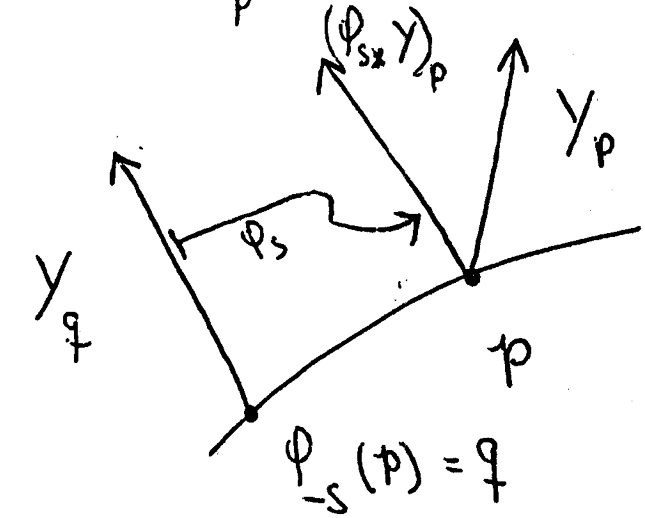
FIP: show ① & ② are equivalent

~~Then $\varphi_* : X_p \to X_{\Phi_s(p)}$ iff $(\varphi_* X_p)(f) = X_p(f \circ \varphi$~~

- Now given another vector field Y, Let $\varphi_{s*}Y$ denote the forward push of Y by the flow φ_s induced by X.

<u>Defn</u>: $L_X Y\big|_p = -\frac{d}{ds}\left\{\varphi_{s*}Y_{\varphi_{-s}(p)}\right\}$

<u>Picture</u>:



$L_X Y$ defined invariantly $\Rightarrow$ indept of coords

Then:
$$L_X Y\big|_p = \lim_{s\to 0}\frac{1}{s}\left\{Y_p - (\varphi_{s*}Y)_p\right\}$$
$$= \lim_{s\to 0} -\frac{1}{s}\left\{(\varphi_{s*}Y)_p - (\varphi_{0*}Y)_p\right\}$$
$$= -\frac{d}{ds}\left\{\varphi_{s*}Y_{\varphi_{-s}(p)}\right\}$$

Note: this is the expression for Lie Bracket that generalizes to arbitrary tensors

<u>Thm</u>: $L_X Y = [X, Y]$ (FIP)

$\Rightarrow$

(14)

<u>Idea</u>: Assume $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ (I.e. map represented in x-coords)

$$\underset{\sim}{y} = \varphi(\underset{\sim}{x})$$

What is φ_*? In $\mathbb{R}^n$, $\varphi_* \equiv \dfrac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}} \equiv n \times n$ matrix

<u>Then</u>: If $\underset{\sim}{x}(s)$ is a curve in $\mathbb{R}^n$, φ maps this to $(\varphi \circ \underset{\sim}{x})(s)$ and so $X = \dot{x}^i \dfrac{\partial}{\partial x^i}$ maps to vector with components

$$\frac{d}{ds}\left(\varphi^j \circ \underset{\sim}{x}(s)\right) = \frac{\partial \varphi^j}{\partial x^i}\, \dot{x}^i(s).$$

<u>Thus</u>:

$$\varphi_* : \underbrace{\dot{x}^i \frac{\partial}{\partial x^i}}_{\text{tangent to preimage curve}} \longmapsto \underbrace{\left(\frac{\partial \varphi^i}{\partial x^j}\, \dot{x}^j\right) \frac{\partial}{\partial x^i}}_{\text{Tangent to image curve}}$$

$$\Leftarrow\quad -\frac{d}{ds}\left\{ \varphi_{s*} Y_{\varphi_{-s}(p)} \right\}\Big|_{s=0} = -\left(\frac{d}{ds}\varphi_{s*}\right)\Big|_{s=0} Y_p - \frac{d}{ds} Y_{\varphi_{-s}(p)} = -\left(\frac{\partial}{\partial x^j}\frac{d}{ds}\varphi^i_s\right) Y^j \frac{\partial}{\partial x^i} + X(Y)^i \frac{\partial}{\partial x^i}$$

$$= -[X, Y]$$

Also: if $f: \mathbb{R}^n \to \mathbb{R}^n$

$$X(f) = \dot{x}^i \frac{\partial}{\partial x^i}(f)$$

But: $$X(f \circ \varphi) = \dot{x}^i \frac{\partial}{\partial x^i}(f \circ \varphi)$$

