DIFFERENTIAL FORMS, INTEGRATION, AND FROBENIUS'S THEOREM

In this appendix we shall collect a number of results related to differential forms and integration. Most of these results require only manifold structure; specifically, they do not require the presence of a metric or a preferred derivative operator. Thus, they are basic results of very general applicability in differential geometry.

B.1 Differential Forms

Let M be an n-dimensional manifold. A differential p-form is a totally antisymmetric tensor of type $(0,p)$, i.e., $\omega_{a_1 \cdots a_p}$ is a p-form if

$$\omega_{a_1 \cdots a_p} = \omega_{[a_1 \cdots a_p]} \tag{B.1.1}$$

We denote the vector space of p-forms at a point x by Λ_x^p and the collection of p-form fields by Λ^p. Note that $\Lambda_x^p = \{0\}$ if $p > n$ and $\dim \Lambda_x^p = n!/p!(n-p)!$ for $0 \leqq p \leqq n$. If we take the outer product of a p-form $\omega_{a_1 \cdots a_p}$ and a q-form $\mu_{b_1 \cdots b_q}$, we will get a tensor of type $(0, p+q)$; but since this tensor will not, in general, be totally antisymmetric, it is not a $(p+q)$-form. However, we can totally antisymmetrize this tensor, thus producing a map $\wedge : \Lambda_x^p \times \Lambda_x^q \to \Lambda_x^{p+q}$ via

$$(\omega \wedge \mu)_{a_1 \cdots a_p b_1 \cdots b_q} = \frac{(p+q)!}{p!q!} \omega_{[a_1 \cdots a_p} \mu_{b_1 \cdots b_q]} \quad . \tag{B.1.2}$$

(If $p + q > n$, this tensor, of course, will be zero.) We define the vector space of all differential forms at x to be the direct sum of the Λ_x^p,

$$\Lambda_x = \bigoplus_{p=0}^{n} \Lambda_x^p \quad . \tag{B.1.3}$$

The map $\wedge : \Lambda_x \times \Lambda_x \to \Lambda_x$ gives Λ_x the structure of a Grassmann algebra[1] over the vector space of one-forms.

If we are given a derivative operator, ∇_a, we could define a map from smooth p-form fields to $(p+1)$-form fields by

$$\omega_{a_1 \cdots a_p} \to (p+1) \nabla_{[b} \omega_{a_1 \cdots a_p]} \quad . \tag{B.1.4}$$

If instead we were given another derivative operator $\tilde{\nabla}_a$, we would obtain the map

$$\omega_{a_1 \cdots a_p} \to (p+1) \tilde{\nabla}_{[b} \omega_{a_1 \cdots a_p]} \quad . \tag{B.1.5}$$

1. See, e.g., Bishop and Crittenden (1964) for the definitior of a Grassmann algebra.

However, according to equation (3.1.14), we have

$$\nabla_{[b}\omega_{a_1\cdots a_p]} - \tilde{\nabla}_{[b}\omega_{a_1\cdots a_p]} = \sum_{j=1}^{p} C^d{}_{[ba_j}\omega_{a_1\cdots|d|\cdots a_p]} = 0 \tag{B.1.6}$$

since $C^c{}_{ab}$ is symmetric in a and b. Thus the map defined by equation (B.1.4) is independent of derivative operator, i.e., it is well defined without the presence of a preferred derivative operator on M. We denote this map by d. In particular, we may use the ordinary derivative, ∂_a, associated with any coordinate system to calculate d.

Since the index structure of differential forms is trivial, it is customary to drop the indices when writing them; e.g., we write $\boldsymbol{\omega}$ instead of $\omega_{a_1\cdots a_p}$ and write $\boldsymbol{\omega} \wedge \boldsymbol{\mu}$ instead of $(\omega \wedge \mu)_{a_1\cdots b_q}$. (The only disadvantage in doing so is that we must remember the dimensionality of the forms with which we are dealing.) We shall use boldface letters for forms to avoid confusion with functions. We denote the $(p + 1)$-form resulting from the action of the map $d\colon \Lambda^p \to \Lambda^{p+1}$ on the p-form $\boldsymbol{\omega}$ by $d\boldsymbol{\omega}$.

