Geodesics of a metric:

- Given a metric $g_{ij}(\underset{\sim}{x}) \equiv$ "$n\times n$ matrix of fn's"

$$\underset{\sim}{x} = (x^1, \dots, x^n), \quad i,j = 1, \dots, n$$

- $g_{ij}(\underset{\sim}{x}) =$ "$n\times n$ matrix of fn's that keep track of the distortion betw lengths in spacetime & lengths in the coordinate system $\underset{\sim}{x}$"

- Length of a curve $\underset{\sim}{x}(\xi)$: $\xi_1 \leq \xi \leq \xi_2$

$$S_{\text{length}}[\underset{\sim}{x}(\cdot)] = \int_{\xi_1}^{\xi_2} \| \dot{\underset{\sim}{x}}(\xi) \| \, d\xi$$

$$\| \dot{\underset{\sim}{x}}(\xi) \| = \sqrt{g_{ij}(\underset{\sim}{x}(\xi)) \, \dot{x}^i(\xi) \, \dot{x}^j(\xi)}$$

- We would like to say geodesics are critical pts of $S_{\text{length}}[\underline{x}(\cdot)]$ but it turns out to be easier to study critical pts of

$$S[\underline{x}(\cdot)] = \int_{\xi_1}^{\xi_2} \|\dot{\underset{\sim}{x}}(\xi)\|^2 \, d\xi$$

<u>Defn</u>: Geodesics are curves that are critical pts of S

<u>Theorem</u>: The critical points of S agree with the critical points of S_{length} <u>except</u> critical points of S are always parameterised wrt a <u>multiple</u> of arclength.

Moreover, you can always reparameterise so the geodesic is parameterised wrt arclength exactly

<u>Partial Proof</u>: We show that critical pts of S are parameterized wrt a multiple of arclength –

For $S[\underline{x}(\cdot)] = \int_{\xi_1}^{\xi_2} \|\dot{\underline{x}}(\xi)\| \, d\xi$

$$L(\underset{\sim}{x}, \dot{\underset{\sim}{x}}) = g_{ij}(\underset{\sim}{x})\, \dot{x}^i \dot{x}^j$$

The energy is constant along soln's of E-L equations – that is, geodesics solve (E-L)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0$$

and Energy is const along soln's –

$$E(\underset{\sim}{x}(\xi)) = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L$$

$$= \dot{x}^i \frac{\partial}{\partial \dot{x}^i}\left\{ g_{ij}(\underset{\sim}{x})\dot{x}^i \dot{x}^j \right\}$$

$$- g_{ij}(\underset{\sim}{x})\dot{x}^i \dot{x}^j$$

$$\left.\begin{array}{l} \text{But} \\ \frac{\partial}{\partial \dot{x}^i}\{\ \} = 2 g_{ij}(\underset{\sim}{x})\dot{x}^j \\ \text{because } g \text{ is symmetric, } g_{ij} = g_{ji} \end{array}\right\} \text{H.W.}$$

Thus

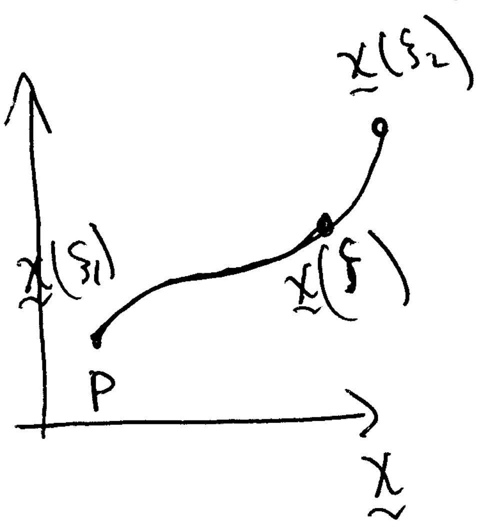
$$E = \dot{x}^i \cdot 2 g_{ij}(\underset{\sim}{x})\dot{x}^j - g_{ij}(\underset{\sim}{x})\dot{x}^i \dot{x}^j$$

$$= g_{ij}(\underset{\sim}{x})\dot{x}^i \dot{x}^j = \|\dot{\underset{\sim}{x}}\|^2$$

