MAPS OF MANIFOLDS, LIE DERIVATIVES, AND KILLING FIELDS

This appendix deals with topics related to the maps induced on tensor fields by maps between manifolds. As will be shown in section C.1, if we have a map, $\phi: M \to N$, between manifolds M and N, we can use ϕ to bring upper index tensor fields from M to N and lower index tensor fields from N to M. If ϕ is a diffeomorphism, all types of tensor fields can be carried from M to N or from N to M. An important special case of this result occurs when $\phi_t: M \to M$ is a one-parameter family of diffeomorphisms generated by a vector field v^a. We can compare a given tensor field with the new tensor field that arises from the action of ϕ_t for small t. As will be shown in section C.2, this gives rise to the notion of the Lie derivative with respect to the vector field v^a. Finally, a vector field which generates a one-parameter group of isometries is called a Killing vector field. Using the general formulas for Lie derivatives, an equation for Killing fields is easily obtained and some important properties of them are derived in section C.3.

C.1 Maps of Manifolds

Let M and N be manifolds (*not* necessarily of the same dimension) and let $\phi: M \to N$ be a C^∞ map. In a natural manner, ϕ "pulls back" a function $f: N \to \mathbb{R}$ on N to the function $f \circ \phi: M \to \mathbb{R}$ obtained by composing f with ϕ. Similarly, in a natural way, ϕ "carries along" tangent vectors at $p \in M$ to tangent vectors at $\phi(p) \in N$—i.e., it defines a map $\phi^*: V_p \to V_{\phi(p)}$—as follows: For $v \in V_p$ we define $\phi^* v \in V_{\phi(p)}$ by

$$(\phi^* v)(f) = v(f \circ \phi) \tag{C.1.1}$$

for all smooth $f: N \to \mathbb{R}$, where we have dropped the vector indices on v and $\phi^* v$ since that notation is inconvenient here. It is easy to check that $\phi^* v$ satisfies the properties required of a tangent vector at $\phi(p)$ and thus equation (C.1.1) properly defines the map ϕ^*. Note that ϕ^* is linear and may be viewed as the "derivative of ϕ" at p. [The matrix of components of ϕ^* in the coordinate bases of a coordinate system $\{x^\nu\}$ at p and a coordinate system $\{y^\mu\}$ at $\phi(p)$ equals the Jacobian matrix of the map ϕ between the coordinates, i.e., $(\phi^*)^\mu{}_\nu = \partial y^\mu / \partial x^\nu$.] By the implicit function theorem, $\phi: M \to N$ will be one-to-one in a neighborhood of p if $\phi^*: V_p \to V_{\phi(p)}$ is one-to-one.

Similarly, we can use ϕ to "pull back" dual vectors at $\phi(p)$ to dual vectors at p. We define the map $\phi_*: V^*_{\phi(p)} \to V^*_p$ by requiring that for all $v^a \in V_p$,

$$(\phi_*\mu)_a v^a = \mu_a(\phi^* v)^a \quad . \tag{C.1.2}$$

We can extend the action of ϕ_* to map tensors of type $(0, l)$ at $\phi(p)$ to tensors of type $(0, l)$ at p by

$$(\phi_* T)_{a_1 \cdots a_l}(v_1)^{a_1} \cdots (v_l)^{a_l} = T_{a_1 \cdots a_l}(\phi^* v_1)^{a_1} \cdots (\phi^* v_l)^{a_l} \quad . \tag{C.1.3}$$

Similarly, we can extend the action of ϕ^* to map tensors of type $(k, 0)$ at p to tensors of type $(k, 0)$ at $\phi(p)$ by

$$(\phi^* T)^{b_1 \cdots b_k}(\mu_1)_{b_1} \cdots (\mu_k)_{b_k} = T^{b_1 \cdots b_k}(\phi_*\mu_1)_{b_1} \cdots (\phi_*\mu_k)_{b_k} \quad . \tag{C.1.4}$$

(By eq. [C.1.2], this is consistent with our original definition of ϕ^* on vectors.) However, in general we cannot extend ϕ^* or ϕ_* to mixed tensors since ϕ^* does not know how to "carry along" lower index tensors, while ϕ_* does not know how to "pull back" upper index tensors.