An important property of the map d is that $d^2 = d \circ d = 0$. This result, known as the *Poincaré lemma,* follows from the fact that we can compute d using an ordinary derivative operator. Indeed, restoring the indices, we have for an arbitrary smooth p-form $\boldsymbol{\omega}$,

$$(d^2\omega)_{bca_1\cdots a_p} = (p + 2)(p + 1)\partial_{[b}\partial_c\omega_{a_1\cdots a_p]} = 0 \quad , \tag{B.1.7}$$

because of the equality of mixed partial derivatives in $\mathbb{R}^n$.

Conversely, it can be shown (see, e.g., Flanders 1963) that if one has a *closed* p-form, i.e., a p-form $\boldsymbol{\alpha}$ satisfying $d\boldsymbol{\alpha} = 0$, then *locally* (i.e., in any open region diffeomorphic to $\mathbb{R}^n$) this form is *exact*, i.e., there exists a $(p - 1)$-form $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} = d\boldsymbol{\beta}$. However, in general this result is not valid globally. Indeed, an important theorem in algebraic topology due to de Rham establishes that the dimension of the vector space of closed p-forms modulo the exact p-forms equals a topological quantity: the pth Betti number of the manifold.[2]

B.2 Integration

Let M be an n-dimensional manifold. At each point $x \in M$, the vector space of n-forms will be one-dimensional. If it is possible to find a continuous, nowhere vanishing n-form field $\boldsymbol{\epsilon} = \epsilon_{[a_1\cdots a_n]}$ on M, then M is said to be *orientable* and $\boldsymbol{\epsilon}$ is said to provide an orientation.[3] Two orientations $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}'$ are considered equivalent if $\boldsymbol{\epsilon} = f\boldsymbol{\epsilon}'$, where f is a (strictly) positive function, so any orientable manifold

2. Roughly speaking, the pth Betti number of M is the number of independent p-dimensional boundaryless surfaces in M which are not themselves boundaries of $(p + 1)$-dimensional regions. For more details, including a complete statement and proof of de Rham's theorem, see, e.g., Warner (1971).

3. For the case where M is an n-dimensional surface in the Euclidean space $\mathbb{R}^{n+1}$, this definition of orientability is equivalent to the more intuitive notion that there exists a consistent (i.e., continuous) choice of normal vector u^a to M: If a continuous nonvanishing u^a exists, then $\boldsymbol{\epsilon} = \bar{\epsilon}_{a_1\ldots a_{n+1}}u^{a_1}$ provides an orientation of M, where $\bar{\boldsymbol{\epsilon}}$ is an orientation of $\mathbb{R}^{n+1}$. Conversely, if M is orientable, then $\bar{\epsilon}^{a_1\cdots a_{n+1}}\epsilon_{a_1\ldots a_n}$ provides a continuous normal vector.

possesses two inequivalent orientations, usually referred to as "right handed" and "left handed." It is easy to check that the manifolds $\mathbb{R}^n$ and S^m are orientable. Indeed, it is not difficult to show that every simply connected manifold is orientable. (As discussed further in chapter 13, a topological space is said to be *simply connected* if every closed curve can be continuously deformed to a point. $\mathbb{R}^n$ and S^m for $m \geq 2$ are simply connected.) Furthermore, the product of any two orientable manifolds is orientable. Thus, we obtain a wide class of examples of orientable manifolds. On the other hand, the Möbius strip [defined as $\mathbb{R}^2$ with the identification $(x, y) = (x + 1, -y)$] provides a simple example of a nonorientable manifold.

We will define the integral of a continuous (or, more generally, a measurable[4]) n-form field $\boldsymbol{\alpha}$ over an n-dimensional orientable manifold (with respect to the orientation $\boldsymbol{\epsilon}$) as follows. We begin by considering an open region $U \subset M$ covered by a single coordinate system ψ. If we expand $\boldsymbol{\epsilon}$ in the coordinate basis of ψ, we will obtain

$$\boldsymbol{\epsilon} = h dx^1 \wedge \ldots \wedge dx^n \tag{B.2.1}$$

(i.e. $\epsilon_{a_1 \cdots a_n} = n!\, h(dx^1)_{[a_1} \cdots (dx^n)_{a_n]}$), where the function h is nonvanishing. If $h > 0$, the coordinate system, ψ, is said to be *right handed* with respect to $\boldsymbol{\epsilon}$; if $h < 0$, ψ is called *left handed*. We may also expand $\boldsymbol{\alpha}$ in the coordinate basis, thereby obtaining

$$\boldsymbol{\alpha} = a(x^1, \ldots, x^n) dx^1 \wedge \ldots \wedge dx^n \quad . \tag{B.2.2}$$

If ψ is right handed, we define the integral of $\boldsymbol{\alpha}$ over the region U by

$$\int_U \boldsymbol{\alpha} = \int_{\psi[U]} a dx^1 \ldots dx^n \quad , \tag{B.2.3}$$

where the right-hand side is the standard Riemann (or Lebesgue) integral in $\mathbb{R}^n$. If ψ is left-handed, we define $\int_U \boldsymbol{\alpha}$ to be minus the right-hand side of equation (B.2.3).

