Differential Forms / Integration (Local meaning)

- General Manifold $\mathcal{M}^n$, no metric (Warner)

  $\Omega^n \subset \mathcal{M}^n$ open, n-dimensional

  $f : \mathcal{M}^n \to \mathbb{R}$

  Assume: Ω^n covered by single coord chart $\underset{\sim}{x}$

  $$\underset{\sim}{x} : \mathcal{U}_{\underset{\sim}{x}} \to \mathbb{R}^n, \quad \underset{\sim}{x}(\mathcal{U}) = \mathcal{U}_{\underset{\sim}{x}}$$

  $$\Omega^n \subset \mathcal{U}_{\underset{\sim}{x}}, \quad \underset{\sim}{x}(\Omega^n) = \Omega_{\underset{\sim}{x}}$$

- Problem: How to give

  $$\text{"} \int_{\Omega_{\underset{\sim}{x}}} f \text{"}$$

  a coordinate independent meaning.

- Note: Under coordinate transformation $\underset{\sim}{x} \to \underset{\sim}{y}$, integral transforms by

  $$\int_{\Omega_{\underset{\sim}{x}}} f \, dx^1 \cdots dx^n = \int_{\Omega_{\underset{\sim}{y}}} f \left| \frac{\partial x}{\partial y} \right| dy^1 \cdots dy^n, \qquad (*)$$

where

$$\left|\frac{\partial x}{\partial y}\right| = \det\left(\frac{\partial x^i}{\partial y^k}\right).$$

E.g., if we interchang x^i, x^j, then integral changes by factor (-1). $\Rightarrow$ (*) has a "double valued" meaning – $\pm$ for oppositely oriented coord frames "

- **Idea**: define $dx^1 \cdots dx^n = dx^1 \wedge \cdots \wedge dx^n$ in such a way that under a change of coordinates,

$$dx^1 \wedge \cdots \wedge dx^n = \left|\frac{\partial x}{\partial y}\right| dy^1 \wedge \cdots \wedge dy^n. \quad (*)$$

Then, if we let

$$\alpha = dx^1 \wedge \cdots \wedge dx^n,$$

then

$$\int_{\Omega} f \alpha$$

is a coord. indept expression. (modulo a $\pm$ sign to be discussed later)

- To achieve (*), it suffices to make $dx^1 \wedge \cdots \wedge dx^n$ a completely anti-symmetric n-tensor:

<u>Defn</u>: $T_{i_1 \cdots i_k} \, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$ is anti-symmetr[ic] $\binom{0}{k}$-tensor, $1 \leq k \leq n$, if

$$T_{i_{\pi(1)} \cdots i_{\pi(k)}} = (-1)^{\pi} \, T_{i_1 \cdots i_k}$$

where π is a permutation of indices $i_1 \cdots i_n$

$(-1)^{\pi} = -1$ if π is <u>odd</u> - it takes odd # of interchanges to take $i_1 \cdots i_k \to \pi(i_1) \cdots \pi(i_k)$

$(-1)^{\pi} = 1$ if π is <u>even</u>.

<u>Immediate</u>: $T_{i_1 \cdots i_k} = 0$ if $i_p = i_q$, so it suffices to sum over sequ's of <u>distinct</u> indices $i_1 \cdots i_k$ in (A)

<u>Lemma</u>: Anti-symmetry is a coord. indept property of a $\binom{0}{k}$ tensor. (HW FIP)

- Explicit examples of $\binom{0}{k}$-tensors:

To construct one, we need only define an anti-symmetric linear operator on k vectors.

$\boxed{k=n}$

Given: $X_1, \dots, X_n$ in TM^n

In x-coords: $X_\ell = X^i_\ell \frac{\partial}{\partial x^i}$

$\Rightarrow$ $n \times n$ coordinate matrix X^i_ℓ

$$X^i_\ell \equiv \begin{bmatrix} | & | & & | \\ X^i_1 & X^i_2 & \cdots & X^i_n \\ | & | & & | \end{bmatrix}^i_\ell$$

<u>Defn</u>: $dx^1 \wedge \cdots \wedge dx^n (X_1, \dots, X_n) = \det(X^i_\ell)$

Clearly: linear, anti-symmetric

<u>Problem</u>: how does it transform under coordinate transformation $\underset{\sim}{x} \to \underset{\sim}{y}$?

- <u>Soln</u>: $\det(X^i_\ell) \equiv \sum_\pi (-1)^\pi X_1^{\pi(1)} \cdots X_n^{\pi(n)}$

"sum over all permutations π of $(1, \ldots, n)$."

$$= \sum_\pi (-1)^\pi dx^{\pi(1)} \otimes \cdots \otimes dx^{\pi(n)} (X_1, \ldots, X_n)$$

$$= T_{i_1 \cdots i_n} dx^{i_1} \otimes \cdots \otimes dx^{i_n} (X_1, \ldots, X_n)$$

where $T_{i_1 \cdots i_n} = (-1)^\pi \quad \pi : (1, \ldots, n) \to (i_1 \cdots i_n)$

Now let $\underset{\sim}{x} \to \underset{\sim}{y}$ be a coord transformation:

$$T = T_{i_1 \cdots i_n} dx^{i_1} \otimes \cdots \otimes dx^{i_n}$$

$$\Rightarrow \quad T = \overline{T}_{\alpha_1 \cdots \alpha_n} dy^{\alpha_1} \otimes \cdots \otimes dy^{\alpha_n}$$

where

$$\overline{T}_{\alpha_1 \cdots \alpha_n} = T_{i_1 \cdots i_n} \frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_n}}{\partial y^{\alpha_n}}$$

But:

$$T_{i_1 \cdots i_n} \frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_n}}{\partial y^{\alpha_n}} = \sum_{\pi} (-1)^{\pi} \frac{\partial x^{\pi(1)}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{\pi(n)}}{\partial y^{\alpha_n}}$$

$$= (-1)^{\bar{\pi}} \det \frac{\partial x}{\partial y} \qquad \bar{\pi} : (1, \cdots, n) \to (\alpha_1 \cdots \alpha_n)$$

$\therefore$ in y-coordinates,

$$T = T_{\alpha_1 \cdots \alpha_n} dy^{\alpha_1} \otimes \cdots \otimes dy^{\alpha_n} = \sum_{\bar{\pi}} (-1)^{\bar{\pi}} \det \frac{\partial x}{\partial y} dy^{\alpha_1} \otimes \cdots \otimes dy^{\alpha_n}$$

$$= \det \frac{\partial x}{\partial y} \sum_{\bar{\pi}} (-1)^{\bar{\pi}} dy^{\alpha_1} \otimes \cdots \otimes dy^{\alpha_n}$$

$$\equiv \det \frac{\partial x}{\partial y} dy^1 \wedge \cdots \wedge dy^n.$$

Conclude:

$$dx^1 \wedge \cdots \wedge dx^n = \det \frac{\partial x}{\partial y} dy^1 \wedge \cdots \wedge dy^n$$

has the desired transformation properties.

- Invariant defn of integral: we call

$$\alpha = a(\underset{\sim}{x})\, dx^1 \wedge \cdots \wedge dx^n, \quad (a \text{ a scalar depending on coords})$$

a volume form, or n-form. Then

$$\int_\Omega f\alpha = \int_{\Omega_x} f(x)\, a(\underset{\sim}{x})\, dx^1 \wedge \cdots \wedge dx^n$$

by which we mean

$$= \int_{\Omega_x} f(x)\, a(\underset{\sim}{x})\, dx^1 \cdots dx^n.$$

Then in y-coords,

$$\alpha = a(\underset{\sim}{x} \circ \underset{\sim}{y}) \det \frac{\partial x}{\partial y}\, dy^1 \wedge \cdots \wedge dy^n$$

$$\Rightarrow \int_\Omega f\alpha = \int_{\Omega_y} f(y)\, a(\underset{\sim}{x} \circ \underset{\sim}{y}) \det \frac{\partial x}{\partial y}\, dy^1 \wedge \cdots \wedge dy^n$$

$$= \int_{\Omega_y} f(y)\, a(\underset{\sim}{x} \circ \underset{\sim}{y}) \det \frac{\partial x}{\partial y}\, dy^1 \cdots dy^n$$

gives the invariant defn of $\int_\Omega f\alpha$.

- Another Way: Define

$$\int_\Omega f\alpha \equiv \int_{\Omega_x} f(x)\, \underbrace{\alpha\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)}_{a(x)\cdot 1} dx^1 \cdots dx^n$$

Then in y-coordinates

$$\int_\Omega f\alpha \equiv \int_{\Omega_y} f(x)\, \underbrace{\alpha\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right)}_{a(x)\det\frac{\partial x}{\partial y}\cdot 1} dy^1 \cdots dy^n$$

In this picture, "$\alpha(X_1, \dots, X_n)$ eats the n-vectors $X_1 \cdots X_n$ and evaluates the α-measure of the n-volume of the ll-opiped spanned by $X_1 \cdots X_n$."

Really: $\alpha_p(X_1, \dots, X_n) \equiv \dfrac{\alpha\text{-Volume of } X_1 \cdots X_n}{\text{coord volume}}$ in coord system with $\frac{\partial}{\partial x^i} = X_i$ at P.

- ORIENTATION: Let σ be an n-form defined on M. Then in coordinates,

$$\sigma = a(\underset{\sim}{x})\, dx^1 \wedge \cdots \wedge dx^n.$$

Defn: an orientation on M is a non-zero n-form defined on M.

I.e., given any coord system $\underset{\sim}{x}$, we say $\underset{\sim}{x}$ is pos oriented wrt α if $a(\underset{\sim}{x}) > 0$. Since in y-coords,

$$\sigma = a(\underset{\sim}{x} \circ \underset{\sim}{y}) \det \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}}\, dy^1 \wedge \cdots \wedge dy^n$$

$$\sigma = \bar{a}(\underset{\sim}{y})\, dy^1 \wedge \cdots \wedge dy^n, \quad \bar{\sigma} = a \det \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}},$$

it follows that $\underline{y}$ is pos oriented wrt σ iff $\det \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}} > 0$, as we want.