As defined in chapter 2, a C^∞ map $\phi: M \to N$ is said to be a diffeomorphism if it is one-to-one, onto, and its inverse is C^∞. If ϕ is a diffeomorphism (which necessarily implies $\dim M = \dim N$), then we can use ϕ^{-1} to extend the definition of ϕ^* to tensors of all types by using the fact that $(\phi^{-1})^*$ goes from $V_{\phi(p)}$ to V_p. If $T^{b_1 \cdots b_k}{}_{a_1 \cdots a_l}$ is a tensor of type (k, l) at p, we define the tensor $(\phi^* T)^{b_1 \cdots b_k}{}_{a_1 \cdots a_l}$ at $\phi(p)$ by,

$$(\phi^* T)^{b_1 \cdots b_k}{}_{a_1 \cdots a_l}(\mu_1)_{b_1} \cdots (\mu_k)_{b_k}(t_1)^{a_1} \cdots (t_l)^{a_l}$$
$$= T^{b_1 \cdots b_k}{}_{a_1 \cdots a_l}(\phi_*\mu_1)_{b_1} \cdots ([\phi^{-1}]^* t_l)^{a_l} \quad . \tag{C.1.5}$$

Similarly, we could extend the map ϕ_* to all tensors. However, it is not difficult to show that $\phi_* = (\phi^{-1})^*$, so we need only consider ϕ^* and $(\phi^{-1})^*$.

If $\phi: M \to M$ is a diffeomorphism and T is a tensor field on M, we can compare T with $\phi^* T$. If $\phi^* T = T$, then even though we have "moved T" via ϕ, it has "stayed the same." In other words, ϕ is a *symmetry transformation* for the tensor field T. In the case of the metric g_{ab}, a symmetry transformation—i.e., a diffeomorphism ϕ such that $(\phi^* g)_{ab} = g_{ab}$—is called an *isometry*.

We have already remarked in chapter 2 that if $\phi: M \to N$ is a diffeomorphism, than M and N have identical manifold structure. If a theory describes nature in terms of a spacetime manifold, M, and tensor fields, $T^{(i)}$, defined on the manifold, then if $\phi: M \to N$ is a diffeomorphism, the solutions $(M, T^{(i)})$ and $(N, \phi^* T^{(i)})$ have physically identical properties. Any physically meaningful statement about $(M, T^{(i)})$ will hold with equal validity for $(N, \phi^* T^{(i)})$. On the other hand, if $(N, T'^{(i)})$ is not related to $(M, T^{(i)})$ by a diffeomorphism and if the tensor fields $T^{(i)}$ represent measurable quantities, then $(N, T'^{(i)})$ will be physically distinguishable from $(M, T^{(i)})$. Thus, the diffeomorphisms comprise the gauge freedom of any theory formulated in terms of tensor fields on a spacetime manifold. In particular, diffeomorphisms comprise the gauge freedom of general relativity.

It is worth noting that an alternative viewpoint on diffeomorphisms can be taken. Above, we have discussed diffeomorphisms without introducing or making any reference to coordinate systems. We have taken an "active" point of view by associating with ϕ a map from tensors at p to tensors at $\phi(p)$. However, if we are given a coordinate system $\{x^\mu\}$ covering a neighborhood, U, of p and a coordinate system $\{y^\mu\}$ covering a neighborhood, V, of $\phi(p)$, we may take the following "passive" point of view. We may use ϕ to define a new coordinate system x'^μ in the neighborhood $O = \phi^{-1}[V]$ of p by setting $x'^\mu(q) = y^\mu(\phi(q))$ for $q \in O$. We then may view the effect of ϕ as leaving p and all tensors at p unchanged, but inducing the coordinate transformation $x^\mu \to x'^\mu$. This "passive" point of view on diffeomorphisms is, philosophically, drastically different from the above "active" viewpoint, but, in practice, these viewpoints are really equivalent since the components of the tensor $\phi^* T$ at $\phi(p)$ in the coordinate system $\{y^\mu\}$ in the active viewpoint are precisely the components of T at p in the coordinate system $\{x'^\mu\}$ in the passive viewpoint.