First, we note that $\int_U \boldsymbol{\alpha}$ is independent of the choice of coordinate system, ψ, covering U; namely, if we had used a different coordinate system ψ' to cover U, then the expansion of $\boldsymbol{\alpha}$ in the new coordinate basis would be

$$\boldsymbol{\alpha} = a' dx'^1 \wedge \ldots \wedge dx'^n \quad . \tag{B.2.4}$$

But it follows from the tensor transformation law, equation (2.3.8), that

$$a' = a \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right) \quad . \tag{B.2.5}$$

The standard law for transformation of integrals in $\mathbb{R}^n$ then shows that our definition, equation (B.2.3), is coordinate independent.

To define the integral of $\boldsymbol{\alpha}$ over all of M, we use the paracompactness property of M. As discussed at the end of appendix A, a paracompact manifold can be covered by a countable collection $\{O_i\}$ of locally finite coordinate neighborhoods such that

4. $\boldsymbol{\alpha}$ is said to be measurable if for all charts its coordinate basis components are Lebesgue measurable functions in $\mathbb{R}^n$.

each $\overline{O}_i$ is compact. Furthermore, a partition of unity $\{f_i\}$ subordinate to this covering will exist. If $\sum_i \int_{\psi_i[O_i]} f_i |a_i| dx^1 \ldots dx^n < \infty$, we say $\boldsymbol{\alpha}$ is integrable and we define

$$\int_M \boldsymbol{\alpha} = \sum_i \int_{O_i} f_i \boldsymbol{\alpha} \quad . \tag{B.2.6}$$

It can be shown that this definition is independent of the choice of covering $\{O_i\}$ and partition of unity $\{f_i\}$ and thus properly defines $\int_M \boldsymbol{\alpha}$.

We can use the above definition of integration on manifolds to define the integral of p-forms on M over well behaved, orientable p-dimensional surfaces in M. First, we must define more precisely the notion of a "well behaved surface." Let S be a manifold of dimension $p < n$. If $\phi: S \to M$ is C^∞, is locally one to one—i.e., each $q \in S$ has an open neighborhood O such that ϕ restricted to O is one-to-one—and $\phi^{-1}: \phi[O] \to S$ is C^∞, then $\phi[S]$ is said to be an *immersed submanifold* of M. If, in addition, ϕ is globally one-to-one (i.e., $\phi[S]$ does not "intersect itself"), then $\phi[S]$ is said to be an *embedded submanifold* of M. (In some references the additional condition is imposed on embedded submanifolds that $\phi: S \to \phi[S]$ is a homeomorphism with the topology on $\phi[S]$ induced from M. Roughly speaking, this additional condition ensures that $\phi[S]$ does not come arbitrarily close to intersecting itself.) We shall use the notion of an embedded submanifold as our precise notion of a "well behaved surface" in M. An embedded submanifold of dimension $(n - 1)$ is called a *hypersurface*.

For an embedded submanifold, there is a natural manifold structure on $\phi[S]$ obtained via ϕ from the manifold structure on S. Thus, at each $q \in \phi[S]$, the tangent space W_q for $\phi[S]$ is defined. This tangent space is naturally identified with a p-dimensional subspace of V_q, the tangent space of q in M. Thus, a p-form $\boldsymbol{\beta}$ in M at q naturally gives rise to a p-form $\tilde{\boldsymbol{\beta}}$ on $\phi[S]$ by restriction of the action of $\boldsymbol{\beta}$ to vectors lying in W_q. The integral of $\boldsymbol{\beta}$ over the surface $\phi[S]$ may then be defined as simply the integral of the p-form $\tilde{\boldsymbol{\beta}}$ over the p-dimensional manifold $\phi[S]$.

