Differential Forms / k-Volume / Integration

- Assume n-dimensional manifold $\mathcal{M}$

Q1: How does one define the integral of a function over a k-dimensional surface in $\mathcal{M}$?

Q2: If you have a metric, how do you define k-dimensional volume on k-surface in $\mathcal{M}$

Introduction / Motivation: k-volume in $\mathbb{R}^n$

- Assume $\mathcal{M} \equiv \mathbb{R}^n$, $T\mathcal{M} \equiv \mathbb{R}^n$

- We use following fact from ele. calculus:

If $X_1 \cdots X_n$ is a basis for $\mathbb{R}^n$, then

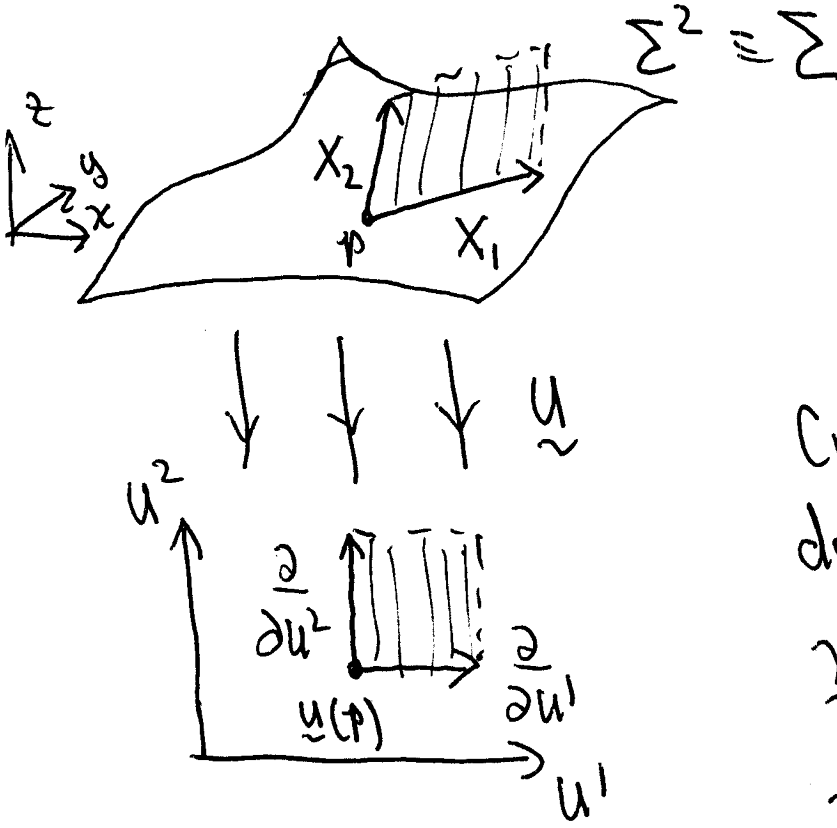
$$n\text{-Vol}[X_1, \dots, X_n] \equiv \text{"n-volume of ||-opiped spanned by } X_1 \cdots X_n\text{"}$$

$$= \det\begin{bmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{bmatrix}.$$

We use the notation:

$\begin{bmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{bmatrix}$ is the matrix whose i-column has entries equal to the components of X_i.

■ Integration on 2-surfaces $\Sigma \subseteq \mathbb{R}^3$



$\underset{\sim}{x} \equiv (x^1, x^2, x^3) \equiv (x, y, z)$

$\underset{\sim}{u} \equiv (u^1, u^2)$

Coordinate system $\underset{\sim}{u}$ determines

$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{u})$

$x^i = x^i(u^1, u^2)$

- The coordinate vector fields $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ at $\underset{\sim}{u}(p)$ correspond to coordinate vector fields X_1, X_2, resp, in $\mathbb{R}^3$ where:

$$\frac{\partial}{\partial u^1} \cong \frac{\partial x^1}{\partial u^1}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial u^1}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial}{\partial x^3} \equiv X_1$$

$$\frac{\partial}{\partial u^2} \cong \frac{\partial x^1}{\partial u^2}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial u^2}\frac{\partial}{\partial x^2} + \frac{\partial x^3}{\partial u^2}\frac{\partial}{\partial x^3} \equiv X_2$$