$$= \dot{x}^i \frac{\partial f}{\partial x^\sigma} \frac{\partial \varphi^\sigma}{\partial x^i}$$

$$= \left(\dot{x}^i \frac{\partial \varphi^\sigma}{\partial x^i}\right) \frac{\partial}{\partial x^\sigma} f$$

$$= (\varphi_* X)(f)$$

Idea: φ_* is represented in each coord system by matrix $\frac{\partial (x^i \circ \varphi)}{\partial x^j}$ which transforms like a $\binom{1}{1}$-tensor, but its not really a tensor because it maps betw $T_p X \to T_{\varphi(p)} X$ different spaces.

Note ②: The "push forward" (lower $*$) is Natural because it applies even when $\dim(M^m) = m < n = \dim M^n$

Eg: $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$p \longmapsto q$

$$X_p = a^i \frac{\partial}{\partial x^i} \longmapsto Y_q = \underbrace{\left(\frac{\partial \varphi^i}{\partial x^j} a^j \right)}_{} \frac{\partial}{\partial x^i}$$

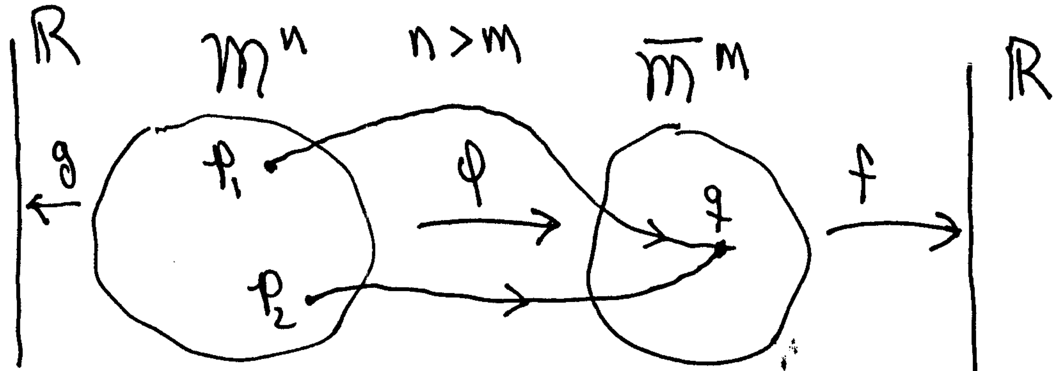
$1 \le i \le m$

$1 \le j \le n$

defined even if $\frac{\partial \varphi^i}{\partial x^j}$ not invertible

$\Rightarrow$ the "pull back" not well-defined for vectors, but is natural for co-vectors —

Picture:



- "Pullback" φ^* makes sense for f

$$\varphi^* f = f \circ \varphi$$

I.e. $\{f : \overline{M} \to \mathbb{R}\} \overset{\varphi^*}{\Longrightarrow} \{\varphi^* f : M \to \mathbb{R}\}$

$$\varphi^* f(p) = f(\varphi(p))$$

- You could not push $g : M \to \mathbb{R}$ forward to a fn on $\overline{M}$ because no unique value for q
- Principle: "Things that act on pullbacks get pushed forward"[1]

Ex: X a v.f. on $M \Rightarrow \varphi_* X$ v.f on $\overline{M}$

$$(\varphi_* X)(f) = X(\varphi \circ f)$$

[1] pulling fns on $\overline{M}$ back to M gives a way to define the action of $X \in TM$ on $f \Rightarrow X$ pushes forward"

pullback of f to M gives $X \in TM$ meaning on fn's f on $\overline{M}$

Frobenius Integrability Thm:

Let M^n be an n-dimensional manifold, and let $X_1, \dots, X_n$ be linearly independent vector fields defined on a nbhd of $p \in M$.