<u>Defn</u>: a manifold M is orientable if $\exists$ an orientation for M.

<u>Homework</u>: show that the sphere <u>is</u> orientable but the torus is <u>not</u>.

Recall: a metric on M is a symmetric nonsingular $\binom{0}{2}$-tensor ~~def~~ which defines a quadratic form on each T_pM: ie.

$$g \equiv g_{ij}\, dx^i \otimes dx^j$$

$$(g^{ij}) = (g_{ij})^{-1}$$

$X = a^i \frac{\partial}{\partial x^i}$, $Y = b^i \frac{\partial}{\partial x^i}$

$$\langle X_p, Y_p \rangle = g(X,Y)_p = g_{ij}\, a^i b^j$$

- Canonical map $T_pM \to T_p^*M$ (raising/lowering)

$$T_pM \ni X \longleftrightarrow \omega \in T_p^*M$$

$$a^i \frac{\partial}{\partial x^i} \longleftrightarrow a_i\, dx^i$$

$$a_i = g_{i\sigma}\, a^\sigma$$

$$g^{i\sigma} a_\sigma = a^i$$

$\Rightarrow \langle X, Y \rangle = \omega(Y)$ $\forall\ Y \in T_pM$

- ~~This extends to a mapping of contravariant indices to covariant ones:~~

$$\cancel{T_{ij}\, dx^i \otimes dx^j \longleftrightarrow T^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \quad \| \quad T^{ij} = T_{\sigma\tau}\, g^{\sigma i} g^{\tau j}}$$

~~etc~~

- The canonical mapping extends to raising & lowering of any indice on a tensor:

$$T^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j \longleftrightarrow T_{ij}\, dx^i \otimes dx^j$$

$$T_{ij} = T^\sigma{}_j\, g_{\sigma i}$$

- <u>Thm</u>: The raising / lowering of two covariant/contravariant indices preserves symmetry / antisymmetry wrt the two indices.

Eg. $T_{ij\cdots}^{\cdots} = T^{\sigma\tau\cdots}_{\cdots}\, g_{i\sigma}\, g_{j\tau}$

then $T_{ij\cdots}^{\cdots} = T_{ji\cdots}^{\cdots}$ iff $T^{\sigma\tau\cdots}_{\cdots} = T^{\tau\sigma\cdots}_{\cdots}$

and $T_{ij\cdots}^{\cdots} = -T_{ji\cdots}^{\cdots}$ iff $T^{\sigma\tau\cdots}_{\cdots} = -T^{\tau\sigma\cdots}_{\cdots}$

FIP

<u>Defn</u>: $\Lambda^k \equiv$ set of all completely antisymmetric $\binom{0}{k}$ - tensors $\equiv$ differential forms of order k

$$\equiv \left\{ T_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} : T_{i_1 \cdots i \cdots j \cdots i_k} = -T_{i_1 \cdots j \cdots i \cdots i_k} \right\}$$

$\Lambda_k \equiv$ set of all completely antisymmetric $\binom{k}{0}$ - tensors $\equiv$ co-differential forms of order k

$$\equiv \left\{ T^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} : T^{i_1 \cdots i \cdots j \cdots i_k} = -T^{i_1 \cdots j \cdots i \cdots k} \right\}$$

- <u>Conclude</u>: The metric g induces a canonical mapping from $\Lambda^k \to \Lambda_k$ by

$$T_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k} \leftrightarrow T^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}$$

$$\boxed{T_{i_1 \cdots i_k} = T^{\sigma_1 \cdots \sigma_k}\, g_{\sigma_1 i_1} \cdots g_{\sigma_k i_k}\,.}$$

This in turn induces an <u>inner product</u> on Λ^k and Λ_k: e.g., $S, T \in \Lambda_k$,

$$S = S^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}$$

$$T = T^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}$$

$$\boxed{\langle S, T \rangle = S_{i_1 \cdots i_k} T^{i_1 \cdots i_k}}$$

- It turns out — this determines k-Vol induced by g (as we shall see) start

☒ $\exists$ a natural volume form associated with a metric g defined on M:

- $\forall p \in M$ $\exists$ coord system defined in a nbhd of p in which

$$g_{ij} = d_{ij} = \mathrm{diag}\{\sigma_1, \dots, \sigma_n\}$$

$$\sigma_1 = \dots = \sigma_r = -1, \quad \sigma_{r+1} = \dots = \sigma_n = 1$$

$$s \equiv \mathrm{sgn}(g) \equiv \sum_i \sigma_i, \quad s = -r + (n-r) = n - 2r$$

Assume g has $\mathrm{sgn}(g) = s$ at every p

eg. Pos. definite case: $\mathrm{sgn}(g) = n$, $d_{ij} = \mathrm{id}_{ij}$

Lorentzian case $n = 4$, $s = 2$, $d_{ij} = \eta_{ij}$

- We derive a volume form that is naturally associated with g:

Fix $p \in M$, and choose a coord. system $\underline{x}$ that is orthonormal at p, i.e.,

$$g_{ij}(p) = d_{ij} \equiv \operatorname{diag}\{\sigma_1 \cdots \sigma_n\}.$$

Note first that

$$\det(d_{ij}) = (-1)^r,$$

and let

$$g \equiv g(\underline{x}) \equiv \det g_{ij}(\underline{x}).$$

Recall that $\det T^i_j$ of a $\binom{1}{1}$-tensor is a coordinate independent number at each p, but <u>not</u> so for $\binom{0}{2}$-tensors; thus $g \equiv g(\underline{x})$ is a coordinate <u>dependent</u> number at each pt p.

In $\underline{x}$-coord's,

$$g = \det g_{ij} = \det d_{ij} = (-1)^r \text{ at } p.$$

- Since $\underline{x}$ is O.N. at p, it is natural to define the volume form so that $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)_p$ span the unit volume, ie

$$\eta = \pm dx^1 \wedge \cdots \wedge dx^n .$$

(We can choose + here to set a positively oriented frame). To see that this is consistent, let $\underset{\sim}{y}$ be any othe coord frame that is O.N. at p, ie

$$\bar{g}_{\alpha\beta}(\underset{\sim}{x}_0) = d_{ij} = \mathrm{diag}\{\sigma_1 \cdots \sigma_n\}.$$

Then

$$\bar{g}_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$$

$$\det \bar{g}_{\alpha\beta} = \det g_{ij} \left(\det \frac{\partial x^i}{\partial y^\alpha}\right)^2$$

$$\Rightarrow \qquad \det \frac{\partial x^i}{\partial y^\alpha} = \pm 1 \quad \left(\begin{array}{c}\pm \text{ depending on} \\ \text{whether } \underset{\sim}{x} \text{ is pos} \\ \text{oriented wrt } y\end{array}\right)$$

Thus in another ON frame $\underline{y}$ that is positively oriented wrt $\underline{x}$ we have

$$\eta = + \det\left(\frac{\partial y^\alpha}{\partial x^i}\right) dy^1 \wedge \cdots \wedge dy^n$$

$$= + dy^1 \wedge \cdots \wedge dy^n .$$

<u>Conclude</u>: The volume form η agrees with the <u>Euclidean</u> <u>coordinate</u> <u>volume</u> in each positively oriented ON-frame at P.

- <u>Theorem</u>: The volume form η which has value

$$\eta = dx^1 \wedge \cdots \wedge dx^n$$

in each positively oriented frame $\underset{\sim}{x}$ at P, $g_{ij}(\underset{\sim}{x}(P)) = \delta_{ij}$, takes the value

$$\eta = \sqrt{|g|}\, dy^1 \wedge \cdots \wedge dy^n$$

in an arbitrary frame positively oriented with $\underset{\sim}{x}$.

Proof: Let y be an arbitrary coordinate system pos. oriented wrt $\underset{\sim}{x}$. Then in ~~y~~ coordinates,

$$\eta = dx^1\wedge\cdots\wedge dx^n = \left(\det \frac{\partial \cancel{x}}{\partial \underset{\sim}{y}}\right) dy^1\wedge\cdots\wedge dy^n.$$

But also

$$\det d_{ij} = \det g_{ij} = \det\left(\bar{g}_{\alpha\beta}\,\frac{\partial y^\alpha}{\partial x^i}\,\frac{\partial y^\beta}{\partial x^j}\right)$$

$$\Rightarrow \qquad \det(d_{ij}) = \det \bar{g}_{\alpha\beta}\,\left|\det\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}}\right|^2$$

In particular, $\left|\det \frac{\partial y}{\partial \underset{\sim}{x}}\right|^2 > 0 \Rightarrow$

$$\operatorname{sign}\{\det d_{ij}\} = \operatorname{sign}\{\det \bar{g}_{\alpha\beta}\} = (-1)^r.$$

Also

$$\boxed{\det\frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{x}} = \sqrt{\frac{(-1)^r}{\det \bar{g}_{\alpha\beta}}} \iff \det\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{y}} = \sqrt{(-1)^r \det \bar{g}_{\alpha\beta}}}$$

Thus

$$\eta = \sqrt{(-1)^r \det \bar{g}_{\alpha\beta}}\; dy^1 \wedge \cdots \wedge dy^n$$

$$\boxed{\eta = \sqrt{|g|}\; dy^1 \wedge \cdots \wedge dy^n}$$

gives a formula for the volume form in any coord frame $\underset{\sim}{y}$ that is positively oriented wrt $\underset{\sim}{x}$

Notation: $\mathcal{E} \equiv dx^1 \wedge \cdots \wedge dx^n$ is a coordinate dependent $\binom{0}{n}$-tensor called the Levi-Civita tensor. It evaluates coordinate n-volume. Thus

$$\eta = \pm\sqrt{g}\,\mathcal{E}$$

where we choose + if $\underset{\sim}{x}$ is positively oriented with the canonical coordinate system (at each p) in which

$$g_{ij} = d_{ij} = \mathrm{diag}\{\sigma_1 \cdots \sigma_n\}.$$

Thus - in ON frames, the metric-volume form agrees with the Euclidean coord. volume form $\mathcal{E}$.

