INTRODUCTION

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Math-280: A Mathematical Introduction
to
Shock Waves

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Math 280 – "Mathematical Theory of Shock waves"

- Compressible Euler Equations: (inviscid)

$p = \rho(\gamma-1)e$
$E = \tfrac{1}{2}\rho u^2 + \rho e$

$$\rho_t + \operatorname{div}(\rho \underset{\sim}{u}) = 0$$
$$(\rho u)_t + \operatorname{div}(\rho u^i \underset{\sim}{u} + p e^i) = 0$$
$$E_t + \operatorname{div}((E+p)\underset{\sim}{u}) = 0$$

Notation: div, $\frac{\partial}{\partial t}u = u_t$, $\frac{\partial}{\partial x^i} = u_{x^i}$
ρ = density
$\underset{\sim}{u}$ = velocity
$E = \frac{\text{energy}}{\text{vol}}$
p = pressure

Derived Conservation Law for Entropy:

$$S_t + \operatorname{div}(Su) = 0$$

$p = p(\rho, E) \Rightarrow$ 5 eqn's in 5 unknowns

$p = p(\rho, S)$ eqn of state

"nonlinear theory of sound waves" & "Explosions"

- History: written down $\approx$ 1750 by Leonid Euler

Problem ① 1687 Newtons Principia ~ give a continuum version of Newtons laws of motion (Newton got it wrong)

Problem ② Explain/Predict sound wave propagation

- Euler & D'Alembert in St. Petersburg Russia ~1750
DeLambert introduce wave eqn $u_{tt} - c^2\Delta u = 0$
"eqn in which everything prop's with sound speed c"

- 1750, Euler introduced the nonlinear theory & linearized the equations to get wave eqn in density $\rho_{tt} - c^2\Delta\rho = 0$.

Q: (open) Do the nonlinear eqn's support periodic soln's $\approx$ sinusoidal soln's of linear wave eqn?

Appl: Compute flow over airfoil –

Accurate to neglect viscosity except on surface layer

- 1-D Case (Shock tube problem)

$$\rho_t + (\rho u)_x = 0$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0$$
$$E_t + [(E+p)u]_x = 0$$

Derived Eqn: $S_t + \operatorname{div}(Su) = 0$

- Need an equation of state to close the system

polytropic: $p = c\, e^{S/c_v} \rho^\gamma$ $\quad 1 < \gamma < 2 \quad p = \rho(\gamma-1)e$

$\gamma \approx 1.4$ models air at STP $\quad p = p(\rho, S)$

Compressible Euler ~ Special case of a general nonlinear system of cons laws:

- Multi-D: $u_t + \mathrm{div} F(u) = 0$ $\qquad \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_t + \mathrm{div}\begin{bmatrix} \vec{F}_1(u) \\ \vdots \\ \vec{F}_n(u) \end{bmatrix}$

$\vec{F}_k = (F_k^1, \dots, F_k^n)$

- 1-D Case: $u_t + f(u)_x = 0$

$u = (u^1, \dots, u^n)$, $f(u) = (f_1(u), \dots, f_n(u))$

$$\underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_t}_{\substack{\text{conserved} \\ \text{quantities}}} + \underbrace{\begin{bmatrix} f_1(u) \\ \vdots \\ f_n(u) \end{bmatrix}_x}_{\substack{\text{nonlinear} \\ \text{fluxes}}} = 0 \iff \underbrace{\begin{bmatrix} \rho \\ m \\ E \end{bmatrix}_t}_{\substack{\text{Euler} \\ \text{conserved} \\ \text{quantities}}} + \underbrace{\begin{bmatrix} \rho u \\ \rho u^2 + P(\rho, E) \\ E(p+u) \end{bmatrix}_x}_{\substack{\text{Euler} \\ \text{fluxes}}} = 0$$