Theorem (Frobenius): The condition

$$[X_i, X_j] = 0 \quad \forall p$$

$\forall\ i, j = 1, \dots, k$, holds iff $\exists$ a coordinate system $\underset{\sim}{y}$ in a nbhd $\mathcal{U}$ of p such that

$$\underset{\sim}{y} : \mathcal{U} \to \mathbb{R}^n$$

$$\underset{\sim}{y}(p) = 0$$

$$\frac{\partial}{\partial y^i} = X_i, \quad i = 1, \dots, k.$$

In particular,

$$\underset{\sim}{y}^{-1}(y^1, \ldots, y^k, 0, \ldots, 0)$$

defines a k-dimensional surface $\mathcal{N}^k$ in $\mathcal{M}$ such that

$$T_q \mathcal{N} = \mathrm{Span}\{X_1, \ldots, X_k\};$$

i.e., each X_i is tangent to $\mathcal{N}^k$.

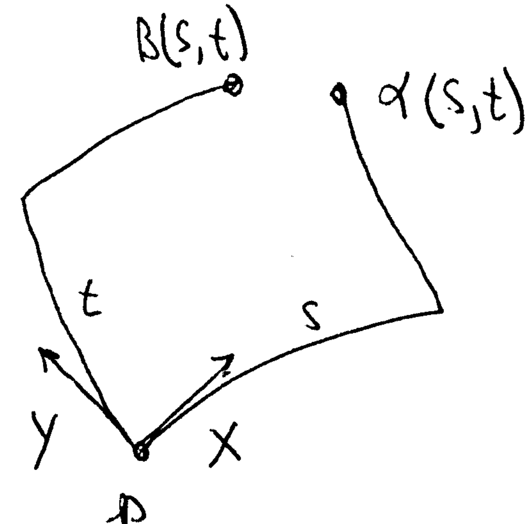
<u>Proof</u>: We do the case $k=2$; the general case will be left to the homework.

Let $X_1 \equiv X$, $X_2 \equiv Y$. Let $\underset{\sim}{x}$ denote a coord system in a nbhd of p in which

$$\underset{\sim}{x}(0) = p$$

$$\left.\frac{\partial}{\partial x^1}\right|_p = \underset{\displaystyle a^i \frac{\partial}{\partial x^i}}{\overset{\shortparallel}{X}}, \quad \left.\frac{\partial}{\partial x^2}\right|_p = \underset{\displaystyle b^i \frac{\partial}{\partial x^i}}{\overset{\shortparallel}{Y}}.$$

Let φ^X_s, φ^Y_t denote the flows for X & Y in $\underset{\sim}{x}$-coordinates. We have:



<u>Thm</u> (A) $\alpha(s,t) = \varphi^Y_t \circ \varphi^X_s(0)$

$\beta(s,t) = \varphi^X_s \circ \varphi^Y_t(0)$

$$[X,Y]^i_{\underset{\sim}{x}_0} = \lim_{s,t\to 0} \frac{\alpha(s,t)^i - \beta(s,t)^i}{st}$$

$\boxed{\text{~~This implies~~}}\ \boxed{\text{~~that if $[X,Y] \neq 0$~~}}$

This implies ($\Leftarrow$) i.e., if $\exists$ such a coord system, then $[X,Y]=0$ because for coord. vector fields $\alpha(s,t)=\beta(s,t)$.

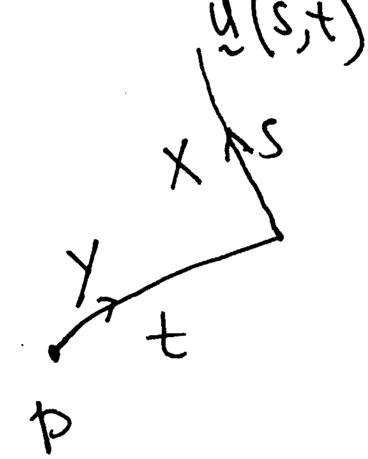
For ($\Rightarrow$): We prove X, Y are tangent to a 2-d surface thru each pt $p \in \mathcal{U} \subseteq M$.

- Define:

$$\underset{\sim}{u}(s,t) = \varphi_s^X(\varphi_t^Y(p))$$

$$\underset{\sim}{u} : \mathbb{R}^2 \longrightarrow M$$

$$\underset{\sim}{u}(0,0) = p$$



- Clearly, $\dfrac{\partial \underset{\sim}{u}}{\partial s} = X$

- If we knew the flow for X commuted with the flow for Y, i.e., if

$$\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X,$$

then

$$\underset{\sim}{u}(s,t) = \varphi_t^Y \circ \varphi_s^X(p)$$

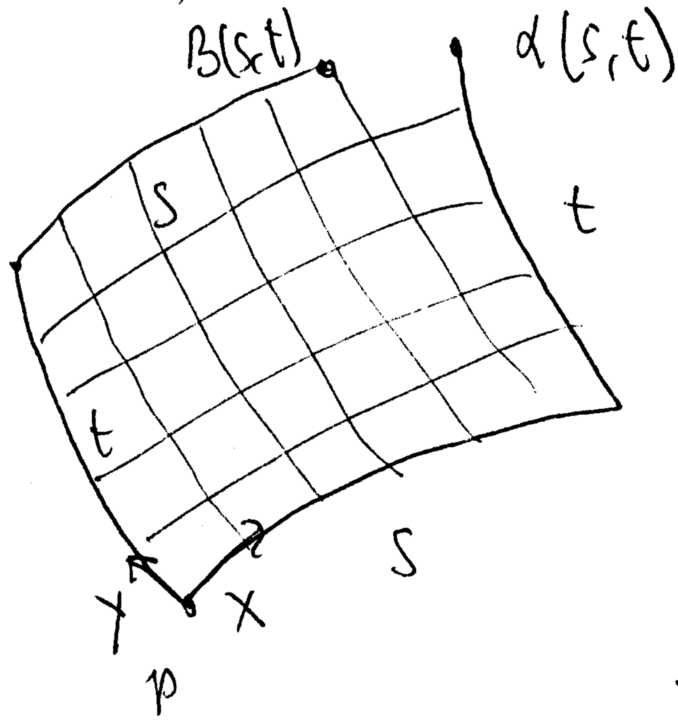
$$\Rightarrow \qquad \frac{\partial \underset{\sim}{u}}{\partial t} = Y$$

- Conclude: To prove that $\underset{\sim}{u}(s,t)$ parameterizes a 2-d surface with tangent vectors X & Y it suffices to prove the following theorem:

<u>Thm</u>: If $[X,Y] \equiv 0$ in $\mathcal{U} \ni p$, then

$$\varphi_s^X \circ \varphi_t^Y = \varphi_t^Y \circ \varphi_s^X \text{ in } \mathcal{U}.$$

<u>Proof</u>: we'd like to prove this as follows:
We need $\alpha(s,t) = \beta(s,t)$ $\forall$ s,t in a nbhd of 0. To do this, try:



Break "surface" up into n^2 small "square" loops and see that 'failure to close' of the larger square is the sum of the failures of the smaller squares to close.

But

$$[X,Y] = \lim_{\frac{s}{n},\frac{t}{n}\to 0} \frac{\alpha\left(\frac{s}{n},\frac{t}{n}\right)-\beta\left(\frac{s}{n},\frac{t}{n}\right)}{\left|\frac{s}{n}\cdot\frac{t}{n}\right|} = 0$$ (23)

$\Rightarrow$

$$\Rightarrow \alpha\left(\frac{s}{n},\frac{t}{n}\right)-\beta\left(\frac{s}{n},\frac{t}{n}\right) = O\left(\left|\frac{s}{n}+\frac{t}{n}\right|^3\right)$$

$$= O\left(\frac{1}{n^3}\right).$$

Thus

$$\alpha(s,t)-\beta(s,t) = \sum_{n^2} O\left(\frac{1}{n^3}\right) \sim O\left(\frac{1}{n}\right)$$

So as $n\to\infty$, get $\alpha(s,t)=\beta(s,t)$.

It is difficult to make this rigorous because its hard to write $\alpha(s,t)-\beta(s,t)$ as a sum of "$\alpha\left(\frac{s}{n},\frac{t}{n}\right)-\beta\left(\frac{s}{n},\frac{t}{n}\right)$" terms. Thus we use a more indirect approach.

start

Thm: If X, Y are v.f.s & $[X,Y]=0$, then $\exists$ integral submanifold

$$\underset{\sim}{u}(s,t) = \varphi_s^X(\varphi_t^Y(p))$$

thru each p st $\frac{\partial \underset{\sim}{u}}{\partial s} = X$, $\frac{\partial \underset{\sim}{u}}{\partial t} = Y$.