▣ k-dimensional volume induced by g:

↦ Gen theory w/o metric:

Defn: Let $\Lambda^k(p)$, $\Lambda_k(p)$ denote the space of completely antisymmetric $\binom{0}{k}$-tensors, $\binom{k}{0}$-tensors, respectively, defined at $p\in M$. Let Λ^k, Λ_k denote the space of completely antisymmetric $\binom{0}{k}$-, $\binom{k}{0}$-tensor fields defined on M, resp.

Lemma ①: $\Lambda^k(p)$ & $\Lambda_k(p)$ are vector spaces at each p FIP.

We now construct a natural basis for $\Lambda^k(p)$, $\Lambda_k(p)$

• Note: The construction of the spaces Λ^k, Λ_k applies to an arbitrary manifold w/o a metric g

- Let $\underset{\sim}{x}$ be a coord system defined in a nbhd of $p \in M$. <u>Define</u>

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k \qquad (*)$$

at each $p \in M$ by:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} (X_1, \ldots, X_k)$$
$$= \det_{i_1 \cdots i_k} \begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix}$$

where $\begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix}$ denotes the $n \times k$ matrix obtained by putting the $\underset{\sim}{x}$-coords of X_i in the ith colm., and $\det_{i_1 \cdots i_k}$ denotes the det taken in the $i_1 \cdots i_k$ rows, ~~re-ordered by~~ in the order $i_1 \cdots i_k$.

- <u>Note</u> $(*)$ is a coord. dependent expression

• Observe: $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ defines a multilinear, antisymmetric function on k-vectors $X_1 \cdots X_k$, and thus defines an element of Λ^k. Moreover, by antisymmetry,

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}(X_1, \cdots, X_k) = 0$$

if $i_\ell = i_m$ or $X_\ell = X_m$. FIP

• We obtain the tensor components of $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ in terms of the basis $\{dx^{i_1} \otimes \cdots \otimes dx^{i_k}\}$ for $\mathcal{Y}^0_k \supset \Lambda^k$ at each $p \in M$:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k}\left[\underset{|}{X_1} \cdots \underset{|}{X_k} \right] = \sum_\pi (-1)^\pi X_1^{i_{\pi(1)}} \cdots X_k^{i_{\pi(k)}}$$

$$\pi : \{1, \cdots, k\} \to \{1, \cdots, k\}$$

$$= \sum_\pi (-1)^\pi dx^{i_{\pi(1)}} \otimes \cdots \otimes dx^{i_{\pi(k)}} (X_1, \cdots, X_k)$$

$$= T_{j_1 \cdots j_k} dx^{j_1} \otimes \cdots \otimes dx^{j_k}$$

where

$$T_{j_1\cdots j_k} = \begin{cases} 1 & \text{if } (j_1\cdots j_k) = (i_1\cdots i_k) \\ (-1)^{\pi} & \text{if } (j_1\cdots j_k) = (i_{\pi(1)}\cdots i_{\pi(k)}) \\ 0 & \text{otherwise.} \end{cases}$$

Note: By defn of $dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ as taking a $k\times k$ det., we have

$$dx^{i_1}\wedge\cdots\wedge \underbrace{dx^{i}\wedge dx^{j}}\wedge\cdots\wedge dx^{i_k} = -dx^{i_1}\wedge\cdots\wedge \underbrace{dx^{j}\wedge dx^{i}}\wedge\cdots\wedge dx^{i_k}$$

Theorem: $\{dx^{i_1}\wedge\cdots\wedge dx^{i_k},\ 0<i_1<i_2<\cdots<i_k\leq n\}$ forms a basis for $\Lambda^k(p)$ at each p. Thus, $\dim \Lambda^k = \binom{n}{k}$, and $T\in\Lambda^k \Rightarrow$

$$T = T_{|i_1\cdots i_k|}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

where $|i_1\cdots i_k|$ means sum over all increasing sequences of indices

- <u>Conclude</u>: every k-form can be written as:

$$T_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}\,, \quad T_{i_{\pi(1)} \cdots i_{\pi(k)}} = (-1)^{\pi} T_{i_1 \cdots i_k}$$

$$T_{|i_1 \cdots i_k|}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad (\star)$$

$$\frac{1}{k!} T_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

I.e. $\Lambda^k \ni T = T_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$

$$= \sum_{|i_1 \cdots i_k|} \sum_{\pi} (-1)^{\pi}\, T_{i_1 \cdots i_k}\, dx^{i_{\pi(1)}} \otimes \cdots \otimes dx^{i_{\pi(k)}}$$

$$= \sum_{|i_1 \cdots i_k|} T_{i_1 \cdots i_k} \sum_{\pi} (-1)^{\pi}\, dx^{i_{\pi(1)}} \otimes \cdots \otimes dx^{i_{\pi(k)}}$$

$$= T_{|i_1 \cdots i_k|}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$= \frac{1}{k!} T_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad \checkmark$$

- Transformation Law:

$$T = T_{|i_1 \cdots i_k|}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \underset{\sim}{x} \longrightarrow \underset{\sim}{y}$$

$$\Rightarrow T = T_{|\alpha_1 \cdots \alpha_k|}\, dy^{\alpha_1} \wedge \cdots \wedge dy^{\alpha_k}$$

where

$$T_{\alpha_1 \cdots \alpha_k} = T_{|i_1 \cdots i_k|} \det{}^{|i_1 \cdots i_k|}_{\alpha_1 \cdots \alpha_k} \frac{\partial x}{\partial y}$$

Pf.

$$T = T_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

$$= T_{i_1 \cdots i_k} \frac{\partial x^{i_1}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{\alpha_k}}\, dy^{\alpha_1} \otimes \cdots \otimes dy^{\alpha_k}$$

$$= \sum_{|i_1 \cdots i_k|} T_{|i_1 \cdots i_k|} \sum_{\pi} (-1)^{\pi} \frac{\partial x^{i_{\pi(1)}}}{\partial y^{\alpha_1}} \cdots \frac{\partial x^{i_{\pi(k)}}}{\partial y^{\alpha_k}}\, dy^{\alpha_1} \otimes \cdots \otimes dy^{\alpha_k}$$

$$= \underbrace{T_{|i_1 \cdots i_k|} \det{}^{|i_1 \cdots i_k|}_{\alpha_1 \cdots \alpha_k} \frac{\partial x}{\partial y}}_{= T_{\alpha_1 \cdots \alpha_k}}\, dy^{\alpha_1} \otimes \cdots \otimes dy^{\alpha_k}$$

Proof of Theorem:

We first show that

$$\{ dx^\lambda \equiv dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_k}, 0 < \lambda_1 < \cdots < \lambda_k < n \}$$

form an independent set of vectors in Λ^k.

I.e., if

$$\alpha_{|\lambda|} dx^\lambda = 0, \quad \text{(sum over } \lambda \text{ increasing)}$$

then

$$\alpha_{|\lambda|} dx^\lambda [\underset{|}{\overset{|}{X_1}} \cdots \underset{|}{\overset{|}{X_k}}] = 0$$

$\forall \{X_1 \cdots X_k\}$. Choose $X_1 \cdots X_k = \frac{\partial}{\partial x^{\lambda_1}}, \cdots, \frac{\partial}{\partial x^{\lambda_k}}$

Then

$$dx^\mu \left(\frac{\partial}{\partial x^{\lambda_1}} \cdots \frac{\partial}{\partial x^{\lambda_k}}\right) = 0 \quad \mu \neq \lambda,$$

so

$$dx^\lambda \left(\frac{\partial}{\partial x^{\lambda_1}} \cdots \frac{\partial}{\partial x^{\lambda_k}}\right) = 1.$$

Thus

$$0 = \alpha_\lambda dx^\lambda \left[\frac{\partial}{\partial x^{\lambda_1}} \cdots \frac{\partial}{\partial x^{\lambda_k}} \right] = \alpha_\lambda .$$

This holds $\forall \lambda \Rightarrow \{dx^\lambda\}$ are indep't.

We now show $\{dx^\lambda\}$ spans Λ^k.

Let $\omega \in \Lambda^k$. Then

$$\omega(X_1 \cdots X_k) = \omega\left(X_1^{i_1} \frac{\partial}{\partial x^{i_1}} \cdots X_k^{i_k} \frac{\partial}{\partial x^{i_k}} \right)$$

$$= X_1^{i_1} \cdots X_k^{i_k} \, \omega\left(\frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}} \right)$$

where each i_r sums from $1 \cdots n$. Now partition $\{1, \ldots, n\}$ into $\binom{n}{k}$ increasing sequences $M_1 \cdots M_{\binom{n}{k}}$, each of k-length.

Thus

$$\omega(X_1 \cdots X_k) = \sum_{i=1}^{\binom{n}{k}} \sum_{\lambda = \pi(\mu_i)} X_1^{\lambda_1} \cdots X_k^{\lambda_k}\, \omega\left(\frac{\partial}{\partial x^{\lambda_1}} \cdots \frac{\partial}{\partial x^{\lambda_k}}\right)$$

where $\mu_i = (i_1 \cdots i_k)$, $\pi(\mu_i) = (i_{\pi(1)} \cdots i_{\pi(k)})$. Thus

$$\omega(X_1 \cdots X_k) = \sum_{i=1}^{\binom{n}{k}} \sum_{\pi} (-1)^{\pi} X_1^{i_{\pi(1)}} \cdots X_k^{i_{\pi(k)}}\, \omega\left(\frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}}\right)$$

$$= \sum_{i=1}^{\binom{n}{k}} dx^{\mu_i}(X_1 \cdots X_k)\, \underbrace{\omega\left(\frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_k}}\right)}_{\alpha_{\mu_i}}$$

$$\omega = \sum_{i=1}^{\binom{n}{k}} \alpha_{\mu_i}\, dx^{\mu_i} \checkmark$$

- **Wedge Product:** Our defn of $dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ is as a det operator,

$$dx^{i_{\pi(1)}}\wedge\cdots\wedge dx^{i_{\pi(k)}} = (-1)^{\pi}\, dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

We can extend $\wedge$ to operation on $\biguplus_k \Lambda^k$ by linearity & antisymmetry:

$$\Lambda^k \ni \alpha = \alpha_{|\lambda|}\, dx^{\lambda}$$

$$\Lambda^l \ni \beta = \beta_{|\mu|}\, dx^{\mu}$$

Defn: $\alpha\wedge\beta \in \Lambda^{k+l}$

(1) $$\alpha\wedge\beta = \alpha_{|\lambda|}\beta_{|\mu|}\, dx^{\lambda}\wedge dx^{\mu} \quad \left(\begin{array}{c}\text{sum over increasing} \\ \lambda, \mu\end{array}\right)$$

(2) $$dx^{\lambda}\wedge dx^{\mu} = dx^{\lambda_1}\wedge\cdots\wedge dx^{\lambda_k}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_l}.$$

Thm: (1) & (2) define $\alpha\wedge\beta$ as a unique ele of Λ^{k+l}, indept of which coord system (1), (2) are performed in.