C.2 Lie Derivatives

Let M be a manifold and let ϕ_t be a one-parameter group of diffeomorphisms. As discussed in section 2.2, ϕ_t will be generated by a vector field, v^a. By the results of the previous section, we can use ϕ_t^* to carry along a smooth tensor field $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$. Comparison of $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ and $\phi_{-t}^* T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ for small t gives rise to the notion of the Lie derivative, $\pounds_v$, with respect to v^a. More precisely, we define $\pounds_v$ by

$$\pounds_v T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = \lim_{t \to 0} \left\{ \frac{\phi_{-t}^* T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} - T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}}{t} \right\} \quad , \qquad \text{(C.2.1)}$$

where all tensors appearing in equation (C.2.1) are evaluated at the same point p. Note that the vector index on v^a is dropped in the symbol $\pounds_v$ since its presence could lead to confusion.

It follows immediately from its definition, equation (C.2.1), that $\pounds_v$ is a linear map from smooth tensor fields of type (k, l) to smooth tensor fields of type (k, l). It also is not difficult to show (see eq. [C.2.4] below) that $\pounds_v$ satisfies the Leibnitz rule on outer products of tensors. Furthermore, since v^a is tangent to the integral curves of ϕ_t, for functions $f: M \to \mathbb{R}$ we have

$$\pounds_v(f) = v(f) \quad . \qquad \text{(C.2.2)}$$

Note also that $\pounds_v T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = 0$ everywhere if and only if for all t, ϕ_t is a symmetry transformation for $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$.

To analyze the action of $\pounds_v$ on an arbitrary tensor field, it is helpful to introduce a coordinate system on M where the parameter t along the integral curves of v^a is chosen as one of the coordinates x^1, so that $v^a = (\partial/\partial x^1)^a$. (This always can be done locally in any region where $v^a \neq 0$.) The action of ϕ_{-t} then corresponds to the coordinate transformation $x^1 \to x^1 + t$, with $x^2, \ldots, x^n$ held fixed. From the parenthetical remark below equation (C.1.1), we have $(\phi^*)^\mu{}_\nu = \delta^\mu{}_\nu$ and hence, the coordinate basis components of $\phi_{-t}^* T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ at the point p whose coordinates are $(x^1, \ldots, x^n)$ are

$$(\phi_{-t}^* T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l})(x^1, \ldots, x^n) = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}(x^1 + t, x^2, \ldots, x^n) \quad . \qquad \text{(C.2.3)}$$

Consequently, the components of the Lie derivative of $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ in a coordinate system adapted to v^a are simply

$$\pounds_v T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = \frac{\partial T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}}{\partial x^1} \quad . \tag{C.2.4}$$

Thus, in particular, ϕ_t will be a symmetry transformation of $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ if and only if the components $T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l}$ in a coordinate system adapted to v^a are independent of the integral curve coordinate x^1.

We can obtain a coordinate independent expression for the Lie derivative of a vector field w^a by noting that in an adapted coordinate system we have by equation (C.2.4),

$$\pounds_v w^\mu = \frac{\partial w^\mu}{\partial x^1} \quad . \tag{C.2.5}$$

On the other hand, since $v^a = (\partial/\partial x^1)^a$ and $w^a = \sum_\mu w^\mu(\partial/\partial x^\mu)^a$, the commutator of v^a and w^a is given by

$$\begin{aligned}[v, w]^\mu &= \sum_\nu \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right) \\ &= \frac{\partial w^\mu}{\partial x^1} \quad .\end{aligned} \tag{C.2.6}$$

Thus, we find that the components of $\pounds_v w^a$ and $[v, w]^a$ are equal in an adapted coordinate system. However, since both of these quantities are defined in a coordinate-independent manner, we obtain

$$\pounds_v w^a = [v, w]^a \quad , \tag{C.2.7}$$

which is the coordinate-independent formula we sought for the Lie derivative of a vector field.