P.f. $\frac{\partial \underset{\sim}{u}}{\partial s} = X$ no problem, but $\frac{\partial \underset{\sim}{u}}{\partial t} = Y$ requires

$$\varphi_s^X \varphi_t^Y = \varphi_t^Y \varphi_s^X$$

Lemma: If $\alpha : M \to M$

$$\alpha_* : TM \to TM$$

and

$$\alpha_* Y_q \longmapsto Y_{\alpha(q)=p} \ \forall q,$$

then α commutes with φ_t^Y:

$$\alpha \circ \varphi_t^Y = \varphi_t^Y \circ \alpha .$$

Proof: Consider the vector field $\alpha_* Y$.

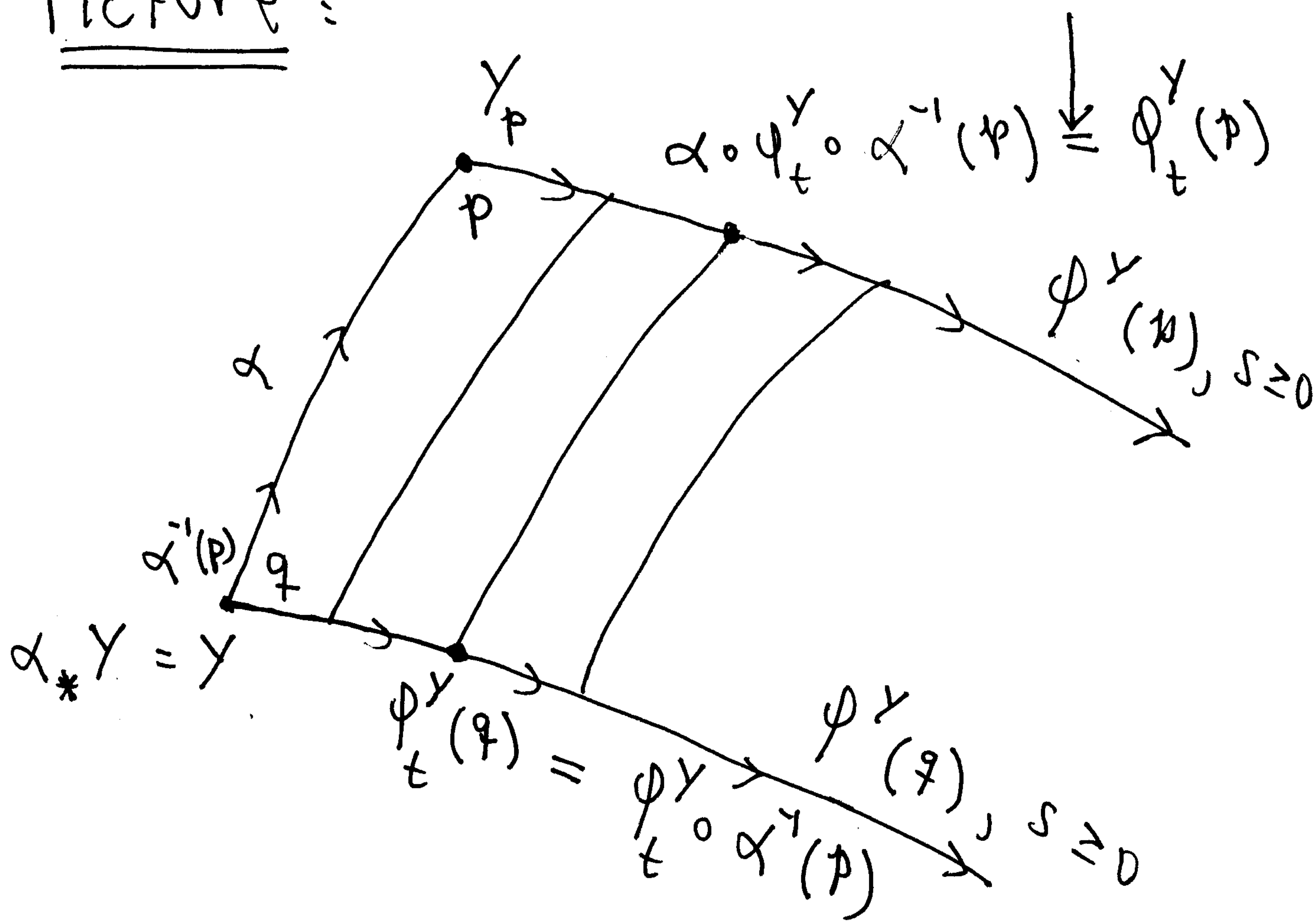
Claim: $\alpha \circ \varphi_t^Y \circ \alpha^{-1}$ is the flow for the vector field $\alpha_* Y_q = Y_p$