(FIP)

<u>Thus</u>: The 2-volume $du^1 du^2$ at $\underset{\sim}{u}(p)$ in the $\underset{\sim}{u}$-coordinate system corresponds to

$$dA = |X_1 \wedge X_2|\, du^1 du^2$$

up on the surface at P, where

$A = 2\text{-Vol}[X_1, X_2] \equiv |X_1 \wedge X_2| =$ area of the $\parallel$-ogram generated in $\mathbb{R}^3$ by X_1 & X_2.

Conclude:

$$\int_{\Sigma} f \, dA = \int_{\underset{\sim}{u}(\Sigma)} f \, |X_1 \wedge X_2| \, du^1 du^2.$$

Thus, to define integration of a function wrt area on Σ it suffices to find a formula for the area $|X_1 \wedge X_2|$ of a $\parallel$-ogram. For 2-vectors in $\mathbb{R}^3$, the cross-product gives a formula for area:

Thus:

$$X_1 \times X_2 = \det\begin{bmatrix} | & | & \frac{\partial}{\partial x^1} \\ X_1 & X_2 & \frac{\partial}{\partial x^2} \\ | & | & \frac{\partial}{\partial x^3} \end{bmatrix}$$

$$= \det_{12}\begin{bmatrix} | & | \\ X_1 & X_2 \\ | & | \end{bmatrix}\frac{\partial}{\partial x^3} - \det_{13}\begin{bmatrix} | & | \\ X_1 & X_2 \\ | & | \end{bmatrix}\frac{\partial}{\partial x^2}$$

$$+ \det_{23}\begin{bmatrix} | & | \\ X_1 & X_2 \\ | & | \end{bmatrix}\frac{\partial}{\partial x^1}$$

$$= D_1 \frac{\partial}{\partial x^3} + D_2 \frac{\partial}{\partial x^2} + D_3 \frac{\partial}{\partial x^3}$$

- What really is important is that $X_1 \times X_2$ is a 3-component object satisfying

$$|X_1 \times X_2|^2 = (D_1^2 + D_2^2 + D_3^2) = 2\text{-Vol}\,[X_1, X_2].$$

- We can get this 3-component object algebraically as follows:

$$X_1 \equiv X_1^i \frac{\partial}{\partial x^i} \equiv a^i \frac{\partial}{\partial x^i}$$

$$X_2 \equiv X_2^i \frac{\partial}{\partial x^i} \equiv b^i \frac{\partial}{\partial x^i}$$

Formally define $\wedge$ as follows: $\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = -\frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^i}$

(anti-symmetry $\Rightarrow \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^i} = 0$)

$$\frac{\partial}{\partial x^i} \wedge \left(a \frac{\partial}{\partial x^j} + b \frac{\partial}{\partial x^k}\right) = a \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + b \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^k}$$

(linearity in each component)

Then

$$\left(a^1\frac{\partial}{\partial x^1}+a^2\frac{\partial}{\partial x^2}+a^3\frac{\partial}{\partial x^3}\right)\wedge\left(b^1\frac{\partial}{\partial x^1}+b^2\frac{\partial}{\partial x^2}+b^3\frac{\partial}{\partial x^3}\right)$$

$$=\left(a^1b^2-a^2b^1\right)\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\right)$$

$$+\left(a^1b^3-a^3b^1\right)\left(\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^3}\right)$$

$$+\left(a^2b^3-a^3b^2\right)\left(\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}\right)$$

$$=\det{}^{\lambda}\begin{bmatrix} | & | \\ X_1 & X_2 \\ | & | \end{bmatrix}\left(\frac{\partial}{\partial x}\right)_{\lambda}$$