The action of the Lie derivative on all other types of tensor fields is determined by equations (C.2.2), (C.2.7) and the Leibnitz rule. For example, for a dual vector field, μ_a, we have by equation (C.2.2)

$$\pounds_v(\mu_a w^a) = v(\mu_a w^a) \quad , \tag{C.2.8}$$

where w^a is an arbitrary field. On the other hand, by the Leibnitz rule and equation (C.2.7), we have,

$$\pounds_v(\mu_a w^a) = w^a \pounds_v \mu_a + \mu_a [v, w]^a \quad . \tag{C.2.9}$$

From the equality of the right sides of equations (C.2.8) and (C.2.9) we obtain a formula which determines $\pounds_v \mu_a$. This formula is most conveniently expressed in terms of a derivative operator. If ∇_a is an arbitrary derivative operator on M, we have by properties (4) and (2) of the definition of derivative operator (see section 3.1)

$$\begin{aligned}v(\mu_a w^a) &= v^b \nabla_b(\mu_a w^a) \\ &= v^b w^a \nabla_b \mu_a + v^b \mu_a \nabla_b w^a \quad .\end{aligned} \tag{C.2.10}$$

On the other hand, we showed previously (see eq. [3.1.2]) that

$$[v, w]^a = v^b \nabla_b w^a - w^b \nabla_b v^a \quad . \tag{C.2.11}$$

Thus, we find

$$v^b w^a \nabla_b \mu_a + v^b \mu_a \nabla_b w^a = w^a \pounds_v \mu_a + \mu_a v^b \nabla_b w^a - \mu_a w^b \nabla_b v^a \quad , \tag{C.2.12}$$

i.e.,

$$\pounds_v \mu_a = v^b \nabla_b \mu_a + \mu_b \nabla_a v^b \quad . \tag{C.2.13}$$

More generally for an arbitrary tensor field $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ we find by induction that

$$\pounds_v T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = v^c \nabla_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} - \sum_{i=1}^{k} T^{a_1 \cdots c \cdots a_k}{}_{b_1 \cdots b_l} \nabla_c v^{a_i}$$
$$+ \sum_{j=1}^{l} T^{a_1 \cdots a_k}{}_{b_1 \cdots c \cdots b_l} \nabla_{b_j} v^c \quad . \tag{C.2.14}$$

Again, we emphasize that equation (C.2.14) holds for any derivative operator ∇_a.

Finally, we already remarked in section C.1 above that if $\phi: M \to M$ is a diffeomorphism, then (M, g_{ab}) and $(M, \phi^* g_{ab})$ represent the same physical spacetime. If we consider a one-parameter family of spacetimes $(M, g_{ab}(\lambda))$, then $(M, \phi^*_\lambda g_{ab}(\lambda))$ represents the same physical one-parameter family, where ϕ_λ is an arbitrary one-parameter group of diffeomorphisms. If, as in sections 4.4 and 7.5, we consider the first order perturbation of $g_{ab}|_{\lambda=0}$ obtained by differentiating $g_{ab}(\lambda)$ with respect to λ at $\lambda = 0$, we find that $\gamma_{ab} = dg_{ab}/d\lambda|_{\lambda=0}$ and $\gamma'_{ab} = d(\phi^*_\lambda g_{ab})/d\lambda|_{\lambda=0}$ represent the same physical perturbation. But, it is not difficult to see that

$$\gamma'_{ab} = \gamma_{ab} - \pounds_v g_{ab} \quad , \tag{C.2.15}$$

where v^a is the vector field which generates ϕ_λ and $g_{ab} = g_{ab}(\lambda = 0)$. Thus, the gauge freedom in perturbations, γ_{ab}, is given by $\pounds_v g_{ab}$, where v^a is an arbitrary vector field. Furthermore, by equation (C.2.14) we have

$$\begin{aligned} \pounds_v g_{ab} &= v^c \nabla_c g_{ab} + g_{cb} \nabla_a v^c + g_{ac} \nabla_b v^c \\ &= \nabla_a v_b + \nabla_b v_a \quad , \end{aligned} \tag{C.2.16}$$

where the second line of equation (C.2.16) holds when ∇_a is the derivative operator associated with g_{ab}. Thus, the gauge transformations of linearized general relativity about a solution g_{ab} are

$$\gamma_{ab} \to \gamma'_{ab} = \gamma_{ab} - \nabla_a v_b - \nabla_b v_a \quad . \tag{C.2.17}$$

This is closely analogous to the gauge freedom $A_a \to A'_a = A_a - \nabla_a \chi$ of electromagnetism.