To see this, we need only show that

$$\frac{d}{dt} f\left(\alpha \circ \varphi_t^Y \circ \alpha^{-1}\right)_p = \left(\alpha_* Y_q\right)(f) = Y_p(f)$$

for every smooth function f

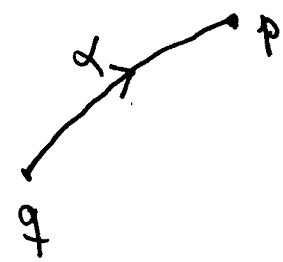
Picture:


Lemma
Y_p
$\alpha \circ \varphi_t^Y \circ \alpha^{-1}(p) = \varphi_t^Y(p)$
p
$\varphi^Y(p), s \geq 0$
α
$\alpha^{-1}(p)$
q
$\alpha_* Y = Y$
$\varphi_t^Y(q) = \varphi_t^Y \circ \alpha^{-1}(p)$
$\varphi^Y(q), s \geq 0$

But

$$(\alpha_* Y_q)_p(f) = \alpha_* Y_{\alpha^{-1}(p)}(f)$$

$\alpha(q) = p$



$$\equiv Y_{\alpha^{-1}(p)}(f \circ \alpha)$$

$$= \lim_{h \to 0} \left\{ \frac{1}{h} \left[f \circ \alpha \left(\varphi^Y_{t+h}(\alpha^{-1}(p)) \right) - f \circ \alpha \left(\varphi^Y_t(\alpha^{-1}(p)) \right) \right] \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{1}{h} \left[f\left(\alpha \circ \varphi^Y_{t+h} \circ \alpha^{-1}(p) \right) - f\left(\alpha \circ \varphi^Y_t \circ \alpha^{-1}(p) \right) \right] \right\}$$

$$= \frac{d}{dt} f\left(\alpha \circ \varphi^Y_t \circ \alpha^{-1} \right) \checkmark$$

Thus $\alpha_* Y_q = Y_p \Rightarrow$ the flow for $\alpha_* Y$, namely φ^Y_t, is $\alpha \circ \varphi^Y_t \circ \alpha^{-1} \Rightarrow$

$$\alpha \circ \varphi^Y_t \circ \alpha^{-1} = \varphi^Y_t \iff \alpha \circ \varphi^Y_t = \varphi^Y_t \circ \alpha$$

Conclude: to prove that $\varphi_t^Y \circ \varphi_s^X = \varphi_s^X \circ \varphi_t^Y$, we only need to show that

$$\varphi^X_{s*} Y_q = Y_{\varphi^X_s(q)} \quad \forall q \, ; \quad (*)$$

i.e., then $\psi \equiv \varphi^X_s$ commutes with the flow for Y.