An important special case of an embedded submanifold arises when $\phi[S]$ is the $(n - 1)$-dimensional boundary, $\dot{N}$, of a closed region $N \subset M$ such that N is a "manifold with boundary." Here, the notion of an n-dimensional *manifold with boundary*, N, can be defined in the abstract in the same way as a manifold (see chapter 2) except that $\mathbb{R}^n$ is replaced by "half of $\mathbb{R}^n$," i.e., by the portion of $\mathbb{R}^n$ with $x^1 \leqq 0$. The boundary, $\dot{N}$, of N is composed of the set of points of N which are mapped into $x^1 = 0$ by the chart maps. Note that these chart maps of N with x^1 set to zero give $\dot{N}$ the structure of an $(n - 1)$-dimensional manifold without boundary. Note also that $\text{int}(N) \equiv N - \dot{N}$ is an n-dimensional manifold without boundary.

If N is an orientable manifold with boundary, then an orientation on N induces a natural orientation on the boundary as follows: We consider the coordinate systems on $\dot{N}$ which arise from deleting the first coordinate, x^1, of a right-handed coordinate system on N in the family of charts that makes N into a manifold with boundary. We wish to define an orientation on $\dot{N}$ which makes these coordinate systems be "right handed." In order to do so, we verify first that the Jacobian, $\det(\partial x'^\mu / \partial x^\nu)$, is positive in the overlap region of any two such coordinate systems. Then, we choose a partition of unity (F_i, U_i) of $\dot{N}$, where each U_i is a coordinate neighborhood of this type. Finally, we define $\bar{\boldsymbol{\epsilon}}$ on $\dot{N}$ by $\bar{\boldsymbol{\epsilon}} = \sum_i F_i \, dx_i^2 \cdots dx_i^n$. Then $\bar{\boldsymbol{\epsilon}}$ is continuous and nonvanishing and thus defines the desired orientation of $\dot{N}$. Having defined the orientation of $\dot{N}$, we may now state one of the most important results concerning integration on manifolds, the proof of which can be found in many references (see, e.g., Flanders 1963).

THEOREM B.2.1 (Stokes's theorem). *Let N be an n-dimensional compact oriented manifold with boundary and let $\boldsymbol{\alpha}$ be an $(n-1)$-form on M which is C^1. Then*

$$\int_{\mathrm{int}(N)} d\boldsymbol{\alpha} = \int_{\dot{N}} \boldsymbol{\alpha} \quad . \tag{B.2.7}$$

Integration of functions on an orientable manifold M can be accomplished if one is given a *volume element*, that is, continuous nonvanishing n-form $\boldsymbol{\epsilon}$. (A volume element differs from an orientation in that orientations are considered equivalent if they differ by positive multiples whereas volume elements are not.) The integral of f over M is defined by

$$\int_M f = \int_M f\boldsymbol{\epsilon} \quad , \tag{B.2.8}$$

where the integral of the n-form $f\boldsymbol{\epsilon}$ was defined previously.[5]

If one is given only the structure of a manifold, M, there is no natural choice of volume element. However, if M has a metric, g_{ab}, defined on it, then a natural choice of $\boldsymbol{\epsilon}$ is specified up to sign (i.e., up to choice of orientation) by the condition

$$\epsilon^{a_1 \cdots a_n} \epsilon_{a_1 \cdots a_n} = (-1)^s n! \quad , \tag{B.2.9}$$

where s is the number of minuses appearing in the signature of g_{ab}. (Thus, $s = 0$ for a Riemannian metric, while $s = 1$ for a Lorentzian metric.) Note that differentiation of equation (B.2.9) using the derivative operator, ∇_a, associated with the metric implies that

$$2\epsilon^{a_1 \cdots a_n} \nabla_b \epsilon_{a_1 \cdots a_n} = 0 \quad , \tag{B.2.10}$$

which, in turn, implies that

$$\nabla_b \epsilon_{a_1 \cdots a_n} = 0 \tag{B.2.11}$$

since $\nabla_b \epsilon_{a_1 \cdots a_n}$ is totally antisymmetric in its last n indices and $\epsilon^{a_1 \cdots a_n}$ is nonvanishing. It is also worth noting that

$$\epsilon^{a_1 \cdots a_n} \epsilon_{b_1 \cdots b_n} = (-1)^s n! \, \delta^{[a_1}{}_{b_1} \delta^{a_2}{}_{b_2} \cdots \delta^{a_n]}{}_{b_n} \quad , \tag{B.2.12}$$

where $\delta^a{}_b$ is the identity map on the tangent space. Equation (B.2.12) follows from the fact that the tensors of type (n, n) on an n-dimensional manifold which are totally antisymmetric in all lower and all upper indices form a one-dimensional vector space and thus must be proportional to the antisymmetrized product of $\delta^a{}_b$ tensors; the