• Said Differently: assuming (2) together with $\alpha \wedge \beta = (-1)^{k+l} \beta \wedge \alpha$ and multilinearity in each slot,

$$\alpha \wedge (c_1 \beta^1 + c_2 \beta^2) = c_1 \alpha \wedge \beta^1 + c_2 \alpha \wedge \beta^2$$

$$(c_1 \alpha^1 + c_2 \alpha^2) \wedge \beta = c_1 \alpha^1 \wedge \beta + c_2 \alpha^2 \wedge \beta,$$

defn (1) is implied.

Problem: It is not so easy to prove directly (and it is a bit remarkable) that (1) defines a coord. indept operation i.e., given $\underline{y}$-coords,

$$\alpha = \bar{\alpha}_{\bar{\lambda}} \, dy^{\bar{\lambda}}$$

$$\beta = \bar{\beta}_{\bar{\mu}} \, dy^{\bar{\mu}}$$

we need

$$\alpha \wedge \beta = \alpha_\lambda \beta_\mu \, dx^\lambda \wedge dx^\mu = \bar{\alpha}_{\bar{\lambda}} \bar{\beta}_{\bar{\mu}} \, dy^{\bar{\lambda}} \wedge dy^{\bar{\mu}}.$$

We'd like to argue that

" $\bar{\alpha}_{\bar{\lambda}} = \alpha_{|\lambda|} \dfrac{\partial x^{\lambda}}{\partial y^{\bar{\lambda}}}$ "

" $\bar{\beta}_{\bar{\mu}} = \beta_{|\mu|} \dfrac{\partial x^{\mu}}{\partial y^{\bar{\mu}}}$ "

$$\Rightarrow \quad \underset{\text{LHS}}{\alpha_{|\lambda|}\beta_{|\mu|}\, dx^{\lambda}\wedge dx^{\mu}} = \underset{\text{RHS}}{\bar{\alpha}_{|\bar{\lambda}|}\,\bar{\beta}_{|\bar{\mu}|}\,\frac{\partial x^{\bar{\lambda}}}{\partial y^{\lambda}}\frac{\partial x^{\bar{\mu}}}{\partial y^{\mu}}\, dy^{\bar{\lambda}}\wedge dy^{\bar{\mu}}}$$

<u>Problem</u>: although $dx^{\lambda}\wedge dx^{\mu} = 0$ unless $\gamma = (\lambda, \mu) = (\lambda_1, \dots, \lambda_k, \mu_1, \dots \mu_l)$ is a reordering of $\{1, \dots, k+l\}$, $\exists$ complicated coeff's when we use antisymmetry to ~~reorder~~ collect all the coeff's of the permutation of the increasing sequence. I.e. how to relate

$$\alpha_{|\lambda|}\beta_{|\mu|}\, dx^{\lambda}\wedge dx^{\mu} = \gamma_{|\hat{\gamma}|}\, dx^{\hat{\gamma}}$$

- To prove theorem we give an invariant formula for $\wedge$; then prove that this agrees on k-forms $dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ and is multilinear & anti-symmetric $\Rightarrow$ (1) holds.

<u>Defn</u> (1) Let $\omega^1,\ldots,\omega^k$ be elements of T^*M. Define $\omega^1\wedge\cdots\wedge\omega^k$ to be the k-form that acts on k-vectors $X_1,\ldots,X_k$ by

$$\omega^1\wedge\cdots\wedge\omega^k(X_1\cdots X_k) = \det\left[\omega^i(X_j)\right]. \quad (3)$$

- Clearly (3) is multilinear, antisymm, and agrees with defn of $dx^{i_1}\wedge\cdots\wedge dx^{i_k}$
- Any k-form that can be expressed as $\omega^1\wedge\cdots\wedge\omega^k$ some $\omega^i\in T^*M$, is called <u>decomposable</u>.

Defn ② Let $\alpha \in \Lambda^k$, $\beta \in \Lambda^l$. Define

$$\alpha \wedge \beta(X_1 \cdots X_k X_{k+1} \cdots X_{k+l}) \tag{4}$$
$$= \sum_{|\lambda||\mu|} (-1)^{\pi(\lambda,\mu)} \alpha(X_{\lambda_1} \cdots X_{\lambda_k}) \beta(X_{\mu_1} \cdots X_{\mu_l})$$

where the sum is over $\lambda = (\lambda_1 \cdots \lambda_k)$, $\mu = (\mu_1 \cdots \mu_l)$ satisfying $\lambda_1 < \cdots < \lambda_k$, $\mu_1 < \cdots < \mu_l$, where $(\lambda, \mu) = (\lambda_1 \cdots \lambda_k, \mu_1 \cdots \mu_l)$ is a reordering of $(1, \ldots, k+l)$ of parity π.

Note: (4) is an invariant defn $\Rightarrow$ coord. indept meaning.

<u>Theorem</u>: Defn (4) has following properties:

① If α, β are decomposible, then (4) agrees with the defn (3) for $\alpha \wedge \beta$.

② <u>Antisymmetry</u>: $\alpha \wedge \beta = (-1)^{k\ell}\, \beta \wedge \alpha$

③ <u>Multilinearity</u>: $\alpha \wedge c_1\beta^1 + c_2\beta^2 = c_1\alpha \wedge \beta^1 + c_2 \alpha \wedge \beta^2$

$$(c_1\alpha^1 + c_2\alpha^2) \wedge \beta = c_1\alpha^1 \wedge \beta + c_2\alpha^2 \wedge \beta.$$

④ <u>Associativity</u>: $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

~~Cor: ②, ③ imply~~ $\cancel{\alpha_{|\lambda|} dx^\lambda \wedge \beta_{|\mu|} dx^\mu = \alpha_{|\lambda|}\beta_{|\mu|} dx^\lambda \wedge dx^\mu}$

<u>Corollary</u>:

⑤ $$\alpha_{|\lambda|}\, dx^\lambda \wedge \beta_{|\mu|}\, dx^\mu = \alpha_{|\lambda|}\beta_{|\mu|}\, dx^\lambda \wedge dx^\mu$$

(Follows directly from ①-③)

Pf of thm:

(1) $\alpha = \omega^1 \wedge \cdots \wedge \omega^k \quad \beta = \omega^{k+1} \wedge \cdots \wedge \omega^{k+\ell}$

$$\alpha \wedge \beta = \sum_{|\lambda||\mu|} (-1)^{\pi(\lambda,\mu)} \det\left[\omega^i(X_{\lambda_j})\right]_{i=1,\dots,k} \det\left[\omega^i(X_{\mu_j})\right]_{i=k+1,\dots,k+\ell}$$

$$\det\left[\omega^i(X_{\lambda_j})\right] = \sum_{\pi} (-1)^{\pi} \omega^1(X_{\lambda_{\pi(1)}}) \cdots \omega^k(X_{\lambda_{\pi(k)}})$$

$$\det\left[\omega^i(X_{\mu_j})\right] = \sum_{\pi'} (-1)^{\pi'} \omega^{k+1}(X_{\mu_{\pi'(1)}}) \cdots \omega^{k+\ell}(X_{\mu_{\pi'(\ell)}})$$

$$\alpha \wedge \beta = \boxed{\cancel{\sum_{|\lambda||\mu|} (-1)^{\pi(\lambda,\mu)}}} = \sum_{|\lambda||\mu|} (-1)^{\pi(\lambda,\mu)} \left(\sum_{\pi} (-1)^{\pi} \omega^1(X_{\lambda_{\pi(1)}}) \cdots \omega^k(X_{\lambda_{\pi(k)}}) \right)$$

$$\cdot \left(\sum_{\pi'} (-1)^{\pi'} \omega^{k+1}(X_{\mu_{\pi'(1)}}) \cdots \omega^{k+\ell}(X_{\mu_{\pi'(\ell)}}) \right)$$

$$= \underbrace{\sum_{|\lambda||\mu|} \sum_{\pi, \pi'}}_{\text{names an arb perm. } \gamma \text{ of } (1,\dots,k+\ell)} (-1)^{\pi} (-1)^{\pi'} (-1)^{\pi(\lambda,\mu)} \omega^1(X_{\lambda_{\pi(1)}}) \cdots \omega^k(X_{\lambda_{\pi(k)}})$$

$$\cdot\, \omega^{k+1}(X_{\mu_{\pi'(1)}}) \cdots \omega^{k+\ell}(X_{\mu_{\pi'(\ell)}})$$

$$= \sum_{\gamma} (-1)^{\gamma} \omega^1(X_{\gamma(1)}) \cdots \omega^{k+\ell}(X_{\gamma(k+\ell)}) = \det\left[\omega^i(X_j)\right] \checkmark$$