- Note: (we use u^i or u_i as convenient)

$$\begin{bmatrix} u^1 \\ \vdots \\ u^n \end{bmatrix}_t + \begin{bmatrix} f_1(u) \\ \vdots \\ f_n(u) \end{bmatrix}_x = 0 \iff u_t + \underbrace{df(u)}_{\substack{n\times n \\ \text{matrix} \\ \text{of partials}}} \cdot u_x = 0$$

$$\uparrow$$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} u_x^1 + \cdots + \frac{\partial f_1}{\partial u_n} u_x^n \\ \vdots \\ \frac{\partial f_n}{\partial u_1} u_x^1 + \cdots + \frac{\partial f}{\partial u_n} u_x^n \end{bmatrix}$$

$$\uparrow$$

$$df \cdot u_x$$

$$\nearrow$$

$$\begin{bmatrix} \frac{\partial f_i}{\partial u_j} \end{bmatrix} \begin{bmatrix} u_x^1 \\ \vdots \\ u_x^n \end{bmatrix}$$

- Problem: look for solutions $u(x,t)$

- Q: what are the issues? $u_t + f(u)_x = 0$

Warmup Problem: Burgers' Eqvn (Inviscid)

$$u_t + \left(\tfrac{1}{2}u\right)^2_x = 0 \qquad f(u) = \tfrac{1}{2}u^2$$

$$u_t + u u_x = 0 \qquad df = u$$

Nonlinearity $\Rightarrow$ shock waves form in finite time

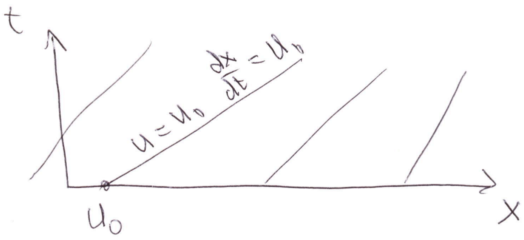
(1) $u_t + u u_x = 0 \iff \nabla_{(1,u)} u(x,t) = 0$

$\iff \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x}\right) u(x,t) = 0$

$\iff$ Solutions are constant in directions $\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ in xt-plane

$\iff$ $u(x,t)$ constant along lines of speed $\frac{dx}{dt} = u$

Picture:

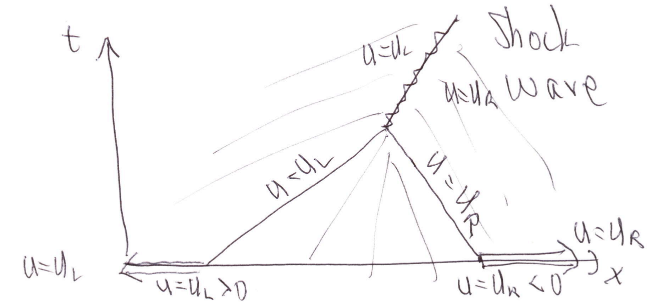


Conclude: The initial value problem makes sense:

$u_t + u u_x = 0$ "constant values are propagated
$u(x,0) = u_0(x)$ forward in time along characteristics $\frac{dx}{dt} = u$ ~
hyperbolic like the wave eqvn except nonlinear"

- Immediate Consequence – "Shock wave discontinuities form in finite time"

Picture:



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Q: How to continue the theory of solutions to include discontinuous soln's that model shock waves?

(2) Shock Formation by PDE methods:
(Blowup of soln's)

- Assume a soln $u(x,t)$ of $u_t + uu_x = 0$ is given. We derive condition under which $u_x \to \infty$ in finite time $\approx$ shock

- Diff eqn (to get eqn for u_x):

$$u_{xt} + uu_{xx} + u_x^2 = 0$$

- Consider a characteristic: $x = x(t)$

$$\dot{x} \equiv \frac{dx}{dt} = u(x(t),t)$$

"u known $\Rightarrow$ $x(t)$ is the solution of an autonomous ODE"

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- Our eqn holds along characteristic:

$$\underbrace{\frac{\partial}{\partial t} u_x(x(t),t) + u \frac{\partial}{\partial x} u_x(x(t),t)} = -u_x^2(x(t),t)$$

$$\frac{d}{dt} u_x(x(t),t) \sim u_{xx} \underbrace{\dot{x}}_{u} + u_{xt} \checkmark$$

