(1) Derive the transformation laws for $\frac{\partial}{\partial x^i}$.

(2) Derive the transformation laws for $dx^i$.

(3) Prove that $\{dx^0, ..., dx^3\}$ is a basis for the $T^*M_p$.

(4) Verify the general transformation law for $\left(\begin{array}{c}1 \\ 1 \end{array}\right)$-tensor- and $\left(\begin{array}{c}0 \\ 2 \end{array}\right)$-tensor-tensors, and write the laws in summation and matrix notation.

(5) Prove that $dx \otimes \frac{\partial}{\partial x^j} |_p$ spans the space of $\left(\begin{array}{c}1 \\ 1 \end{array}\right)$-tensor-tensors at $p$.

(6) Prove that if $g_{ij}$ is defined in each coordinate system so that (at a given $p \in M$) we have

$$g_{ij} a^i b^j = <X, Y> = g_{\alpha\beta} a^\alpha b^\beta,$$

for all vectors $X = a_i \frac{\partial}{\partial x^i} = a_\alpha \frac{\partial}{\partial y^\alpha}$, $Y = b_j \frac{\partial}{\partial x^j} = b_\beta \frac{\partial}{\partial y^\beta} \in T_pM$, then $g_{ij}$ must transform by

$$g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}.$$

Write the matrix version of this transformation law.

(7) Prove that the dimension of the space of multi-linear functionals on the cross product $V_1 \times V_2$ of two vector spaces, equals the product of the dimensions of the vector spaces $V_k$.

(8) Prove by counterexample that symmetry is not a covariant (i.e., coordinate independent) property of $(1, 1)$-tensors, but is a covariant property of $(0, 2)$-tensors.

(9) Let $X_1, ..., X_n$ be a basis of tangent vectors in $T_pM$. Prove that there exists a coordinate system $x : U_\kappa \to R$, $p \in U_\kappa$ such that

$$X_i = \frac{\partial}{\partial x^i} |_p.$$

(10) Prove that if $g$ is a Riemannian metric and $X \in T_pM$, then $<X, X> = 0$ if and only if $X = 0$.

(11) Prove that $\Gamma_{ikj} + \Gamma_{jki} = g_{ij,k}$. 

(12) Prove that in special relativity, $(g_{ij} \equiv \eta_{ij} = Diag \{-1, 1, 1, 1\})$, the map that takes the components of a vector (with respect to the coordinate basis $\frac{\partial}{\partial x^j}$) in $T_0M$ to the corresponding point in spacetime named by the Minkowski coordinate system $x^i$, is exactly the exponential map as we defined it.
(13) For a given vector $X \in T_pM$, define the orthogonal projection of vector $Y$ onto $X$ by

$$\text{Proj}_X Y = \frac{\langle X, Y \rangle}{\langle X, X \rangle} X.$$ 

In the case of 1+1 special relativity $g_{ij} = \eta_{ij}$, show that this is the correct definition by decomposing $Y$ into $Y = aX + bX^\perp$ and seeing that $\text{Proj}_X Y = aX$ “projects $Y$ onto $X$ along the vector $g$-orthogonal to $Y$”. (Note that $Y$ can be anything, but explain geometrically why $\text{Proj}_X Y$ is undefined when $X$ is lightlike).

(14) Show that for Lorentz transformation, $L(\theta)L(\bar{\theta}) = L(\theta + \bar{\theta})$, and use this to prove the relativistic velocity addition law

$$\bar{v} = \frac{v + \bar{v}}{1 + \frac{v \cdot \bar{v}}{c^2}},$$

where $v$ is the velocity of the barred frame as measured in the unbarred frame, and $\bar{v}$ is the velocity of the double barred frame as measured in the barred frame, and $\bar{v}$ is the velocity of the double barred frame as measured in the unbarred frame.

(15) Prove that the Proper Lorentz Transformations $A$ are characterized by $A^0_0 \geq 1$ and $\text{Det}(A) > 0$. Give a careful proof that if $A(\xi)$ is a family of Lorentz transformations that depend continuously on the parameter $\xi \in \mathbb{R}$ and $A(0)$ is proper, then $A(\xi)$ is proper for every $\xi$.

(16) Prove that the set of coordinate transformations of the form $x^i = A^i_\alpha y^\alpha + a^i$, where $A^i_\alpha$, $a^i$ are constants and $A^i_\alpha$ satisfy $\eta_{ij} = A^i_\alpha A^j_\beta \eta_{\alpha\beta}$, forms a group under composition, (the Poincare or Inhomogeneous Lorentz group).

(17) Show that in coordinates, $[X, Y] = XY - YX = \left( X^\sigma Y^j_{\sigma} - Y^\sigma X^j_{\sigma} \right) \frac{\partial}{\partial x^\sigma}$ where $\left( X^\sigma Y^j_{\sigma} - Y^\sigma X^j_{\sigma} \right)$ transforms contra-variantly.