<u>Conclude</u>: $\|\dot{\underset{\sim}{x}}(\xi)\|^2$ is constant along geodesics $\Rightarrow$ $\|\dot{x}(\xi)\| = \text{const}$ along geodesics $\Rightarrow$ critical pts of S are parameterized wrt a const. times arclength-

- Recall arclength parameterization:

<u>Defn</u>: a curve $\underset{\sim}{x}(\xi)$ is parameterized wrt arclength if $\|\dot{\underset{\sim}{x}}(\xi)\| = 1$. Then



Length from ξ_1 to ξ along $\underset{\sim}{x}(\cdot)$ is

$$L(\xi) = \int_{\xi_1}^{\xi} \|\dot{\underset{\sim}{x}}(s)\| \, ds = \int_{\xi_1}^{\xi} ds$$

$$= \xi - \xi_1 \quad \text{if } \|\dot{\underset{\sim}{x}}\| = 1$$

<u>Theorem</u>: The (E-L) equations for

$$S[\underset{\sim}{x}(\cdot)] = \int_{\xi_1}^{\xi_2} \|\underset{\sim}{x}(\xi)\|^2 d\xi$$

are equivalent to the geodesic equation

$$\ddot{x}^i + \Gamma^i_{jh} \dot{x}^j \dot{x}^h = 0$$

That is, geodesic $\underset{\sim}{x}(\xi)$ satisfy

$$\ddot{x}^i(\xi) + \Gamma^i_{jh}(\underset{\sim}{x}(\xi)) \dot{x}^j(\xi) \dot{x}^h(\xi) = 0$$

$$\Gamma^i_{jh} = \frac{1}{2} g^{i\sigma} \left\{ -g_{jh,\sigma} + g_{\sigma j,h} + g_{h\sigma,j} \right\}$$

$$g_{ij,h}(\underset{\sim}{x}) = \frac{\partial}{\partial x^h} g_{ij}(\underset{\sim}{x})$$

$$(E\text{-}L) \qquad \frac{d}{d\xi}\frac{\partial L}{\partial \dot{x}^h} - \frac{\partial L}{\partial x^h} = 0$$

$$L(\underset{\sim}{x}, \dot{\underset{\sim}{x}}) = g_{ij}(\underset{\sim}{x})\,\dot{x}^i\dot{x}^j$$

$$\frac{\partial L}{\partial \dot{x}^k} = 2\,g_{kj}(\underset{\sim}{x})\,\dot{x}^j$$

$$\frac{\partial L}{\partial x^h} = g_{ij,k}\,\dot{x}^i\dot{x}^j$$

$$\frac{d}{d\xi}\left(2\,g_{kj}(\underset{\sim}{x})\,\dot{x}^j\right) - g_{ij,h}\,\dot{x}^i\dot{x}^j = 0$$

$$2\,g_{kj,\ell}\,\dot{x}^\ell\dot{x}^j + 2g_{hj}\,\ddot{x}^j - g_{ij,h}\,\dot{x}^i\dot{x}^j = 0$$

$$k = 1, \ldots, n$$

$$2\,g_{ki,j}\,\dot{x}^i\dot{x}^j + 2g_{k\ell}\,\ddot{x}^\ell - g_{ij,h}\,\dot{x}^i\dot{x}^j = 0$$

$$g_{k\ell}\ddot{x}^{\ell} + \frac{1}{2}\left\{-g_{ij,k} + 2g_{ki,j}\right\}\dot{x}^{i}\dot{x}^{j} = 0$$

HW.