To this end, suppose $[X, Y] \equiv 0$. Then

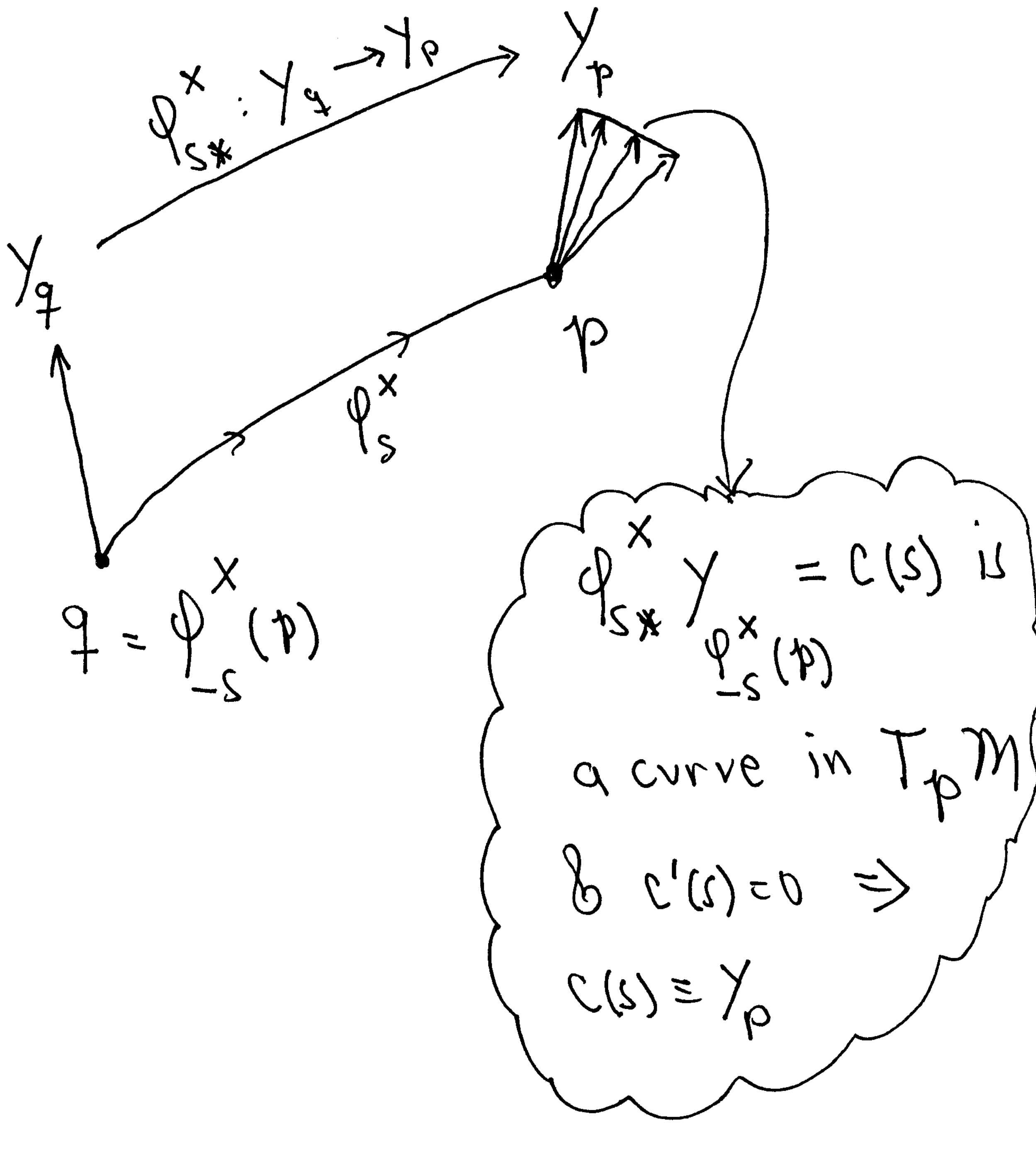
$$\forall p \qquad 0 = \lim_{h \to 0} \frac{1}{h} \left[Y_p - \left(\varphi^X_{h*} Y \right)_p \right] = L_X Y \Big|_p$$

Then given any $p \in \mathcal{U} \subseteq \mathcal{M}$, define the curve

$$c(s) = \left(\varphi^X_{s*} Y \right)_p \in T_p \mathcal{M}$$

We show $c'(s) \equiv 0$. This is all we need, because then $c(s) = c(0) \equiv Y_p$, and hence $(*)$ follows!

Picture:



$\varphi^X_{s*} : Y_q \to Y_p$
Y_p
Y_q
p
φ^X_s
$q = \varphi^X_{-s}(p)$
$\varphi^X_{s*} Y_{\varphi^X_{-s}(p)} = c(s)$ is
a curve in T_pM
& $c'(s) = 0 \Rightarrow$
$c(s) \equiv Y_p$

But

$$c'(s) = \lim_{h\to 0} \frac{1}{h}\left[c(s+h) - c(s)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\left(\varphi^X_{s+h\,*} Y\right)_p - \left(\varphi_{s*} Y\right)_p\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\left(\varphi^X_{s*} \circ \varphi^X_{h*} Y\right)_p - \left(\varphi_{s*} Y\right)_p\right]$$

$$= \lim_{h\to 0} \varphi^X_{s*}\left[\frac{1}{h}\left\{\left(\varphi^X_{h*} Y\right)_{\varphi_{-s}(p)} - Y_{\varphi_{-s}(p)}\right\}\right]$$

$$= \varphi^X_{s*}\left(L_X Y\right)_{\varphi_{-s}(p)} = 0 \checkmark$$

Thus $\varphi^X_s \circ \varphi^Y_t = \varphi^Y_t \circ \varphi^X_s$ & the Frobenius theorem is proved for case $k=2$. ∎