Proof of Thm

② $(-1)^{kl}$ is the parity associated with moving l numbers from the right to the left of k numbers in a permutation w.o. otherwise changing the ordering:

$$(\lambda_1 \cdots \lambda_k\, \mu_1 \cdots \mu_l) \to (\mu_1\, \underbrace{\lambda_1 \cdots \lambda_k}_{k}\, \mu_2 \cdots \mu_l)$$

$$\to (\mu_1\, \mu_2\, \underbrace{\lambda_1 \cdots \lambda_k}_{k}\, \mu_3 \cdots \mu_l)$$

$$\vdots$$

$$\to (\mu_1 \cdots \mu_l\, \lambda_1 \cdots \lambda_k).$$

Thus: $\beta \wedge \alpha (X_1 \cdots X_{k+l}) = \sum_{|\mu||\lambda|} (-1)^{\pi(\lambda\mu)} \alpha(X_{\lambda_1} \cdots X_{\lambda_k}) \cdot \beta(X_{\mu_1} \cdots \mu_l)$

HW

$$= \sum_{|\lambda||\mu|} (-1)^{kl} (-1)^{\pi(\mu\lambda)} \alpha(X_{\lambda_1} \cdots X_{\lambda_k}) \beta(X_{\mu_1} \cdots X_{\mu_l}) = (-1)^{kl} \alpha \wedge \beta$$

✓

3 $\alpha \wedge (C_1 B^1 + C_2 B^2)$

$$= \sum_{|\lambda||\mu|} (-1)^{\pi(\lambda,\mu)} \alpha(X_{\lambda_1} \cdots X_{\lambda_k}) \Big\{ C_1 B^1(X_{\mu_1} \cdots X_{\mu_\ell}) + C_2 B^2(X_{\mu_1}, \cdots, X_{\mu_\ell}) \Big\}$$

$$= C_1 \sum_{|\lambda||\mu|} (-1)^{\pi(\lambda,\mu)} \alpha(X_{\lambda_1} \cdots X_{\lambda_k}) B^1(X_{\mu_1} \cdots X_{\mu_\ell})$$

$$+ C_2 \sum_{|\lambda||\mu|} (-1)^{\pi(\lambda,\mu)} \alpha(X_{\lambda_1} \cdots X_{\lambda_k}) B^2(X_{\mu_1} \cdots X_{\mu_\ell})$$

$$= C_1 \alpha \wedge B_1 + C_2 \alpha \wedge B_2 .$$

From Here we conclude 5 holds, so it suffices to use 5 to verify 4

HW

(4) $\alpha = \alpha_{|\lambda|} dx^\lambda \quad \beta = \beta_{|\mu|} dx^\mu \quad \gamma = \gamma_{|\tau|} dx^\tau$ (34)

$$(\alpha \wedge \beta) \wedge \gamma = \left(\alpha_{|\lambda|} dx^\lambda \wedge \beta_{|\mu|} dx^\mu\right) \wedge \gamma_{|\tau|} dx^\tau$$

$$= \left(\alpha_{|\lambda|} \beta_{|\mu|} dx^\lambda \wedge dx^\mu\right) \wedge \gamma_{|\tau|} dx^\tau$$

HW.

$$= \alpha_{|\lambda||\mu||\tau|} \left(dx^\lambda \wedge dx^\mu\right) \wedge dx^\tau$$

$$= \alpha_{|\lambda||\mu||\tau|} dx^\lambda \wedge \left(dx^\mu \wedge dx^\tau\right)$$

$$= \alpha \wedge (\beta \wedge \gamma)$$

By orig defn as a determinant ✓

Another way to wedge product: (Spivak)

<u>Defn</u>: $T \in \mathcal{T}^0_k$

$$\mathrm{Alt}(T)(X_1 \cdots X_k) = \frac{1}{k!} \sum_{\pi} (-1)^{\pi}\, T(X_{\pi(1)} \cdots X_{\pi(k)})$$

<u>thm</u>

1. $\mathrm{Alt}(T) = T$ iff $T \in \Lambda^k$
2. $\mathrm{Alt}(T) \in \Lambda^k$ "the anti-symmetrization of T"
3. $S \wedge T = \frac{(k+l)!}{k!\, l!} \mathrm{Alt}(S \otimes T)$ $\quad S \in \mathcal{T}^0_k,\ T \in \mathcal{T}^0_l$

- Metric g induces inner product $\langle\,,\rangle$ on $\Lambda^k \Lambda_k$ that determines k-volumes:

- Notation: Given coord system $\underset{\sim}{x}$:

$$dx^\lambda \equiv dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_k}$$

$$\frac{\partial}{\partial x^\lambda} \equiv \frac{\partial}{\partial x^{\lambda_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\lambda_k}}$$

where $\{dx^\lambda\}$, $\{\frac{\partial}{\partial x^\lambda}\}$ are bases for Λ^k, Λ_k resp.

- Eg., $X_1 \wedge \cdots \wedge X_k \equiv a_{|\lambda|} dx^\lambda$ defines a unique ele. of Λ_k by antisymmetry / multi-lin. of $\wedge$

- We wish to identify $X_1 \wedge \cdots \wedge X_k$ with k-parallelopiped spanned by $X_1 \cdots X_k$.

Goal: define metric on Λ_k so

$$\langle X_1 \wedge \cdots \wedge X_k, X_1 \wedge \cdots \wedge X_k \rangle \equiv k\text{-vol}[X_1 \cdots X_k]$$ as measured by g.

- Not all ele's of Λ_k can be written as $X_1 \wedge \cdots \wedge X_k$: There are special:

Defn: $\omega \in \Lambda^k$, $\alpha \in \Lambda_k$ are decomposible if $\exists\ \omega^1, \ldots, \omega^k \in \Lambda^1$, $\alpha_1 \cdots \alpha_k \in \Lambda_1$ st

$$\omega = \omega^1 \wedge \cdots \wedge \omega^k$$

$$\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k,$$

respectively.

Lemma 1 Let $X_1, \ldots, X_k \in T_pM$, $\{X_i\}$ indept. Let $\mathrm{Span}\{X_i\} = \mathrm{Span}\{N_i\}$, $i = 1, \ldots, k$. Then

$$X_1 \wedge \cdots \wedge X_k = \det(X^i_j)\, N_1 \wedge \cdots \wedge N_k$$

where $X_i = X_i^\sigma N_\sigma$.

Pf. $$X_1 \wedge \cdots \wedge X_k = X_1^{\sigma} N_\sigma \wedge \cdots \wedge X_k^{\sigma} N_\sigma$$

$$= \sum_{\pi} (-1)^{\pi} X_1^{\pi(1)} \cdots X_k^{\pi(k)} N_1 \wedge \cdots \wedge N_k$$

$$= \det(X^i_j) N_1 \wedge \cdots \wedge N_k \checkmark$$

<u>Cor</u>: $X_1 \wedge \cdots \wedge X_k \neq 0$ iff $\{X_i\}$ are indept.

Pf. Choose $N_1 \cdots N_\ell$ indept s.t. $\operatorname{Span}\{N_1 \cdots N_\ell\} = \operatorname{Span}\{X_1 \cdots X_k\}$.

Then as above

$$X_1 \wedge \cdots \wedge X_k = X_1^{\sigma} N_\sigma \wedge \cdots \wedge X_k^{\sigma} N_\sigma$$

$$= \det\underbrace{(X^i_j)}_{\ell \times k} N_1 \wedge \cdots \wedge N_\ell$$

$$= 0 \text{ if } \ell < k \checkmark$$

<u>Cor</u>: $\Lambda^k = \Lambda_k = 0$ for $k > n$.

☒ Assume a metric g with components in ON-frame given by

$$g_{ij} \equiv d_{ij} = \mathrm{diag}\{\sigma_1 \cdots \sigma_n\}$$

$$\sigma_1 = \cdots = \sigma_r = -1, \quad \sigma_{r+1} = \cdots = \sigma_n = 1, \quad \mathrm{sgn}(g) = n - r$$

- this induces volume form

$$\eta = \sqrt{|\det g|}\, dx^1 \wedge \cdots \wedge dx^n$$

- The metric also induces a map

$$T^*M \longleftrightarrow TM$$

$$X^i \frac{\partial}{\partial x^i} \longleftrightarrow X_i dx^i$$

$$X^i = g^{i\sigma} X_\sigma$$

$$g^{ij} = (g_{ij})^{-1}$$

- This extends to a map (by raising / lowering)

$$y^k_0 \leftrightarrow y^0_k$$

$$T^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \leftrightarrow T_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

$$T_{i_1 \cdots i_k} = T^{\sigma_1 \cdots \sigma_k} g_{\sigma_1 i_1} \cdots g_{\sigma_k i_k}$$

Lemma 2 This induces a linear mapping

$$L : \Lambda^k \leftrightarrow \Lambda_k$$

$$a_{|\lambda|} dx^\lambda \leftrightarrow a^{|\lambda|} \frac{\partial}{\partial x^\lambda}$$

$$a_\lambda \equiv a_{\lambda_1 \cdots \lambda_k} = a^{\sigma_1 \cdots \sigma_k} g_{\sigma_1 \lambda_1} \cdots g_{\sigma_k \lambda_k} \quad (*)$$

Pf. By (☆) page 20,

$$a_{|\lambda|} dx^\lambda = a_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

where $a_\lambda = a_{\lambda_1 \cdots \lambda_k}$ for λ increasing & $a_{i_1 \cdots i_k}$ transform by $(*)$ ✓

start

- <u>Defn</u>: Let $\omega^1, \omega^2 \in \Lambda^k$; $\alpha_1, \alpha_2 \in \Lambda_k$.

$$\langle \omega^1, \omega^2 \rangle_* = \omega^1 (L\omega^2)$$

$$\langle \alpha_1, \alpha_2 \rangle_* = (L^{-1}\alpha_1)(\alpha_2)$$

- <u>Lemma</u> 3 In coordinates:

$$\langle a_{|\lambda|} dx^\lambda, b_{|\lambda|} dx^\lambda \rangle_* = a^\lambda b_\lambda = k! a^{|\lambda|} b_\lambda$$

$$\langle a^{|\lambda|} \frac{\partial}{\partial x^\lambda}, b^{|\lambda|} \frac{\partial}{\partial x^\lambda} \rangle_* = a_\lambda b^\lambda = k! a_{|\lambda|} b^\lambda$$