5. Integration of functions on a nonorientable manifold can be defined by choosing a continuous "n-form modulo sign" $\boldsymbol{\epsilon}'$ and performing integrals of $f\boldsymbol{\epsilon}'$ over each of the local coordinate neighborhoods by choosing the sign of $\boldsymbol{\epsilon}'$ which makes the coordinate system "right handed" with respect to it.

constant of proportionality is fixed by the normalization condition (B.2.9). Contraction of equation (B.2.12) over j of its indices yields

$$\epsilon^{a_1 \cdots a_j a_{j+1} \cdots a_n} \epsilon_{a_1 \cdots a_j b_{j+1} \cdots b_n} = (-1)^s (n-j)!\, j!\, \delta^{[a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n} \quad . \tag{B.2.13}$$

Equation (B.2.9) implies that the components of $\boldsymbol{\epsilon}$ in a right handed orthonormal basis are

$$\epsilon_{\mu_1 \cdots \mu_n} = \begin{cases} (-1)^P & \text{if all } \mu_i \text{ are distinct} \\ 0 & \text{otherwise} \quad , \end{cases} \tag{B.2.14}$$

where P is the signature of the permutation $(1, \ldots, n) \to (\mu_1, \ldots, \mu_n)$. In a coordinate basis, the components of $\boldsymbol{\epsilon}$ satisfy

$$\sum_{\mu_1, \ldots, \nu_n} g^{\mu_1 \nu_1} \ldots g^{\mu_n \nu_n} \epsilon_{\mu_1 \cdots \mu_n} \epsilon_{\nu_1 \cdots \nu_n} = (-1)^s n! \tag{B.2.15}$$

But the left-hand side of this expression is just $(n!)(\epsilon_{12 \cdots n})^2$ times the determinant of the matrix $(g^{\mu\nu})$, and $\det(g^{\mu\nu}) = 1/\det(g_{\mu\nu})$. Thus, choosing the plus sign appropriate for a right handed coordinate system, we find

$$\epsilon_{12 \cdots n} = [(-1)^s \det(g_{\mu\nu})]^{1/2} = \sqrt{|g|} \quad , \tag{B.2.16}$$

where $g = \det(g_{\mu\nu})$. Thus, in any (right handed) coordinate basis, the natural volume element defined by equation (B.2.9) takes the form

$$\boldsymbol{\epsilon} = \sqrt{|g|}\, dx^1 \wedge \ldots \wedge dx^n \quad . \tag{B.2.17}$$

Using the natural volume element $\boldsymbol{\epsilon}$ associated with a metric, we can convert Stokes's theorem, equation (B.2.7), into a "Gauss's law" form. Let N be an oriented, compact n-dimensional manifold with boundary. Let g_{ab} be a metric on N with associated volume element $\boldsymbol{\epsilon}$. Given any C^1 vector field v^a, we obtain an $(n-1)$-form $\boldsymbol{\alpha}$ by

$$\alpha_{a_1 \cdots a_{n-1}} = \epsilon_{b a_1 \cdots a_{n-1}} v^b \quad . \tag{B.2.18}$$

We have

$$\begin{aligned} (d\alpha)_{c a_1 \cdots a_{n-1}} &= n \nabla_{[c} (\epsilon_{|b| a_1 \cdots a_{n-1}]} v^b) \\ &= n \epsilon_{b[a_1 \cdots a_{n-1}} \nabla_{c]} v^b \quad , \end{aligned} \tag{B.2.19}$$

where equation (B.2.11) was used. On the other hand any totally antisymmetric tensor of type $(0, n)$ must be proportional to $\boldsymbol{\epsilon}$, so

$$\epsilon_{b[a_1 \cdots a_{n-1}} \nabla_{c]} v^b = h \epsilon_{c a_1 \cdots a_{n-1}} \quad . \tag{B.2.20}$$

The function h may be evaluated by contracting with $\epsilon^{c a_1 \cdots a_{n-1}}$ and using equation (B.2.13). We obtain

$$\nabla_b v^b = nh \quad . \tag{B.2.21}$$

Thus, we find

$$d\boldsymbol{\alpha} = (\nabla_a v^a) \boldsymbol{\epsilon} \quad , \tag{B.2.22}$$

and thus Stokes's theorem states that

$$\int_{\text{int}(N)} \nabla_a v^a = \int_{\dot{N}} \epsilon_{ba_1 \cdots a_{n-1}} v^b \quad , \tag{B.2.23}$$

where the natural volume element ϵ on N is understood in the integral on the left-hand side of equation (B.2.23).