$$g_{k\ell}\ddot{x}^{\ell} + \frac{1}{2}\left\{-g_{ij,k} + g_{ki,j} + g_{kj,i}\right\}\dot{x}^{i}\dot{x}^{j}$$

$$g^{\sigma k} g_{k\ell} = \delta^{\sigma}_{\ell} = \text{identity matrix}$$

$$g^{\sigma h} g_{\sigma\ell}\ddot{x}^{\ell} = \delta^{\sigma}_{\ell}\ddot{x}^{\ell} = \ddot{x}^{\sigma}$$

$$\therefore \quad \ddot{x}^{\sigma} + \frac{1}{2} g^{\sigma k}\left\{ \quad \right\}\dot{x}^{i}\dot{x}^{j} = 0$$

Switch σ & h

$$\boxed{\ddot{x}^{h} + \frac{1}{2} g^{k\sigma}\left\{ \quad \right\}\dot{x}^{i}\dot{x}^{j} = 0} \checkmark$$

9 Geodesics of the Schwartzschild metric

- Recall: in Einstein theory, trajectories of the planets follow geodesics of the Schwarzschild metric (Neglecting effects of other planets)

$$d(c\tau)^2 = ds^2 = -A\,d(ct)^2 + A^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$g_{00} = -A\,,\quad g_{11} = A^{-1}\,,\quad g_{22} = r^2\,,\quad g_{33} = r^2\sin^2\theta$$

$$x^0 = ct\,,\quad x^1 = r\,,\quad x^2 = \theta\,,\quad x^3 = \varphi$$

$$A = 1 - \frac{2GM_0}{c^2 r} = 1 - \frac{2M}{r}\,,\quad M = \frac{GM_0}{c^2}$$

- Given a curve $\underset{\sim}{x}(\xi)$, say the trajectory of a planet plotted in $\underset{\sim}{x} = (ct, r, \theta, \varphi)$ coordinates, The length of the curve ~~ref to~~ measured by g is

$$S_{\text{length}}[\underset{\sim}{x}(\cdot)] = \int_{\xi_1}^{\xi_2} \|\dot{\underset{\sim}{x}}(\xi)\|\, d\xi$$

- The trajectory of the planet is a critical point (geodesic) of

$$S[\underset{\sim}{x}(\cdot)] = \int_{\xi_1}^{\xi_2} \| \dot{\underset{\sim}{x}}(\xi) \|^2 \, d\xi$$

$$L(\underset{\sim}{x}, \dot{\underset{\sim}{x}}) = -A(r)\, c^2 \dot{t}^2 + A(r)^{-1} \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2$$

$$A = 1 - \frac{2M}{r}$$

Rather than calculate the equations for geodesics from

$$\ddot{x}^h + \Gamma^h_{ij} \dot{x}^i \dot{x}^j = 0, \qquad (\Gamma)$$

ie, rather than calc Γ^h_{ij}, ~~directly~~, we go to E-L equations directly. <u>Note</u>: $(\Gamma) \Rightarrow$! soln with i-condns $\underset{\sim}{x}(\xi_1)$, $\dot{\underset{\sim}{x}}(\xi_1)$.

(1) Conservation of Energy:

$$\boxed{\|\dot{\underset{\sim}{x}}(\xi)\|^2 = -Ac^2\dot{t}^2 + A^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 = 1}$$

(we set i-condition for trajectory at $\|\dot{x}(\xi_1)\| = 1$, & this is maintained since $\|\dot{x}(\xi)\| = \text{const}$) ξ is now <u>arclength</u>

(2) $(i=0)$ $$\frac{d}{d\xi}\frac{\partial L}{\partial c\dot{t}} - \frac{\partial L}{\partial t} = \frac{d}{d\xi}\left(-2Ac\dot{t}\right) = 0$$

$$\boxed{Ac\dot{t} = \text{Const}}$$

(3) $(i=2)$ $$\frac{d}{d\xi}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = \boxed{\frac{d}{d\xi}\left(2r^2\dot{\theta}\right) - 2r^2\sin\theta\cos\theta\,\dot{\varphi}^2 = 0}$$

(4) $(i=3)$ $$\frac{d}{d\xi}\frac{\partial L}{\partial\dot{\varphi}} - \frac{\partial L}{\partial\varphi} = \boxed{\frac{d}{d\xi}\,2r^2\sin^2\theta\,\dot{\varphi} = 0}$$

- <u>Choose</u>: i-condition so $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$.