Pf. For the 2nd: $\alpha_1 = a^{|\lambda|} \frac{\partial}{\partial x^\lambda} \Rightarrow$

$$L^{-1}\alpha_1 = a_{|\lambda|} dx^\lambda$$

But

$$(a_{|\lambda|} dx^\lambda)(b^{|\lambda|} \tfrac{\partial}{\partial x^\lambda}) = (a_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k})(b^{i_1 \cdots i_k} \tfrac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \tfrac{\partial}{\partial x^{i_k}})$$

$$= a_{i_1 \cdots i_k} b^{i_1 \cdots i_k} = a_\lambda b^\lambda = \sum_{|\lambda|} \sum_{\pi} (-1)^{\pi(\lambda)} a_{|\lambda|} (-1)^{\pi(\lambda)} b^\lambda = k! a_{|\lambda|} b^\lambda$$

- Let $\underset{\sim}{x}$ be an ON basis. In this case

$$\left| \left\langle \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}}, \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \right\rangle \right| = k!$$

because the metric induced mapping $\Lambda^k \longleftrightarrow \Lambda_k$ yields $dx^\lambda \longleftrightarrow \pm \frac{\partial}{\partial x^\lambda}$

<u>Lemma</u> 4 if $\underset{\sim}{x}$ is O.N., then $(d_{ij} = \text{diag}(\sigma_1 \cdots \sigma_n))$

$$dx^\lambda \longleftrightarrow \sigma_{\lambda_1} \cdots \sigma_{\lambda_k} \frac{\partial}{\partial x^\lambda}.$$

<u>Proof</u>: $\alpha = \frac{\partial}{\partial x\lambda} \Rightarrow \alpha = a^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}$

where

$$a^{i_1 \cdots i_k} = \begin{cases} (-1)^\pi & \pi : (i_1 \cdots i_k) \rightarrow (\lambda_1 \cdots \lambda_k) = \lambda \\ 0 & \text{o.w.} \end{cases}$$

Thus

$$a_{i_1 \cdots i_k} = a^{j_1 \cdots j_k} d_{j_1 i_1} \cdots d_{j_k i_k}$$

$$= \sigma_{i_1} \cdots \sigma_{i_k} a^{i_1 \cdots i_k} \quad \leftarrow \text{no sum}$$

$$\Rightarrow \boxed{\alpha = \frac{\partial}{\partial x\lambda} \longleftrightarrow (\sigma_{\lambda_1} \cdots \sigma_{\lambda_k}) dx^\lambda} \quad \checkmark$$

In order that $\{dx^\lambda\}$, $\{\frac{\partial}{\partial x^\lambda}\}$ be O.N. frame for Λ^k, Λ_k resp., we redefine $\langle , \rangle_*$

<u>Defn</u>:
$$\langle \omega^1, \omega^2 \rangle = \frac{1}{k!} \langle \omega^1, \omega^2 \rangle_*$$

$$\langle \alpha_1, \alpha_2 \rangle = \frac{1}{k!} \langle \alpha_1, \alpha_2 \rangle_*$$

so that

$$\langle \alpha_{|\lambda|} dx^\lambda, \beta_{|\lambda|} dx^\lambda \rangle = \alpha_{|\lambda|} \beta^\lambda$$

$$\langle \alpha^{|\lambda|} \frac{\partial}{\partial x^\lambda}, \beta^{|\lambda|} \frac{\partial}{\partial x^\lambda} \rangle = \alpha^{|\lambda|} \beta_\lambda$$

FIP

Let $X_1, \dots, X_k$ be linearly indept vectors in $T_p\mathcal{M}$, and let $N_1 \cdots N_k\ N_{k+1} \cdots N_n$ be unit orthogonal vectors satisfying

$$\langle N_i, N_i \rangle = \sigma_{\pi(i)} = \tau_i ,$$

$$\mathrm{Span}\{N_1, \dots, N_k\} = \mathrm{Span}\{X_1, \dots, X_k\}.$$

(I.e., we cannot ensure than $N_1 \cdots N_n$ is an O.N. basis since τ_i depends on sign $\langle X_i, X_i \rangle$, and we are not free to make $\tau_i = \sigma_i$.)

- Theorem (Main)

① $\det \langle X_i, X_j \rangle = \tau_1 \cdots \tau_k \left[\det X^i_j\right]^2$ where

$$X_j = X^i_j N_i$$

② $|X_1 \wedge \cdots \wedge X_k|^2 = \langle X_1 \wedge \cdots \wedge X_k, X_1 \wedge \cdots \wedge X_k \rangle = \det\langle X_i, X_j \rangle$

③ $\eta(X_1, \dots, X_k, N_{k+1}, \dots, N_n)^2 = |\det \langle X_i, X_j \rangle|$

where $\eta = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n$ is the vol-form for g

• **Note**: $k\text{-Vol}[X_1 \cdots X_k] \equiv k$-volume of ∥-opiped spanned by $X_1 \cdots X_k$ induced by g is defined as

Defn: $k\text{-Vol}[X_1 \cdots X_k]^2 = |\eta(X_1, \dots, X_k, N_{k+1}, \dots, N_n)|$

> 0

Thus: ① & ② of Thm $\Rightarrow$

$$k\text{-Vol}[X_1 \cdots X_k]^2 = \pm |X_1 \wedge \cdots \wedge X_k|^2$$

when g is not pos def, the norm of a vector can be negative

We can thus define the signed $k\text{-vol}[X_1 \cdots X_k]^2$:

Defn: The signed $k\text{-vol}[X_1 \cdots X_k]^2$ is

$$|X_1 \wedge \cdots \wedge X_k|^2 = \det \langle X_i, X_j \rangle.$$

- Thus the final picture emerges for the metric on Λ_k that defines k-dim volume induced by g:

- Fix k-parallelopiped $X_1 \wedge \cdots \wedge X_k \in \Lambda_k$
- Choose $\underset{\sim}{x}$ to be an arbitrary o.n. frame at p:

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = d_{ij} \equiv \mathrm{diag}\{\sigma_1 \cdots \sigma_n\}$$

$\sigma_1 = \cdots = \sigma_r = -1$, $\sigma_{r+1} = \cdots = \sigma_n = 1$. Let

$$X_i = X_i^j \frac{\partial}{\partial x^j}$$

- Then:

$$X_1 \wedge \cdots \wedge X_k = X_1^{i_1} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge X_k^{i_k} \frac{\partial}{\partial x^{i_k}}$$

$$= X_1^{i_1} \cdots X_k^{i_k} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}}$$

$$= \sum_{|\lambda|} \sum_{\pi} (-1)^{\pi} X_1^{\pi(\lambda_1)} \cdots X_k^{\pi(\lambda_k)} \frac{\partial}{\partial x^{\lambda_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\lambda_k}}$$

$$= \sum_{|\lambda|} \det{}^{\lambda} \begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix} \frac{\partial}{\partial x^{\lambda_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\lambda_k}}$$

- Let
$$p^\lambda \equiv \det^\lambda \begin{bmatrix} | & & | \\ X_1 & \cdots & X_k \\ | & & | \end{bmatrix}.$$

Then $p^\lambda \equiv$ k-volume of the projection of $[X_1 \cdots X_k]$ onto the $x_{\lambda_1} \cdots x_{\lambda_k}$-coordinate axes as measured in the coordinate system x.

I.e., p^λ is the coordinate k-volume of this projection.

- Thus: $X_1 \wedge \cdots \wedge X_k = p^{|\lambda|} \dfrac{\partial}{\partial x^\lambda}$

Sum repeated up-down increasing sequences

- Therefore:

$$|X_1 \wedge \cdots \wedge X_k|^2 = \langle X_1 \wedge \cdots \wedge X_k, X_1 \wedge \cdots \wedge X_k \rangle$$
$$= (p^{|\lambda|})^2 \left\langle \frac{\partial}{\partial x^\lambda}, \frac{\partial}{\partial x^\lambda} \right\rangle$$
$$\equiv \text{signed k-vol} [X_1 \cdots X_k]^2$$

and

$$\left\langle \frac{\partial}{\partial x^\lambda}, \frac{\partial}{\partial x^\lambda} \right\rangle = \sigma_{\lambda_1} \cdots \sigma_{\lambda_k} \equiv \sigma_\lambda$$

Here: $\sigma_\lambda = \sigma_{\lambda_1} \cdots \sigma_{\lambda_k} = \pm 1$ depending on the signature of the metric

Conclude:

$$\boxed{\begin{aligned} |X_1 \wedge \cdots \wedge X_k|^2 &= \sigma_{|\lambda|} \left(P^\lambda\right)^2 \qquad (*) \\ &= \left(k\text{-vol}\,[X_1 \cdots X_k] \text{ induced by } g\right) \end{aligned}}$$

- In particular, if $d_{ij} = \delta_{ij}$, $\sigma_1 = \cdots = \sigma_n = 1$ the pos def case, then we obtain the standard projection thm in o.n. basis:

$$|X_1 \wedge \cdots \wedge X_k|^2 = \sum_{|\lambda|} \left(P^\lambda\right)^2$$

For the non-pos def. case, if we choose our $\overbrace{\text{O.N.}}^{\text{orthogonal unit}}$ frame so that

$$\frac{\partial}{\partial x^i} = N_i, \quad i = 1, \ldots, k \quad (\text{at } p),$$

and $\operatorname{Span}\{X_i\} = \operatorname{Span}\{N_i\}$, $\overbrace{\quad}^{\langle N_i, N_i\rangle = \tau_i = \sigma_{\pi(i)} = \pm 1}$ then (*) reduces to

$$|X_1 \wedge \cdots \wedge X_k|^2 = \overbrace{\tau_1 \cdots \tau_k}^{\sigma_\lambda} (p^\lambda)^2, \quad \lambda = (1, \ldots, k)$$

$$p^\lambda = \det(X^i_j), \quad X_i = X^j_i N_j$$
$$i, j = 1, \ldots, k.$$

Conclude: "In ON frames aligned with the ∥-opiped, metric k-volume $= \pm$ coordinate k-volume of the ∥-opiped. In ON frames not-aligned with the ∥-opiped, the $\pm$ signes in $|X_1 \wedge \cdots \wedge X_k|^2 = \sigma_{|\lambda|}(p^\lambda)^2$ account for "Lorentz Contraction" in non-pos definite metrics!"