$\therefore$
$$\frac{d}{dt} u_x(x(t),t) = -u_x^2(x(t),t)$$

holds along characteristic $x = x(t)$. Let

$$V(t) = u_x(x(t),t)$$

$\Rightarrow$ $\dot{V} = -V^2$ Ricotti Eqn

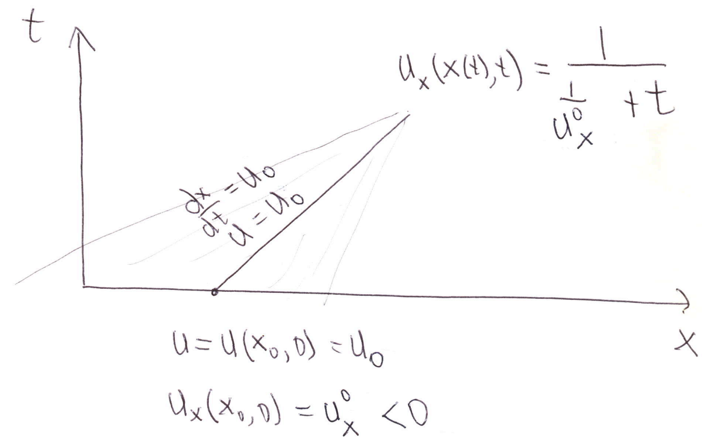
$$\int_{V_0}^{V} \frac{dv}{v^2} = -\int_0^t dt \iff -\frac{1}{V}\Big]_{V_0}^{V} = -t$$

$$\iff \frac{1}{V} - \frac{1}{V_0} = t \iff \boxed{V = \frac{1}{\frac{1}{V_0} + t}}$$

Conclude: $u_x(x(t),t) = \dfrac{1}{\frac{1}{u_x(x_0,0)} + t}$

$\therefore$ if $u_x(x_0,0)$ is negative, then $u_x \to \infty$ as $t \to \dfrac{1}{u_x(x_0,0)}$ $\Rightarrow$ "blowup in derivative"

Picture:



• Theorem: Let $u(x,t)$ denote a smooth solution of Burgers' Eqn $u_t + uu_x = 0$ defined in $t \geq 0$. Then the derivative "blows up" in time

$$T = \frac{1}{\max(-u_x^0)}$$

u_x^0 = largest negative value of u_x at time $t=0$.

Pf.
$$u_x(x(t),t) = \frac{1}{\frac{1}{u_x(x_0,0)} + t}$$

Put in the max neg value of u_x initially & get shortest time to blowup u_x

$$T = \frac{1}{|\text{Max neg value of } u_x(x,0)|}$$

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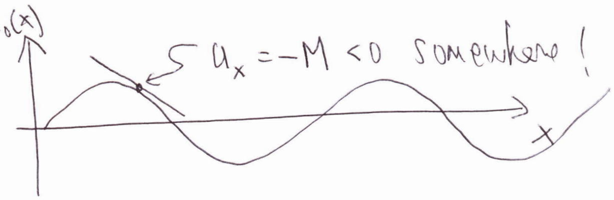
<u>Corollary</u>: Periodic solutions of $u_t + uu_x = 0$ become discontinuous in finite time.

Eg:



(FIP) Show that the same argument applies to any convex cons. law $u_t + f(u)_x = 0$,

$$f''(u) > D > 0.$$

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Q: Does this result extend to compressible Euler equations?

Ans①: In 1950's Lax showed that a similar blowup result holds for 2x2 system ≈ p-system

<u>Ans②</u>: In 1970, Glimm & Lax proved: periodic soln's of ~~compressible~~ 2x2 systems ≈ p system form shock waves <u>and</u> <u>decay</u> like $O\left(\frac{1}{t}\right)$

<u>Ans③</u>: For 3x3 Euler - still <u>unknown</u>

Q: How to continue the notion of solution to allow for shock-waves?