Then equation (3) is identically satisfied if $\theta \equiv \frac{\pi}{2}$, $\dot{\theta} \equiv 0$ all along trajectory. Since then θ drops out of the other 3 equations, this is a consistent ansatz:

<u>Conclude</u>: By aligning $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$ initially, uniqueness of soln's $\Rightarrow$ $\theta \equiv \frac{\pi}{2}$ along trajectory $\Rightarrow$ motion lies in a fixed plane, $\theta = \frac{\pi}{2}$ ✓

- With $\theta = \frac{\pi}{2}$, our equations are

(1) $-Ac^2\dot{t}^2 + A^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1$

(2) $Ac\dot{t} = \text{Const.}$

(3) drops out

(4) $\frac{d}{d\xi} r^2\dot{\phi} = 0$

- Eqn (1), (2), (4) are 3 2nd order eqns in (ct, r, φ).

- Eqn (4) gives "equal area in equal time" like Newton. Set

(4) $r^2\dot{\phi} = H = \frac{H_0}{c}$ $\left(\text{because } \dot{} \equiv \frac{d}{ds} \equiv \frac{d}{c\tau}\right)$

We now substitute (3), (4) into (1), diff, replace $u = \frac{1}{r}$ and $\frac{d}{d\theta} \equiv '$ to get (E):

$$-Ac^2\dot{t}^2 + A^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = 1$$

$$(c\dot{t}) = \frac{C_1}{A}$$

$$-\cancel{A}\frac{C_1^2}{A^{\cancel{2}}} + A^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -1$$

$$-C_1^2 + \dot{r}^2 + Ar^2\dot{\phi}^2 = -A$$

$$r^2\dot{\phi} = H_0/c = H$$

$$-C_1^2 + \dot{r}^2 + AH\dot{\phi} = -A \qquad \dot{\phi} = \frac{H}{r^2}$$

$$\boxed{-C_1^2 + \dot{r}^2 + \frac{AH^2}{r^2} = -A} \qquad A = 1 - \frac{2M}{r}$$

$$M = \frac{GM_0}{c^2}$$

$$u = \frac{1}{r} \quad , \quad ' \equiv \frac{d}{d\varphi}$$

$$\dot{f} = \frac{df}{d\xi} = \frac{df}{d\varphi}\frac{d\varphi}{d\xi} = \frac{df}{d\varphi}\dot{\varphi} = \frac{f' H}{r^2} = f' H u^2$$

$$-C_1^2 + (r' H u^2)^2 + A H^2 u^2 = A$$

$$r' = \left(\frac{1}{u}\right)' = -\frac{u'}{u^2}$$

$$A = 1 - \frac{2M}{r} = 1 - 2Mu$$

$$-C_1^2 + \left(-\frac{u'}{\cancel{u^2}} H \cancel{u^2}\right)^2 + (1-2Mu) H^2 u^2 = (1-2Mu)$$

$$-C_1^2 + H^2 (u')^2 + H^2 u^2 - 2MH^2 u^3 = 1 + 2Mu$$

- Diff wrt ϕ again to get 2nd order ODE

$$\cancel{2}H^2 u' u'' + \cancel{2}H^2 u \cancel{u'} - \overset{3}{\cancel{6}} M H^2 u^2 \cancel{u'} = + 2M \cancel{u'}$$

$$H^2 u'' + H^2 u - 3MH^2 u^2 = +M$$

$$u'' + u \quad \frac{M}{H^2} = 3Mu^2 \qquad M = \frac{GM_0}{c^2}$$

$$H = \frac{H_0}{c}$$

Really: $$u'' + u \quad \frac{GM_0}{\cancel{c^2}} \frac{\cancel{c^2}}{H_0^2} = \frac{3GM_0}{c^2} u^2$$

$$\boxed{u'' + u \quad \frac{GM_0}{H_0^2} = \frac{3GM_0}{c^2} u^2}$$

$$H_0 = \frac{\text{Area}}{\text{second}}$$