C.3 Killing Vector Fields

If $\phi_t: M \to M$ is one-parameter group of isometries, $\phi^*_t g_{ab} = g_{ab}$, the vector field ξ^a which generates ϕ_t is called a *Killing vector field*. As already remarked below equation (C.2.2), the necessary and sufficient condition for ϕ_t to be a group of isometries is $\pounds_\xi g_{ab} = 0$. Thus, according to equation (C.2.16), the necessary and sufficient condition that ξ^a be a Killing field is that it satisfy *Killing's equation*

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \quad , \tag{C.3.1}$$

where ∇_a is the derivative operator associated with g_{ab}.

One of the most useful properties of Killing vector fields is given in the following proposition.

PROPOSITION C.3.1. Let ξ^a be a Killing vector field and let γ be a geodesic with tangent u^a. Then $\xi_a u^a$ is constant along γ.

Proof. We have

$$\begin{aligned} u^b \nabla_b (\xi_a u^a) &= u^b u^a \nabla_b \xi_a + \xi_a u^b \nabla_b u^a \\ &= 0 \quad , \end{aligned} \tag{C.3.2}$$

since the first term vanishes by Killing's equation (C.3.1) and the second term vanishes by the geodesic equation. □

Since in general relativity timelike geodesics represent the spacetime motions of freely falling particles and null geodesics represent the paths of light rays, proposition C.3.1 can be interpreted as saying that every one-parameter family of symmetries gives rise to a conserved quantity for particles and light rays. This conserved quantity enables one to determine the gravitational redshift in stationary spacetimes and is extremely useful for integrating the geodesic equation when symmetries are present (see section 6.3).

Another useful formula relates the second derivative of a Killing field to the Riemann tensor. By definition of the Riemann tensor, we have

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = R_{abc}{}^d \xi_d \quad . \tag{C.3.3}$$

On the other hand, by Killing's equation, we can rewrite equation (C.3.3) as

$$\nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a = R_{abc}{}^d \xi_d \quad . \tag{C.3.4}$$

If we write down the same equation with cyclic permutations of the indices (abc), and then add the (abc) equation to the (bca) equation and subtract the (cab) equation, we obtain

$$\begin{aligned} 2\nabla_b \nabla_c \xi_a &= (R_{abc}{}^d + R_{bca}{}^d - R_{cab}{}^d)\xi_d \\ &= -2R_{cab}{}^d \xi_d \quad , \end{aligned} \tag{C.3.5}$$

where the symmetry property (3.2.14) of the Riemann tensor was used in the last equality. Thus, for any Killing field ξ^a, we obtain the formula

$$\nabla_a \nabla_b \xi_c = -R_{bca}{}^d \xi_d \quad . \tag{C.3.6}$$

An important consequence of equation (C.3.6) is that a Killing field, ξ^a, is completely determined by the values of ξ^a and $L_{ab} \equiv \nabla_a \xi_b$ at any point $p \in M$; namely, if we are given (ξ^a, L_{ab}) at p, then (ξ^a, L_{ab}) at any other point q is determined by integration of the system of ordinary differential equations

$$v^a \nabla_a \xi_b = v^a L_{ab} \quad , \tag{C.3.7}$$

$$v^a \nabla_a L_{bc} = -R_{bca}{}^d \xi_d v^a \quad , \tag{C.3.8}$$

along any curve connecting p and q, where v^a denotes the tangent to the curve. Immediate corollaries of this result are (i) if a Killing field and its derivative vanish at a point, then the Killing field vanishes everywhere, and (ii) on a manifold of dimension n, there can be at most $n + n(n-1)/2 = n(n+1)/2$ linearly independent Killing fields [and, thus, at most an $n(n+1)/2$ parameter group of isometries], since this is the dimension of the space of initial data for (ξ^a, L_{ab}).

It is worth noting that if we contract equation (C.3.6) over a and b, we find

$$\nabla^a \nabla_a \xi_c = -R_c{}^d \xi_d \quad . \tag{C.3.9}$$

Thus, in a vacuum spacetime, $R_c{}^d = 0$, ξ^a satisfies the source-free Maxwell equation (4.3.15) for a vector potential in the Lorentz gauge. (There is a sign difference in the Ricci tensor term between eqs. [4.3.15] and [C.3.9], so Maxwell's equation is not satisfied when $R_{ab} \neq 0$.) The Lorentz gauge condition $\nabla_a \xi^a = 0$ is also satisfied because of Killing's equation, and thus all Killing fields in vacuum spacetimes give rise to solutions of Maxwell's equation. Some solutions of physical interest can be obtained in this way (Wald 1974*b*).