The right-hand side of equation (B.2.23) can be reexpressed as follows. The metric g_{ab} on N induces a tensor field h_{ab} on $\dot{N}$ by restriction of g_{ab} to vectors tangent to $\dot{N}$. If h_{ab} is nondegenerate—which will be the case if $\dot{N}$ is not a null surface—we may use it to define a volume element $\tilde{\epsilon}$ on $\dot{N}$. It is not difficult to show that

$$\frac{1}{n}\, \epsilon_{a_1 \cdots a_n} = n_{[a_1} \tilde{\epsilon}_{a_2 \cdots a_n]} \quad , \tag{B.2.24}$$

where n^b is the unit normal to $\dot{N}$ and is chosen to be "outward pointing" if spacelike and "inward pointing" if timelike in order that $\tilde{\epsilon}$ be of the orientation class used in Stokes's theorem. Contracting v^a into both sides of equation (B.2.24) and restricting the resulting $(n-1)$-forms to vectors tangent to $\dot{N}$, we obtain

$$\epsilon_{ba_1 \cdots a_{n-1}} v^b = (n_b v^b) \tilde{\epsilon}_{a_1 \cdots a_{n-1}} \quad , \tag{B.2.25}$$

where we view both sides of this equation as forms on $\dot{N}$. Thus, if $\dot{N}$ is not null, we can express Stokes's theorem in the form.

$$\int_{\text{int}(N)} \nabla_a v^a = \int_{\dot{N}} n_a v^a \tag{B.2.26}$$

for all C^1 vector fields v^a where the natural volume elements ϵ and $\tilde{\epsilon}$ on int(N) and $\dot{N}$, respectively, are understood. Of course, if $\dot{N}$ is null, equation (B.2.23) still applies. Furthermore, in the null case, if we choose any $\tilde{\epsilon}$ on $\dot{N}$ in the orientation class used in Stokes's theorem and define n^a to be the normal to $\dot{N}$ such that equation (B.2.24) holds, then Stokes's theorem again takes the form (B.2.26).

B.3 Frobenius's Theorem

Let M be an n-dimensional manifold. An issue which arises frequently is the following: At each point $x \in M$ we are given a subspace $W_x \subset V_x$ of the tangent space V_x with dim $W_x = m < n$. The subspace W_x is required to vary smoothly with x in the sense that for each $x \in M$ we can find an open neighborhood O of x such that in O, W is spanned by C^∞ vector fields. We denote the collection of subspaces W_x by W. We wish to know whether we can find integral submanifolds of W, i.e., whether through each point x we can find an embedded submanifold S such that the tangent space to this submanifold at each $y \in S$ coincides with W. An important special case of this general problem arises when one has a metric on M and wishes to know if a vector field ξ^a is orthogonal to a family of hypersurfaces (see, e.g., section 6.1), i.e., whether the $(n - 1)$-dimensional subspaces, W, orthogonal to ξ^a are integrable.

If the subspaces W are one-dimensional, the above problem reduces to that of finding integral curves of a smooth vector field v^a. As discussed in section 2.2, such integral curves always can be found. However, if dim $W > 1$, it is possible for the W-planes to "twist around" so that integral submanifolds cannot be found. To see that this is the case, we note that if we could find integral submanifolds, we could span W in a neighborhood of any point by coordinate vector fields $X_1^a, \ldots, X_m^a$ in M such that $[X_\mu, X_\nu] = 0$. Since any two vector fields Y^a, Z^a which lie in W can be expressed as linear combinations of these coordinate vector fields, this implies that for all Y^a, $Z^a \in W$ we have

$$[Y, Z] = \sum_{\mu,\nu} [f_\mu X_\mu, g_\nu X_\nu] = \sum_{\mu,\nu} (f_\mu X_\mu(g_\nu) - g_\mu X_\mu(f_\nu)) X_\nu \in W \quad . \tag{B.3.1}$$

If W satisfies the property that $[Y, Z]^a \in W$ for all Y^a, $Z^a \in W$, then W is said to be *involute*. We have just shown that a necessary condition for W to possess integral submanifolds is that it be involute. Conversely, it can be shown (see, e.g., Bishop and Crittenden 1964) that this condition is also sufficient. This result is known as Frobenius's theorem.

