Maps between Manifolds:

- Recall: vector field X

flow $\Psi_s : M \to M$ (locally)

in $\underset{\sim}{x}$-coordinates:

$$\phi_s : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$\underset{\sim}{x} \longmapsto \phi_s(\underset{\sim}{x})$$

Induces a derivative map:

$$\phi_{s*} : T\mathbb{R}^n \longrightarrow T\mathbb{R}^n$$
$$a^i \frac{\partial}{\partial x^i} \longmapsto \frac{\partial \phi_s^i}{\partial x^\sigma} a^\sigma \frac{\partial}{\partial x^i}$$

This defines a coord. indept map

$$\Psi_{s*} : TM \longrightarrow TM \quad \text{(all for } s \text{ suff small)}$$

More generally: consider any map

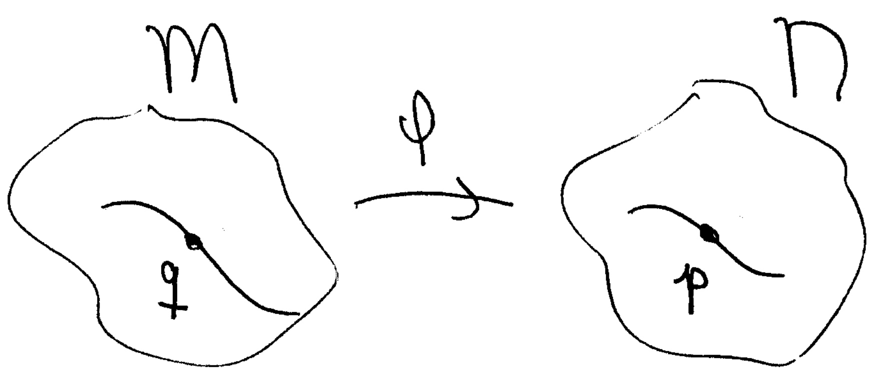
$$\phi : \mathcal{M}^m \to \mathcal{N}^n$$

(we don't even need to assume 1-1).

We can define the derivative map ϕ_* as follows:

- For any curve $c(\xi)$ in $\mathcal{M}$, $c(0)=q$, the map ϕ defines a curve $(\phi \circ c)(\xi)$ in $\mathcal{N}$,

$$(\phi \circ c)(0) = \phi(q) = p$$



Note: if ϕ is not 1-1, it doesn't go the other way: curves in $\mathcal{N}$ do not determine unique curves in $\mathcal{M}$

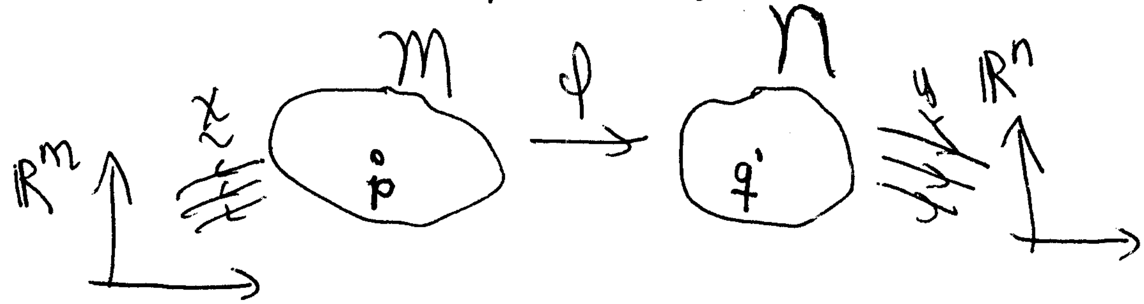
• Define the map ϕ_* by:

$$\phi_{q*} : T_q\mathcal{M} \longrightarrow T_p\mathcal{M}$$

$$X_q \longmapsto \left(\phi_{q*}X\right)_p = Y_p$$

$$T_q\mathcal{M} \ni X_q = \frac{d}{d\xi}c(0) \longmapsto \frac{d}{d\xi}(\phi\circ c)(0) = Y_p \in T_p\mathcal{N}$$

• If $\underset{\sim}{x} : \mathcal{U} \to \mathbb{R}^m$, $q \in \mathcal{U}$ and $\underset{\sim}{y} : \phi(\mathcal{U}) \to \mathbb{R}^n$ are coordinate systems,



Then $\underset{\sim}{y} = \underset{\sim}{y}^{-1}\circ\phi\circ\underset{\sim}{x}\,(\underset{\sim}{x})$ defines a map

$$\underset{\sim}{y}^{-1}\circ\phi\circ\underset{\sim}{x} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$\underset{\sim}{x} \longmapsto \left(\underset{\sim}{y}^{-1}\circ\phi\circ\underset{\sim}{x}\right)(\underset{\sim}{x}) = \underset{\sim}{y}.$$

Thm: $\phi_{q*} : a^i\frac{\partial}{\partial x^i} \longmapsto \frac{\partial y^\alpha}{\partial x^i}a^i\frac{\partial}{\partial y^\alpha}$ (FIP)

Cor: $n = m$ and $\frac{\partial y^\alpha}{\partial x^i}$ nonsingular $\Rightarrow$ ϕ_{q*} is an isomorphism. HW

- <u>Alternatively</u>: $X_{\varphi} \in T_{\varphi} M$ operates on scalar functions

$$X_{\varphi}(f) = a^i \frac{\partial}{\partial x^i}\left(f \circ \underset{\sim}{x}^{-1}\right)\Big|_{\varphi}$$

But if $f : N \to \mathbb{R}$, then f determines a unique function on M:

$$f \circ \phi : M \to \mathbb{R}$$

(Note: if f not 1-1, it doesn't take functions on M uniquely to functions on N - that's why the lower-$*$ maps <u>naturally</u> push vectors <u>forward</u> - the map can be inverted if ϕ 1-1)