In the case of a hypersurface orthogonal Killing vector field, χ^a, we can obtain a simple formula for $\nabla_a \chi_b$. By Frobenius's theorem B.3.2, there exists a vector field v^a such that

$$\nabla_a \chi_b = \nabla_{[a} \chi_{b]} = \chi_{[a} v_{b]} \quad . \tag{C.3.10}$$

Assuming that χ^a is not null, we may choose v^a to be orthogonal to χ^a. Contracting equation (C.3.10) with χ^b, we obtain

$$\frac{1}{2} \nabla_a (\chi^b \chi_b) = -\frac{1}{2} v_a \chi^b \chi_b \quad . \tag{C.3.11}$$

Hence, by solving equation (C.3.11)for v^a and substituting the result in equation (C.3.10), we find that an arbitrary hypersurface orthogonal Killing field χ^a with $\chi^a \chi_a \neq 0$ satisfies

$$\nabla_a \chi_b = -\chi_{[a} \nabla_{b]} \ln |\chi^c \chi_c| \quad . \tag{C.3.12}$$

Finally, we mention two generalizations of the notion of Killing vector fields. First, a *conformal isometry*, ϕ, on a manifold, M, with metric, g_{ab}, is defined to be a diffeomorphism $\phi: M \to M$ for which there is a function Ω such that $\phi^* g_{ab} = \Omega^2 g_{ab}$. (The fact that ϕ is a diffeomorphism implies that Ω is nonvanishing. The case $\Omega = 1$, of course, corresponds to an ordinary isometry.) The infinitesimal generator, ψ^a, of a one-parameter group, ϕ_t, of conformal isometries is called a *conformal Killing vector field*. Clearly, the Lie derivative of g_{ab} with respect to ψ^a must be proportional to g_{ab}. Thus, ψ^a satisfies

$$\nabla_a \psi_b + \nabla_b \psi_a = \alpha g_{ab} \quad , \tag{C.3.13}$$

where ∇_a is the derivative operator associated with g_{ab}. Taking the trace of equation (C.3.13), we evaluate the function α, thus obtaining

$$\nabla_a \psi_b + \nabla_b \psi_a = \frac{2}{n}(\nabla^c \psi_c) g_{ab} \quad , \tag{C.3.14}$$

where $n = \dim M$. Equation (C.3.14) is known as the *conformal Killing equation*.

In Proposition C.3.1, we proved that for any geodesic with tangent u^a and for any Killing field ξ^a, the inner product, $\xi_a u^a$, is constant along the geodesic. The same calculation for a conformal Killing field yields

$$u^b \nabla_b (\psi_a u^a) = \frac{1}{n}(\nabla^c \psi_c) u^b u_b \quad . \tag{C.3.15}$$

Thus, in general, $\psi_a u^a$ is *not* constant along a geodesic. However, for a null geodesic we have $u^b u_b = 0$, so the right-hand side of equation (C.3.15) vanishes. Thus, conformal Killing fields give rise to constants of motion for null geodesics.

The second generalization we mention of a Killing vector is that of a Killing tensor. A *Killing tensor field* of order m on a manifold M with derivative operator ∇_a is defined to be a totally symmetric m-index tensor field, $K_{a_1 \cdots a_m} = K_{(a_1 \cdots a_m)}$, which satisfies the equation

$$\nabla_{(b} K_{a_1 \cdots a_m)} = 0 \quad . \tag{C.3.16}$$

Although equation (C.3.16) is a natural generalization of Killing's equation (C.3.1), it should be noted that (aside from Killing vectors or Killing tensors formed from products of Killing vectors) Killing tensor fields do not arise in any natural way from groups of diffeomorphisms of M. However, Killing tensors share with Killing vectors the property of giving rise to constants of the motion: A repetition of the proof of Proposition C.3.1 shows that for any geodesic γ with tangent u^a, the quantity $K_{a_1 \cdots a_m} u^{a_1} \ldots u^{a_m}$ is constant along γ. The Kerr metric (see chapter 12) possesses a nontrivial Killing tensor K_{ab}, and the constant of motion to which it gives rise (together with the constants obtained from the two Killing vectors) enables one to obtain all the geodesics explicitly.