Maps between Manifolds:

- Recall vector field $X$

  flow $\Phi_s : M \rightarrow M$ (locally)

  in $x$-coordinates:

  $$\phi_s : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

  $$x \mapsto \phi_s(x)$$

Induce a derivative map:

$$\phi_s^* : T\mathbb{R}^n \rightarrow T\mathbb{R}^n$$

$$\frac{\partial^2}{\partial x^i} \mapsto \frac{\partial \phi_s^i}{\partial x^a} \frac{\partial}{\partial x^a}$$

This defines a coord. indept. map

$$\Psi_s^* : TM \rightarrow TM$$ (all $s$ suff. small)
Move generally: consider any map
\[ \phi : M^m \to N^n \]
(We don't even need to assume \(1-1\)).
We can define the derivative map \( \phi_* \) as follows:

- For any curve \( C(\xi) \) in \( M \), \( C(0) = q \), the map \( \phi \) defines a curve \( (\phi \circ C)(\xi) \) in \( N \),
  \[ (\phi \circ C)(0) = \phi(q) = p \]

Note: if \( \phi \) is not \( 1-1 \), it doesn't go the other way: curves in \( N \) do not determine unique curves in \( M \).
Define the map \( \phi^*_q \) by:

\[
\phi^*_q : T_q M \rightarrow T_p N
\]

\[
X_q \mapsto (\phi^*_q X)_p = Y_p
\]

\[
T_q M \ni X_q = \frac{d}{d\xi} c(\xi) \rightarrow \frac{d}{d\xi} (\phi \circ c)(\xi) = Y_p \in T_p N
\]

- If \( x : U \rightarrow \mathbb{R}^m, \, \phi \in U \) and \( y : \phi(U) \rightarrow \mathbb{R}^n \) are coordinate systems,

Then \( y \circ \phi \circ x : (x) \) defines a map

\[
y^\dagger \circ \phi \circ x : \mathbb{R}^m \rightarrow \mathbb{R}^n
\]

\[
x \mapsto (y^\dagger \circ \phi \circ x)(x) = y
\]

Thm: \( \phi^*_q : \frac{\partial}{\partial x^i} \mid_{x^q} \rightarrow \frac{\partial y^a}{\partial x^i} \frac{\partial^2}{\partial y^a \partial y^b} \)

Con: \( n = m \) and \( \frac{\partial y^a}{\partial x^i} \) nonsingular \( \Rightarrow \phi^*_q \) is an isomorphism.
Alternatively: $X_p \in T_p M$ operates on scalar functions

$$X_p(f) = \frac{d}{dt} \bigg|_{t=0} (f \circ \phi^{-1})(t)$$

But if $f: \mathbb{N} \rightarrow \mathbb{R}$, then $f$ determines a unique function on $M$:

$$f \circ \phi: M \rightarrow \mathbb{R}$$

(Note: if $f$ not 1-1, it doesn't take functions on $M$ uniquely to functions on $\mathbb{N}$ - that's why the lower-* maps naturally push vectors forward - the map can be inverted if $\phi$ is)

**Thm** 

$(\phi_* X)_p$ is the element of $T_p M$ that operates on smooth functions by

$$(\phi_* X)_p(f) = X_p(f \circ \phi)$$  (FIP)
The pullback map $\phi^*$:

Let $\phi: M^m \to N^n$

For every $w^p \in T^*_p N$, define

$$\phi^*: T^*_p M \to T^*_p N$$

$$(\phi^* w^p) \mapsto w^p.$$ 

Now $w^p \in T^*_p N$ is defined as a linear functional on $T_p N$; i.e., it:

$$w^p = b^x \frac{\partial}{\partial x^y}$$

in $y$-coordinates, then for $X^p = a^x \frac{\partial}{\partial y^x}$

$$w^p (X^p) = b^x a^x.$$ 

Thus we can define $(\phi^* w^p)_p$ as the

1-form that acts on $X^p$ the way $w^p$ acts

on $\phi^* X^p$. 