Theory of Distributions - (Invented for linear equations)

"Multiply by a test fn & integrate by parts"

$$u_t + f(u)_x = 0$$

$$\iint_{\substack{-\infty < x < \infty \\ t \geq 0}} u_t(x,t)\,\varphi(x,t) + f(u)_x\,\varphi(x,t)\,dx\,dt$$

$\varphi(x,t)$ smooth with compact supp ($\varphi \in C^\infty$ & $\varphi(x,t) = 0$ off a bounded set)

Integrate by parts: (FIP)

$$\iint_{\substack{-\infty < x < \infty \\ t \geq 0}} u\,\varphi_t + f(u)\,\varphi_x\,dx\,dt + \int_{-\infty}^{\infty} u(x,0)\,\varphi(x,0)\,dx = 0 \qquad (*)$$

Hint: Use divergence Thm to derive a formula for integration by parts -

Div Thm: $\vec{f}(\underset{\sim}{x})$, $\underset{\sim}{x} \in \mathbb{R}^n$, $\displaystyle\iint_{\partial\Omega} \vec{f}\cdot\vec{n}\,ds = \iint_{\Omega} \operatorname{div} f\, d^n\underset{\sim}{x}$

$$\Rightarrow \iint_{\Omega} f g_{x_i}\, d^n\underset{\sim}{x} = -\iint_{\Omega} f_{x_i} g\, d^n\underset{\sim}{x} + \int_{\partial\Omega} f g n^i\, dS$$

• Issues ①
: requires total deriv on every term!

$u_t + u u_x = 0$ no weak form

$u_t + \left(\frac{1}{2}u^2\right)_x = 0$ has weak form

$\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x = 0$ has weak form but differnt from Burgers'!

Important to choose the physical conserved quantities in weak formulation

Eg: Euler:
Correct conserved quantity:

$$\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix}_t + \Big(\quad\Big)_x = 0$$

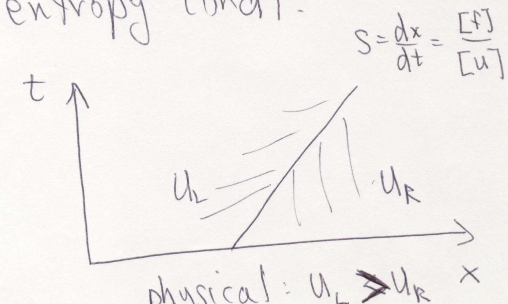
Incorrect

$$\begin{pmatrix}\rho\\ \rho u\\ S\end{pmatrix}_t + \Big(\quad\Big)_x = 0$$

• Issue ②: Shock waves introduce dissipation, time irreversibility, increase of entropy

Weak ~~equn~~'s formulation requires an additional entropy condt:

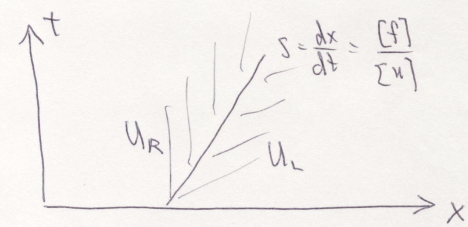
Eg: Burgers'



physical: $u_L > u_R$

Both meet weak equn's

"Lax Char Condt — char's should impinge on the shock"



unphysical: $u_R > u_L$

Topics In Class:

① Derivation of Euler Equns ≈ Continuum Mech's

- Polytropic Equn of state
- Linearize to get DeLambert's wave eqn

(Ref: Hughs & Marsden, Gurtin, Dafermos..)

② Basic Theory of Cons. Laws

- nxn system $u_t + f(u)_x = 0$
- genuine nonlinearity / linearly degenerate characteristic fields
- rarefaction waves, contact discontinuities / shock waves
- Entrop Condition
- Riemann Problem (local & global for gas dynamics)
- Hugoniot Locus
- Initial Value Problem (Glimm's Method)
- Shock Profiles

③ Shock Waves in General Relativity

- Intro to GR
- Israel Condt's for shock matchy
- Construction of exact soln (Te-Sm)
- Cosmology

Ref: Smoller Ch 17 (Shock waves & Reaction Diffusion Equn's)

Leveque – "Numerical Methods for Cons Laws" Part I

D Serre – "Systems of Cons Laws"

Dafermos / Bressan

④ Periodic Solutions – Dissipation Free Transmission of Sound Waves (open problem)
– jt work with Young –