(18) If $c(\xi)$ is a curve in manifold $\mathcal{M}^n$ with tangent vector $X$, and $x$ and $y$ are coordinates systems which overlap, use the chain rule and our conventions to deconstruct the meaning of:

$$\frac{d}{d\xi} \left( \frac{\partial x^i}{\partial y^\alpha} \right) = \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} X^\beta.$$ 

(19) In the summation convention, when the contraction of two indices expresses the multiplication of a matrix by it’s inverse, like

$$\frac{\partial x^i}{\partial y^\alpha} \frac{\partial y^\alpha}{\partial x^j},$$

we set it equal to $= \delta^j_i$, and then set $i = j$. Assuming $i, j, k, ...$ refer to $x$-coordinates and $\alpha, \beta, \gamma, ...$ to $y$-coordinates, use the transformation law for $g^{ij}$ and the summation conventions to directly verify the following identity: (Put in every step, and do not introduce matrix multiplication.)

$$g^{\gamma\sigma} g_{ij} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta} \frac{\partial x^i}{\partial y^\gamma} = \frac{\partial y^\gamma}{\partial x^j} \frac{\partial^2 x^i}{\partial y^\alpha \partial y^\beta}. $$
(20) Write the following canonical boost to velocity \( \mathbf{v} \in \mathbb{R}^3 \), \( ||v|| = v \), as a \( 4 \times 4 \) matrix, and prove that it is indeed a Lorentz transformation:

\[
\begin{pmatrix}
\gamma & -\frac{\mathbf{v} \cdot \mathbf{v}}{c^2} & 0 & 0 \\
-\frac{\mathbf{v} \cdot \mathbf{v}}{c^2} & I + \frac{\gamma-1}{c^2} \mathbf{v} \mathbf{v}^T & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{pmatrix}
\]  

(1)

(Hint: Lorentz transformations map ON bases to ON bases.)

(21) In the proof that \( B = A \tilde{A}^{-1} \) is a pure rotation if \( A, \tilde{A} \) are boosts with the same velocity, we used \( \tilde{A}^{-1} = \eta \tilde{A}^\tau \eta \) to conclude that \( B^\tau_0 = A^\tau_0 \eta_{\sigma\tau} A^\sigma_0 \eta_{00} \), (sum on \( \sigma \)). Verify this using the definition of matrix multiplication, and \( \eta^{-1} = \eta = \text{diag}\{-1,1,1,1\} \).

(22) Use the summation convention (instead of matrix multiplication) to verify that:

\[ g^{\gamma\sigma} \eta_{ij} \frac{\partial^2}{\partial x^j} \frac{\partial}{\partial y^i} = \frac{\partial y^\gamma}{\partial x^i} \frac{\partial^2}{\partial y^\gamma} \frac{\partial y^\sigma}{\partial y^\beta} = \tilde{\Gamma}_{\alpha\beta}^{\gamma} \]

so long as \( y \) is the transformation of an Locally Inertial Frame \( x \), so

\[ g^{\gamma\sigma} = \eta^{ij} \frac{\partial y^\gamma}{\partial x^i} \frac{\partial y^\sigma}{\partial x^j} \]

(23) In the case of \( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \)-tensors, express the statements that (1) Covariant differentiation \( \nabla_X \) commutes with contraction (2) Covariant differentiation \( \nabla_X \) commutes with raising and lowering of indices. The prove both statements.

(24) In the case of \( \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \)-tensor-tensors, prove the Liebniz rule for tensor products:

\[ \nabla_X (Y \oplus \omega) = \nabla_X Y \oplus \omega + Y \oplus \nabla_X \omega, \]

where

\[ Y = Y^i \frac{\partial}{\partial x^i}, \text{ and } \omega = \omega_j dx^j. \]

(25) Use that \( \omega_{\alpha,i} = \omega_{\alpha,i} - \Gamma_{\alpha i}^{\sigma} \omega_{\sigma} \) to prove that in coordinates

\[ (\nabla_j \nabla_i \omega)_{\alpha} - (\nabla_i \nabla_j \omega)_{\alpha} = R_{\alpha \beta i j}^{\gamma} \omega_{\gamma} \]

(26) Prove that if \( R_{\alpha \beta i j}^{\gamma} Z^\sigma \) transforms like a \( \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \)-tensor for every vector \( Z^\sigma \), then \( R_{\alpha \beta i j}^{\gamma} \) transforms as a \( \left( \begin{array}{c} 1 \\ 3 \end{array} \right) \)-tensor.

(27) The Riemann curvature tensor for a metric is a “curl” plus a “commutator”,

\[ R_{\beta i j}^{\alpha} = \Gamma_{\beta i j}^{\alpha} - \Gamma_{\beta i j}^{\alpha} + \Gamma_{\tau i}^{\alpha} \Gamma_{\beta j}^{\tau} - \Gamma_{\tau i}^{\alpha} \Gamma_{\beta j}^{\tau} \]
Argue that in Riemann normal coordinates, the commutator should in general be a lower order term relative to the curl.

(28) Prove that anti-symmetry and symmetry under pair exchange is a coordinate independent property of tensors.

(29) Complete the argument that the Riemann curvature tensor for a metric has (no more than) twenty independent entries.

(30) Prove that for the Einstein equations $G = \kappa T$, if $T = 0$, then $G_{ij} = 0$ if and only if $R_{ij} = 0$. 