THEOREM B.3.1 (Frobenius's theorem; vector form). *A necessary and sufficient condition for a smooth specification, W, of m-dimensional subspaces of the tangent space at each $x \in M$ to possess integral submanifolds is that W be involute, i.e., for all Y^a, $Z^a \in W$ we have $[Y, Z]^a \in W$.*

Frobenius's theorem also has a dual formulation in terms of differential forms. Given $W_x \subset V_x$ as above, we can consider the one-forms $\boldsymbol{\omega} \in V_x^*$ which satisfy

$$\omega_a X^a = 0 \tag{B.3.2}$$

for all $X^a \in W_x$. It is not difficult to see that such $\boldsymbol{\omega}$'s span an $(n - m)$-dimensional subspace, $T_x^* \subset V_x^*$, of the dual tangent space at x. Conversely, an $(n - m)$-dimensional subspace T_x^* of V_x^* defines an m-dimensional subspace W_x of V_x via equation (B.3.2). Thus, we may reformulate our above question in terms of T^*: Under what conditions does a smooth specification, T^*, of $(n - m)$-dimensional subspaces of one-forms at each point have the property that the associated tangent subspaces W (consisting at each x of all vectors X^a satisfying $\omega_a X^a = 0$ for all $\omega_a \in T_x^*$) admit integral submanifolds?

According to Frobenius's theorem, integral submanifolds will exist if and only if for all $\omega_a \in T^*$ and all Y^a, $Z^a \in W$ (so that $\omega_a Y^a = \omega_a Z^a = 0$), we have

$$\omega_a [Y, Z]^a = 0 \quad . \tag{B.3.3}$$

To see what this implies for ω_a, we substitute our expression (3.1.2) for the commutator in terms of an arbitrary derivative operator ∇_a to obtain

$$\begin{aligned} 0 &= \omega_a (Y^b \nabla_b Z^a - Z^b \nabla_b Y^a) \\ &= -Z^a Y^b \nabla_b \omega_a + Y^a Z^b \nabla_b \omega_a \\ &= 2 Y^a Z^b \nabla_{[b} \omega_{a]} \quad . \end{aligned} \tag{B.3.4}$$

However, equation (B.3.4) can hold for Y^a and Z^a in the subspace annihilated by T^* if and only if $\nabla_{[a}\omega_{b]}$ can be expressed as

$$\nabla_{[a}\omega_{b]} = \sum_{\alpha=1}^{n-m} \mu^{\alpha}{}_{[a} v^{\alpha}{}_{b]} \quad , \tag{B.3.5}$$

where each $\boldsymbol{v}^{\alpha}$ is an arbitrary one-form and each $\boldsymbol{\mu}^{\alpha} \in T^*$. Thus, we can reformulate Frobenius's theorem in terms of differential forms as follows:

THEOREM B.3.2 (Frobenius's theorem; dual formulation). *Let T^* be a smooth specification of an $(n - m)$-dimensional subspace of one-forms. Then the associated m-dimensional subspace W of the tangent space admits integral submanifolds if and only if for all $\boldsymbol{\omega} \in T^*$ we have $d\boldsymbol{\omega} = \sum_{\alpha} \boldsymbol{\mu}^{\alpha} \wedge \boldsymbol{v}^{\alpha}$, where each $\boldsymbol{\mu}^{\alpha} \in T^*$.*

The dual formulation of Frobenius's theorem gives a useful criterion for when a vector field ξ^a is hypersurface orthogonal. Letting T^* be the one-dimensional subspace spanned by $\xi_a = g_{ab}\xi^b$, we see that ξ^a will be hypersurface orthogonal if and only if $\nabla_{[a}\xi_{b]} = \xi_{[a}v_{b]}$ (where we have set $\mu_a = \xi_a$ since T^* is one-dimensional). This latter condition is equivalent to $\xi_{[a}\nabla_b\xi_{c]} = 0$, and thus we see that the necessary and sufficient condition that ξ^a be hypersurface orthogonal is

$$\xi_{[a}\nabla_b\xi_{c]} = 0 \quad . \tag{B.3.6}$$