Defn: \((\phi^* w^p)(x^q) = w^p(\phi^*_q x^q)\)

Thm: in coordinates,
\[
\phi^*_p = \left( \frac{\partial y^i}{\partial x^j} b_x \right) dx^i \leftarrow b_x dy^i
\]

(Note: The pullback is natural for forms because when \(\phi\) not 1-1, \(\frac{\partial y^i}{\partial x^j}\) is defined, but not \(\frac{\partial x^i}{\partial y^j}\).)
Pullback $\phi^*$ & push forward $\phi_*$ can be extended to $(k)$-tensors if $\phi$ is 1-1 invertible.  

Eq $\phi_t$ the flow for $X, T \in (1)$-tensors

$$T(x, w^q) = T_{x}^{i} x_{i} w_{j}$$ multi-linear at $p$

$\phi_t(q) = p$ ; $\phi_t : T_q M \rightarrow T_p M$ ; $\phi_t^* : T_p^* M \rightarrow T_q^* M$

$$p = \phi_t q$$

$q = \phi_t^{-1}(p) = \phi_t(p)$

**Defn**: $(\phi T_q)(x, w^q) = T_q(\phi x, \phi^* w^q)$

$$(\phi^* T_p)(x, w^q) = T_p(\phi^* x, \phi^* w^q)$$

"pushforward/pullback of $T$ defined by pushforward/pullback of inputs $x, w"
Thus we can define Lie derivate of $T$:

$$L_y T = \frac{d}{dt} \left[ \varphi_t^* T_{\psi_s}(p) \right]$$

$$L_y T(x_p, y_p) = \frac{d}{dt} (\varphi_t^* T_{\psi_s}(p))(x_p, w^p)$$

$$= \frac{d}{dt} \left[ T^{\varphi_t}_{\psi_t}(p)(\varphi_t^* x_p, \varphi_t^* w^p) \right]$$

$$= \frac{d}{dt} \left[ T^{\varphi_t}_{\psi_t}(p) \left( \frac{\partial \varphi_t^b}{\partial \psi_t^a} x^a_p, \frac{\partial \varphi_t^b}{\partial \psi_t^a} w^p \right) \right]$$

$$= \frac{d}{dt} T^{\varphi_t}_{\psi_t}(p)(x^a_p, w^p)$$

$$+ T^{\varphi_t}_{\psi_t}(p) \left( \frac{d}{dt} \left( \frac{\partial \varphi_t^b}{\partial \psi_t^a} x^a_p \right), \frac{d}{dt} \frac{\partial \varphi_t^b}{\partial \psi_t^a} w^p \right)$$

Now if $Y = \frac{\partial}{\partial x^i}$, $\varphi_t = (x^1_t, x^2_t, \ldots, x^n_t)$, $\frac{\partial \varphi_t^b}{\partial x^a} = \delta^b_a$.

So

$$L_y T(x_p, y_p) = \frac{d}{dt} T^{\varphi_t}_{\psi_t}(p)(x^a_p, w^p) = 0$$

"in coordinates where $Y = \frac{\partial}{\partial x^i}$, $L_y T$ is just deriv of components w.r.t $x^i$. " $\Rightarrow L_y T = 0$ means constant components in those coords."
Conclude: If \( L_y g = 0 \) for (\text{\textcircled{1}}) tensor metric \( g \), then in coordinates where \( y = \frac{2}{\partial x^1} \),

\[
g_{ij}(x', \ldots, x^n) = g_{ij}(x^1 + t, x^2, \ldots, x^n)
\]

\( \Rightarrow \) all angles & lengths indep. of \( x^1 \)

\( \Rightarrow \Phi_t \) is an isometry —

By expressing Lie Derivative in terms of Covariant derivative (next)

Thm: (Killing Vector Field) \( L_y g = 0 \) iff

\[
Y_{ij;j} + Y_{j;i} = 0
\]