<u>Thm</u> $(\phi_{\varphi *} X)_p$ is the element of $T_p N$ that operates on smooth functions by

$$(\phi_{\varphi *} X)_p (f) = X_{\varphi}(f \circ \phi) \qquad \text{(FIP HW)}$$

The pullback map ϕ^*:

Let
$$\phi : M^m \to N^n$$
For every $\omega \in T_p^*N$, define
$$\phi^* : T^*_{\phi}M \leftarrow T_p^*N$$
$$(\phi^*\omega^p)_{\phi} \longleftarrow \omega^p .$$

Now $\omega^p \in T_p^*N$ is defined as a linear functional on T_pN ; i.e., if
$$\omega^p = b_\alpha dy^\alpha$$
in $\underset{\sim}{y}$-coordinate, then for $Y_p = a^\alpha \frac{\partial}{\partial y^\alpha}$,
$$\omega^p(Y_p) \equiv b_\alpha a^\alpha .$$
Thus we can define $(\phi^*\omega^p)_{\phi}$ as the 1-form that acts on X_{ϕ} the way ω^p acts on $\phi_* X_{\phi}$:

<u>Defn:</u> $\left(\phi_p^* \omega^p\right)(X_\phi) = \omega^p(\phi_{\phi *} X_\phi)$

<u>Thm:</u> in coordinates,

$$\phi_p^* : \left(\frac{\partial y^\alpha}{\partial x^i} b_\alpha\right) dx^i \longleftarrow b_\alpha dy^\alpha$$

FIP HW

(<u>Note</u>: The pullback is <u>natural</u> for 1-forms because when ϕ not 1-1, $\frac{\partial y^\alpha}{\partial x^i}$ is defined, but <u>not</u> $\frac{\partial x^i}{\partial y^\alpha}$)

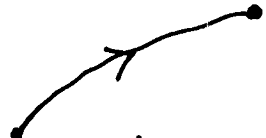
☐ Pullback φ^* & push forward φ_* can be extended to $\binom{k}{\ell}$-tensors if φ is 1-1 invertible

Eg φ_t the flow for X, $T \in \binom{1}{1}$-tensors

$$T(X_p, \omega^p) = T^i_j \, X^i_p \, \omega^p_j \quad \text{multi-linear at } p$$

$$\varphi_t(q) = p \; ; \; \varphi_{t*} : T_q M \to T_p M \; ; \; \varphi^*_t : T^*_p M \to T^*_q M$$

$p = \varphi_t q$



$q = \varphi_t^{-1}(p) = \varphi_{-t}(p)$

<u>Defn</u>: $\left(\varphi_{t*} T_q\right)(X_p, \omega^p) = T_q\big(\underbrace{\varphi_{-t*} X_p}, \varphi^*_t \omega^p\big)$

$$\left(\varphi^*_t T_p\right)(X_q, \omega^q) = T_p\left(\varphi_{t*} X_q, \varphi^*_{-t} \omega^q\right)$$

"pushforward/pullback of T defined by pushforward/pullback of inputs X, ω"

- Thus we can define Lie Deriv of T:

$$L_Y T = \frac{d}{dt}\left[\varphi_{t*} T_{\varphi_{-s}(p)}\right]$$

Lie Deriv & Killing Vector Field

$$L_Y T(X_p, Y_p) = \frac{d}{dt}\left(\varphi_{t*} T_{\varphi_{-s}(p)}\right)(X_p, \omega^p)$$

$$= \frac{d}{dt}\left[T_{\varphi_{-t}(p)}\left(\varphi_{-t*} X_p, \varphi_t^{*} \omega^p\right)\right.$$

$$= \frac{d}{dt}\left[T^i_j(\varphi_{-t}(p))\left(\frac{\partial \varphi^j_{-t}}{\partial x^\sigma} X^\sigma_p, \frac{\partial \varphi^k_t}{\partial x^i} \omega^p_k\right)\right.$$

$$= \frac{d}{dt} T^i_j(\varphi_{-t}(p))(X^j_p, \omega^p_i)$$

$$+ T^i_j(p)\left(\frac{d}{dt}\left(\frac{\partial \varphi^j_{-t}}{\partial x^\sigma}\right) X^\sigma_p, \frac{d}{dt}\frac{\partial \varphi^k_t}{\partial x^i}\omega^p_k\right)$$

Now if $Y = \frac{\partial}{\partial x^1}$, $\varphi_t = (x^1+t, x^2, \ldots, x^n)$, $\frac{\partial \varphi^j}{\partial x^i} = \delta^j_i$ $\Rightarrow = 0$

so $\boxed{L_Y T(X_p, Y_p) = \frac{d}{dt} T^i_j(\varphi_{-t}(p))(X^j_p, \omega^p_i)}$

"in coordinates where $Y = \frac{\partial}{\partial x^i}$, $L_Y T$ is just deriv of components wrt x^i" $\Longleftrightarrow$ $L_Y T = 0$ means constant components in those coords.

- **Conclude**: If $L_Y g = 0$ for $\binom{0}{2}$-tensor metric g, then in coords where $Y = \frac{\partial}{\partial x^1}$,

$$g_{ij}(x^1, \ldots, x^n) = g_{ij}(x^1 + t, x^2, \ldots, x^n)$$

$\Rightarrow$ all angles & lengths indept of x^1

$\Rightarrow$ φ_t is an isometry —

By expressing Lie Derivative in terms of Covariant derivative (next)

Thm: (Killing Vector Field) $L_Y g = 0$ iff

$$\boxed{Y_{i;j} + Y_{j;i} = 0}$$