$\lambda=(\lambda_1,\lambda_2)$ an increasing sequence from $\{1,2,3\}$

$\det^{\lambda}\begin{bmatrix} | & | \\ X_1 & X_2 \\ | & | \end{bmatrix}$ = "determinant of matrix obtained by deleting all but the λ_1, λ_2 rows"

$\left(\frac{\partial}{\partial x}\right)_{\lambda}=\frac{\partial}{\partial x^{\lambda_1}}\wedge\frac{\partial}{\partial x^{\lambda_2}}$ "Sum repeated up-down multiindices λ"

Then $|X_1 \times X_2| = 2\text{-Vol}\,[X_1, X_2] \Rightarrow$

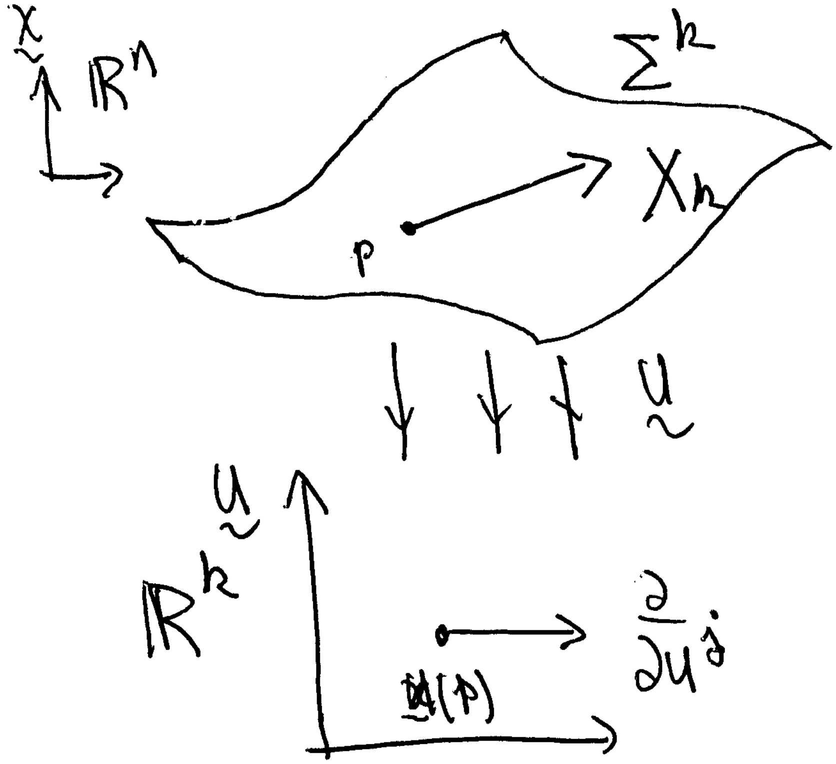
$$|X_1 \wedge X_2|^2 = \sum_{\lambda} \left(\det{}^{\lambda} \begin{bmatrix} | & | \\ X_1 & X_2 \\ | & | \end{bmatrix} \right)^2$$

- Q1: How to generalize this to k-dim. surfaces in $\mathbb{R}^n$

  Q2: How to generalize this to k-dim surface area generated by an arbitrary metric g.

<u>Theorem</u>: The procedure generalizes to a formula for the k-volume of the k-parallelopiped generated by $X_1 \cdots X_k \in \mathbb{R}^n$.

- I.e., Let Σ^k be a k-surface in $\mathbb{R}^n$



$$\underset{\sim}{x} = (x^1, \dots, x^n)$$
$$\underset{\sim}{u} = (u^1, \dots, u^k)$$

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{u})$$
$$x^i = x^i(u^1, \dots, u^k)$$

$$\frac{\partial}{\partial u^j} \equiv X_j = \frac{\partial x^\sigma}{\partial u^j} \frac{\partial}{\partial x^\sigma}$$

$$\int_{\Sigma^k} f \, dA = \int_{\underset{\sim}{u}(\Sigma^k)} f \, |X_1 \wedge \cdots \wedge X_k| \, du^1 \cdots du^k$$

where $|X_1 \wedge \cdots \wedge X_k| \equiv k\text{-Vol}\,[X_1 \cdots X_k] \equiv k\text{-vol}$ of k-parallelopiped spanned by $X_1 \cdots X_k$.