☒ Proof of Theorem (Main):

- For ① note

$$\det\langle X_i, X_j\rangle = \det\langle X_i^{\ell} N_\ell, X_j^m N_j\rangle$$
$$= \det\left(X_i^{\ell} X_j^m \langle N_\ell, N_m\rangle\right)$$
$$= \det\left(H^t \cdot \begin{bmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_n \end{bmatrix} H\right), \quad H = (X_j^i)$$
$$= \left[\det(X_j^i)\right]^2 \tau_1 \cdots \tau_n \quad \checkmark$$

- pf of ②: $\left(|X_1 \wedge \cdots \wedge X_n|^2 = \det\langle X_i, X_j\rangle\right)$

We have

$$X_1 \wedge \cdots \wedge X_n = \det(X_j^i) N_1 \wedge \cdots \wedge N_n,$$

so that

$$|X_1 \wedge \cdots \wedge X_n|^2 = \langle X_1 \wedge \cdots \wedge X_n, X_1 \wedge \cdots \wedge X_n\rangle$$
$$= (\det X_j^i)^2 \langle N_1 \wedge \cdots \wedge N_n, N_1 \wedge \cdots \wedge N_n\rangle$$
$$= (\det X_j^i)^2 \tau_1 \cdots \tau_n = \det\langle X_i, X_j\rangle (\tau_1 \cdots \tau_n)^2$$
$$= \det\langle X_i, X_j\rangle \quad \checkmark$$

• Proof of ①: $\eta(X_1 \cdots X_\mu, N_{\mu+1} \cdots N_n)^2 =$

$$= \det\langle X_i, X_j \rangle$$

$$\eta(X_1, \ldots, X_\mu, N_{\mu+1}, \ldots, N_\mu)$$

$$= \eta\left(X_1^{i_1} N_{i_1}, \ldots, X^{i_\mu} N_{i_\mu}, N_{\mu+1}, \ldots, N_n\right)$$

$$= \det\left(X_j^i\right) \eta(N_1, \ldots, N_n)$$

$$\boxed{\cancel{= (-1)^{\pi} \det X_j^i = (-1)^{\pi} \hat{i}_1 \cdots \hat{i}_\mu \det\langle X_{i_1}, X_{i_1}\rangle}}$$

Thus:

$$\eta(X_1, \ldots, X_\mu, N_{\mu+1}, \ldots, N_n)^2 = (\det X_j^i)^2$$

$$= \tau_1 \cdots \tau_k \det\langle X_i, X_j \rangle$$

Summary:

- Given M^n, no metric, coord syst $\underset{\sim}{x} : \mathcal{U} \to \mathbb{R}^n$

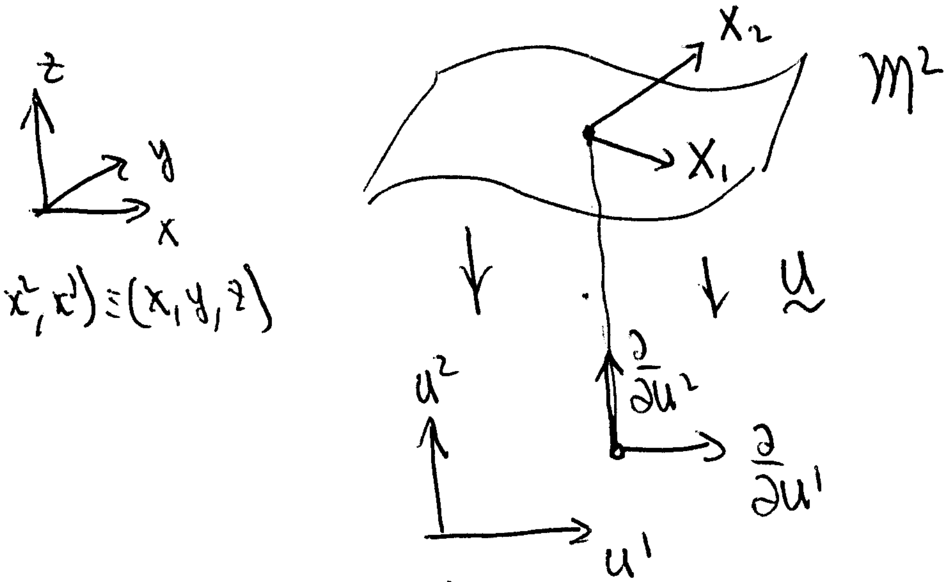
  η an n-form $\eta \in \Lambda^n$

  $\dim \Lambda^n = 1 \Rightarrow \eta = \eta(x)\, dx^1 \wedge \cdots \wedge dx^n$

  Defn: $$\int_{\mathcal{U}} \eta \equiv \int_{\underset{\sim}{x}(\mathcal{U})} \eta(x)\, dx^1 \cdots dx^n$$

  Thm: the integral transforms properly

- What about generalizing the $\mathbb{R}^3$ picture?



$\underset{\sim}{x} = (x^1, x^2, x^3) \equiv (x, y, z)$

$\underset{\sim}{u}^{-1} \equiv \underset{\sim}{x}(u)$
$= (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2))$

$$X_i = \underset{\sim}{u}^{-1}_{*} \frac{\partial}{\partial u^i} = \frac{\partial x^j}{\partial u^i} \frac{\partial}{\partial x^j} \in \mathbb{R}^3$$

$|X_1 \times X_2|$ = area of ∥-ggram $[X_1, X_2]$ in $\mathbb{R}^3$

$$\boxed{\int_{M^2} ds = \int_{\mathbb{R}^2} |X_1 \times X_2| \, du^1 du^2}$$

- To generalize, let $M^n \supset M^k$, g a metric on M^n

and $\underset{\sim}{u} : M^k \to \mathbb{R}^k$ a coord syst on M^k.

$$\underset{\sim}{u}^{-1}_{*}\left(\frac{\partial}{\partial u^i}\right) = X_i \in TM^k \subseteq TM^n$$

$|X_1 \wedge \cdots \wedge X_k| \equiv$ "k-vol of ∥-opiped inherited from g on M^n"

<u>Defn</u>: $$\int_{M^2} ds \equiv \int_{\mathbb{R}^k} |X_1 \wedge \cdots \wedge X_k| \, du^1 \cdots du^k$$

- **Defn**: M^n orientable if $\exists$ non-vanishing k-form on M^k. If $\underset{\sim}{u}$ global coord system, then $X_1 \wedge \cdots \wedge X_n \neq 0 \Rightarrow X_1 \wedge \cdots \wedge X_n$ is a ~~orientation~~ non-vanishing k-form on M^k.

Let: $\theta = \dfrac{X_1 \wedge \cdots \wedge X_n}{|X_1 \wedge \cdots \wedge X_n|}$ denote the "unit" k-form on M^n. We call θ a (normalized) orientation on M^n.

Let $\omega \in \Lambda^n(M^n)$.

- **Defn**: $$\int_{M^n} \omega = \int_{M^k} \langle \omega, \theta \rangle \theta$$

component of ω in direction of θ

- We can extend to more than one coord syst. by "partition of unity".

☒ <u>Application</u>: Spacetime.

- In Einstein's theory of gravity, gravitational effects are determined by a Lorentzian metric g defined on the ~~...~~ 4-d manifold Spacetime: I.e., in a coordinate system $\underset{\sim}{x}$ that is orthonormal at p,

$$\frac{\partial}{\partial x^0} \equiv \frac{\partial}{\partial ct} \quad (\text{at } p)$$

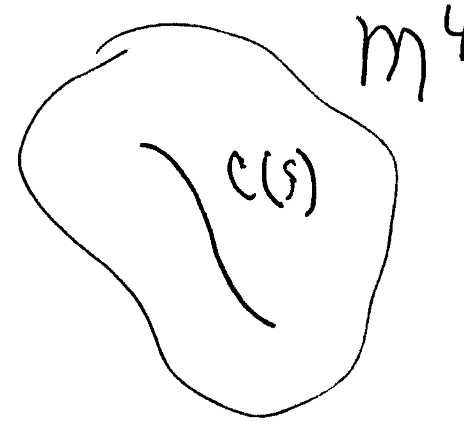
$$\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x}, \frac{\partial}{\partial x^2} = \frac{\partial}{\partial y}, \frac{\partial}{\partial x^3} = \frac{\partial}{\partial z}, \quad (\text{at } p)$$

and

$$g_{ij} = \mathrm{diag}(-1, 1, 1, 1).$$

- The metric g determines <u>proper time</u>: along time-like curves and <u>length in meters</u> along spacelike curves. We assume c = speed of light in $\frac{\text{meters}}{\text{sec}}$, so that x^0 measures time in meters,

$$x^0 = ct, \quad t \equiv \text{seconds}, \; c \equiv \text{speed of light in } \frac{\text{meters}}{\text{sec}}$$



I.e. a curve $c(\xi)$ thru spacetime is <u>timelike</u> if

$$\left\langle \frac{dc}{d\xi}, \frac{dc}{d\xi} \right\rangle < 0$$

all along the curve, and is spacelike if

$$\left\langle \frac{dc}{d\xi}, \frac{dc}{d\xi} \right\rangle > 0$$

all along the curve.

- Timelike curves correspond to the <u>world lines</u> of physical bodies which move at less than the speed of light
- If c is timelike,

$$\Delta\tau = \int_{\xi_1}^{\xi_2} \sqrt{-\left\langle \frac{dc}{d\xi}, \frac{dc}{d\xi} \right\rangle}\, d\xi$$

is the proper time change as measured by observer traversing the path $c(\xi)$, $\xi_1 \leq \xi \leq \xi_2$.

- If c is spacelike,

$$\Delta s = \int_{\xi_1}^{\xi_2} \sqrt{\left\langle \frac{dc}{d\xi}, \frac{dc}{d\xi} \right\rangle}\, d\xi$$

is the "length of the curve in meters".