Using the algebraic properties of $\wedge$ (multi-linearity, anti-symmetry), we are led (formally) to

$$X_1 \wedge \cdots \wedge X_k = \det{}^{\lambda} \begin{bmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{bmatrix} \left(\frac{\partial}{\partial x}\right)_{\lambda} \quad (*)$$

"Sum over all increasing sequences $\lambda = (\lambda_1 \cdots \lambda_k)$ of k indices in $\{1, \ldots, n\}$".

$\det^{\lambda} \begin{bmatrix} | & & | \\ X_1 & \cdots & X_n \\ | & & | \end{bmatrix}$ = 'det of the $k \times k$ matrix obtained by deleting all but colums $\lambda_1 \cdots \lambda_k$'

$$\frac{\partial}{\partial x^{\lambda}} = \frac{\partial}{\partial x^{\lambda_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\lambda_k}}.$$

I.e., $X_1 \wedge \cdots \wedge X_n = X_1^i \frac{\partial}{\partial x^i} \wedge \cdots \wedge X_n^i \frac{\partial}{\partial x^i}$ together with algebraic operations leads to $(*)$

Theorem :

$$|X_1 \wedge \cdots \wedge X_k|^2 = (k\text{-Vol}[X_1, \dots, X_k])^2$$

$$= \sum_\lambda \left(\det{}^\lambda \begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix} \right)^2 .$$

Conclude : If we treat $\left(\frac{\partial}{\partial x}\right)_\lambda$ as an ortho-normal basis for the $\binom{n}{k}$ dimensional space $\Lambda_k \equiv \text{Span}\left\{ \left(\frac{\partial}{\partial x}\right)_\lambda \right\}$, and define the inner product on Λ_k by

$$\alpha \cdot \beta = \sum_\lambda a^\lambda b^\lambda$$

$$\alpha = a^\lambda \left(\frac{\partial}{\partial x}\right)_\lambda , \quad \beta = b^\lambda \left(\frac{\partial}{\partial x}\right)_\lambda ,$$

Then the norm so generated on k-multi-vectors $X_1 \wedge \cdots \wedge X_k$ gives k-volume in $\mathbb{R}^n$:

$$|X_1 \wedge \cdots \wedge X_k| = \sqrt{(X_1 \wedge \cdots \wedge X_k) \cdot (X_1 \wedge \cdots \wedge X_k)}$$

$$= k\text{-Vol}[X_1, \ldots, X_k] \equiv \text{"the } k\text{-vol of the } \|\text{-opiped spanned by } X_1 \cdots X_k\text{."}$$

<u>Conclude</u>: The Euclidean inner product on $\mathbb{R}^n$ is inducing an inner product on Λ_k that determines k-volumes (

<u>Said Differently</u>: The natural generalization of the cross-product is an $\binom{n}{k}$ component object whose length gives k-volume in $\mathbb{R}^n$.

- <u>Note</u>: For surfaces of co-dimension 1, $k = n-1$, $\binom{n}{n-1} = n$, and we can identify $X_1 \wedge \cdots \wedge X_{n-1}$ with a <u>vector</u> in $\mathbb{R}^n$:

Let

$$\hat{\lambda}_i = (\lambda_1, \ldots, \overset{\text{deleted}}{\downarrow}\!\!\!\cancel{\lambda_i}, \ldots, \lambda_n),$$

$\hat{\lambda}_i$ has λ_i missing —

$$\frac{\partial}{\partial x^i} \longleftrightarrow (-1)^{\pi_i} \cancel{\lambda_i} \left(\frac{\partial}{\partial x}\right)_{\hat{\lambda}_i}$$

where

$$\left(\frac{\partial}{\partial x}\right)_{\hat{\lambda}_i} \wedge \frac{\partial}{\partial x^i} = (-1)^{\pi_i} \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$$

defines π_i. Then

$$\cancel{X_1 \times \cdots \times X_{n-1} = \det \begin{bmatrix} | & & | \\ X_1 & \cdots & X_{n-1} \\ | & & | \end{bmatrix} = \sum_i (-1)^{\pi_i} \det}$$

$$X_1 \times \cdots \times X_{n-1} \equiv \det \begin{bmatrix} | & & | & \frac{\partial}{\partial x^1} \\ X_1 & \cdots & X_{n-1} & \vdots \\ | & & | & \frac{\partial}{\partial x^n} \end{bmatrix}$$

$$= \sum_i (-1)^{\pi_i} \det^{\hat{\lambda}_i} \begin{bmatrix} | & & | \\ X_1 & \cdots & X_{n-1} \\ | & & | \end{bmatrix}$$

defines a vector in $\mathbb{R}^n$ whose length is the $(n-1)$-Vol $[X_1 \cdots X_{n-1}]$. FIP

Proof of Theorem

Main Lemma:

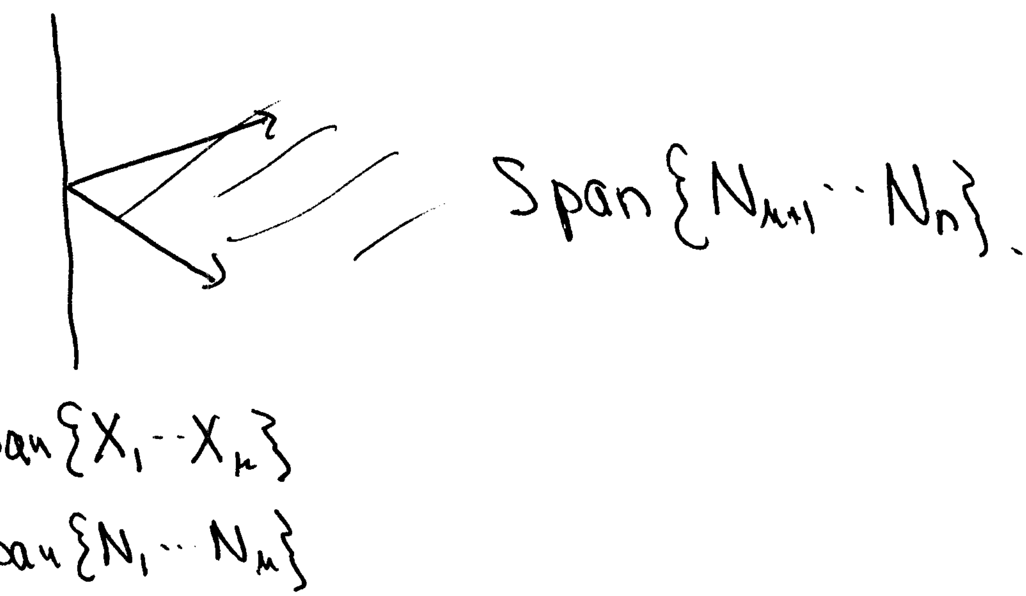
$$|X_1 \wedge \cdots \wedge X_k|^2 = \det(X_i \overset{\substack{\text{Euclidean} \\ \text{Dot Product} \\ \downarrow}}{\cdot} X_j) \qquad (**)$$

We prove this (later) by defining $X_1 \wedge \cdots \wedge X_k$ and $\omega^1 \wedge \cdots \wedge \omega^k$ carefully as anti-symmetric $\binom{k}{0}$- and $\binom{0}{k}$-tensors, resp., and seeing that LHS $(**)$ and RHS $(**)$ can be interpreted as linear operators that agree on a basis $\Rightarrow$ equal.