- Light-like curves correspond to trajectories of light rays.

▲ Assumption of special relativity: $\exists$ a coordinate system $\underline{x}$ on spacetime in which

$$g_{ij} = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & & 1 \end{bmatrix}$$

at every point.

We show next quarter that in general, the best we can do is to make

$$g_{ij}(p) = \mathrm{diag}(-1, 1, 1, 1)$$

$$\frac{\partial}{\partial x^k} g_{ij}(p) = 0,$$

and the 2nd derivatives of g_{ij} measure the curvature of spacetime.

Einstein's equations determine the metric g_{ij} from the energy and momentum densities and their fluxes:

$$G = \kappa T$$

- 2-d Special Relativity: We construct the ON Frames and c.f. the projection thm.

Assume: Lorentzian coords $\underset{\sim}{x}$: $g_{ij} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$x^0 \equiv t, \quad x^1 \equiv x.$$

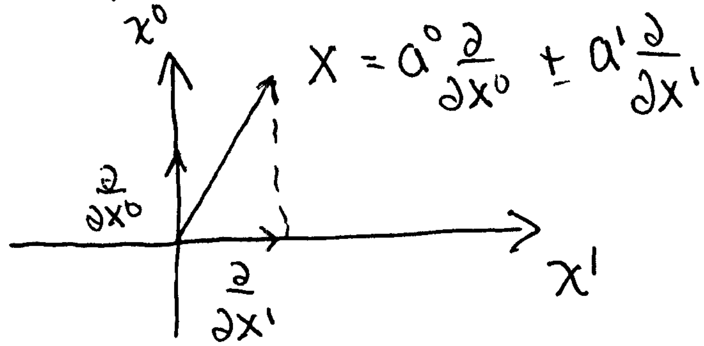
~~Another coordinate system y is O.N. if~~

$$\cancel{g_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}.$$

Q: What are the other ON frames X_0, X_1?

Ans: Let X be vector $X = a^0 \frac{\partial}{\partial x^0} + a^1 \frac{\partial}{\partial x^1}$

Since metric everywhere the same, we can identify components with pts in the space:

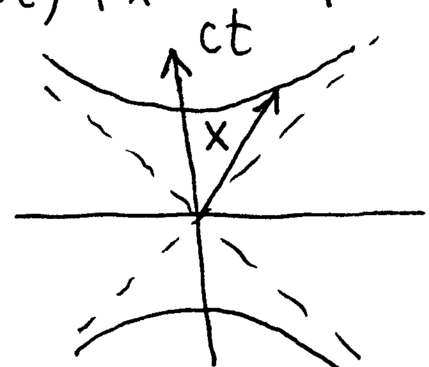


Ie. set $a^0 \equiv ct$, $a^1 \equiv x$, $X = ct \frac{\partial}{\partial x^0} + x \frac{\partial}{\partial x^1}$

Let X be timelike, unit length

$$\langle X, X\rangle = (ct, x)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} ct \\ x \end{bmatrix} = -(ct)^2 + x^2 = -1$$

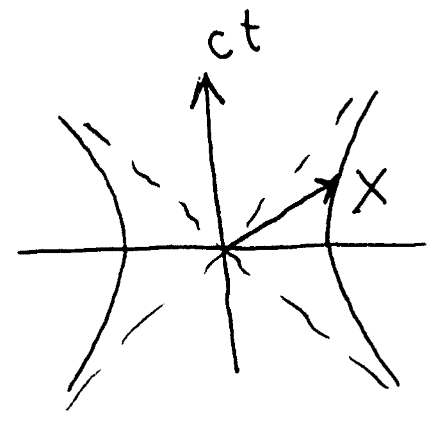
$\Rightarrow X$ lies on unit hyperbola$_1$



Let X be spacelike, unit length

$$\langle X, X\rangle = -(ct)^2 + x^2 = 1$$

$\Rightarrow X$ lies on unit hyperbola$_2$

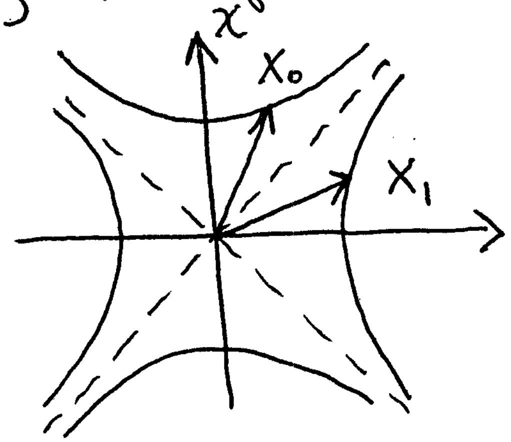


Thus: Assume $\langle X_0, X_1\rangle = 0$, $\langle X_0, X_0\rangle = -1$, $\langle X_1, X_1\rangle = 1$

$$0=\langle X_0, X_1\rangle = (ct_0, x_0)\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}ct_1\\ x_1\end{bmatrix} = (-ct_0, x_0)\cdot(ct_1, x_1) = 0$$

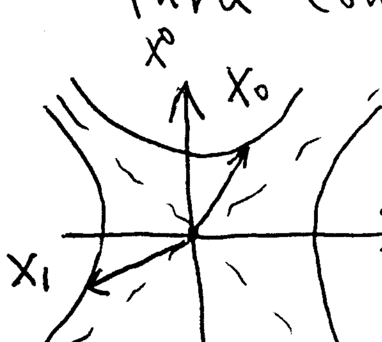
$\Rightarrow$ "$(ct_1, x_1) \parallel \pm(x_0, ct_0)$ $\Rightarrow$ X_1 is the reflection of X_0 in line $x = \pm ct$

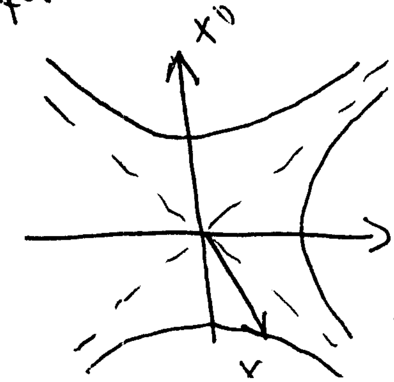
If we assume X_0 timelike, positive directed and $\{X_0, X_1\}$ positively oriented, then



<u>Note</u> ① cannot get to neg oriented frame thru cont transformations

② Cannot get to time inverted or space inverted thru cont sequ. of tran's.

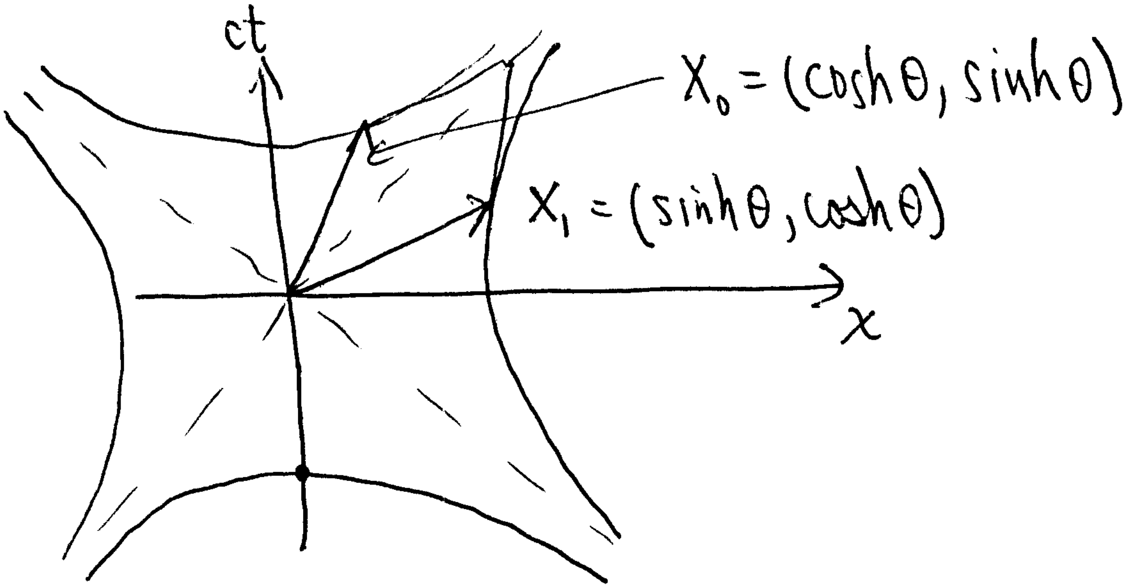




time inversion

Space inversion

Note: $\cosh^2\theta - \sinh^2\theta = 1$, thus



Coordinate area of $\parallel$-ogram is

$$X_0 \times X_1 = \begin{vmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ e_1 & e_2 & e_3 \end{vmatrix}$$

$$\begin{vmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cosh^2\theta - \sinh^2\theta = 1$$

$\Rightarrow$ $\underset{\sim}{x}$-Coordinate volume of orthonormal bases is $\underline{1}$ $\Rightarrow$ Volume form $\eta = dx^1 \wedge dx^0 = c\,dx \wedge dt$ agrees with coord volume in ON-frames ✓

- Consider 4-d Minkowski spacetime:

Global coordinates on spacetime $\underset{\sim}{x} = (x^0, \dots, x^3)$

$x^0 = ct$ (x^i in meters $i = 1, 2, 3$, t in seconds, c = speed of light meters/sec)

Metric: $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$

Given 3 spacetime vectors

$$X_i = X_i^\sigma \frac{\partial}{\partial x^\sigma} \equiv (X_i^0, \dots, X_i^3)$$

$$3\text{-Vol}[X_1, \dots, X_3]^2 = |X_1 \wedge \dots \wedge X_3|^2$$

$$= \det \langle X_i, X_j \rangle$$

$$= \det\left(\underbrace{-X_i^0 X_j^0} + X_i^1 X_j^1 + X_i^2 X_j^2 + X_i^3 X_j^3\right)$$

minus sign accounts for Lorentz contraction