Assuming $(*)$, we obtain the theorem as follows:

- Let $X_1 \cdots X_k$ be k-linearly indep't vectors in $\mathbb{R}^n$. Choose $N_1, \dots, N_n \in T\mathbb{R}^n$ an ortho-normal basis for $\mathbb{R}^n$ such that

$$X_1, \dots, X_k \in \mathrm{Span}\{N_1, \dots, N_k\}$$



Then we can define the k-volume of the ‖-opiped spanned by $X_1, \dots, X_k$ to be

$$k\text{-Vol}[X_1, \dots, X_k] \equiv_{df} \det \begin{bmatrix} | & & | & | & & | \\ X_1 & \cdots & X_k & N_{k+1} & \cdots & N_n \\ | & & | & | & & | \end{bmatrix}$$

(we have to start somewhere — take this as intuitively obvious)

Let A be the $n \times n$ matrix defined by

$$A = \begin{bmatrix} — N_1 — \\ \vdots \\ — N_n — \end{bmatrix}$$

so that A is orthogonal $\Rightarrow$

$$\langle AX, AY \rangle = \langle X, Y \rangle$$

Eucl. Inner Prod

$$\langle AX, AY \rangle = \langle X, A^{tr} A X \rangle$$

$$\Rightarrow A^{tr} A = id \Rightarrow \det A = \pm 1.$$

Thus:

$$\pm \det \begin{bmatrix} | & & | & | & & | \\ X_1 & \cdots & X_k & N_{k+1} & \cdots & N_n \\ | & & | & | & & | \end{bmatrix}$$

$$= \det A \begin{bmatrix} | & & | & | & & | \\ X_1 & \cdots & X_k & N_{k+1} & \cdots & N_n \\ | & & | & | & & | \end{bmatrix}$$

$$= \det \begin{bmatrix} | & & | \\ AX_1 & \cdots & AN_n \\ | & & | \end{bmatrix} = \det \begin{bmatrix} \underbrace{\bar{A} H}_{k \times k \text{ block}} & 0 \\ 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{bmatrix}$$

where $\bar{A}$ is the $k \times n$ matrix obtained by deleting the last $n-k$ colums of A, and

$$H = \begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix}_{n \times k}, \text{ so that } \underset{(k\times n)(n \times k)}{\bar{A}\,H} = k \times k \text{ matrix}$$

On the other hand,

$$\det X_n \circ X_e$$

$$= \det \underset{(k\times n)}{H^{tr}} \cdot \underset{(n\times k)}{H}$$

$$= \det \underset{(k\times n)}{H^{tr}} \underset{(n\times n)}{A^{tr}} \underset{(n\times n)}{A} \underset{(n\times k)}{H}$$

$$= \det (AH)^{tr} \cdot \underbrace{\underset{n\times n}{(A}\,\underset{n\times k}{H)}}_{n\times k} = \det \underset{k\times k}{(\bar{A}H)^{tr}} \underset{k\times k}{(\bar{A}H)}$$

$$= \det(\bar{A}H)^2$$

$$AH = \begin{bmatrix} \boxed{\bar{A}H}_{\,k\times k} \\ 0 \end{bmatrix}_{(n\times k)}$$

<u>Thus</u>: $\det(X_n \cdot X_e) = \text{Vol}\,[X_1, \dots, X_n]$ ✓

- <u>Note</u>:

$$k\text{-Vol}[X_1, \ldots, X_k]^2 = \sum_\lambda \left(\det{}^\lambda \begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix}\right)^2 \qquad (*)$$

Note that

$$\det{}^\lambda \begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix} \equiv k\text{-volume of the}$$

projection of the k-parallelopiped onto the $(x^{\lambda_1} \cdots x^{\lambda_k})$ - axes. Thus (*) says: "the k-vol of a ∥-opiped squared is the sum of the squares of the $\binom{n}{k}$ projections of the volume onto k-coord axes."