SECTION 11
Numerical Approximations and Entropy

Math-280: A Mathematical Introduction to Shock Waves

Blake Temple, UC-Davis
1. **Numerical Approx to (CL) & Entropy**
   - Approx soln on grid
     \[ x_j = jh \]
     \[ t_n = nh \]

   Let \( u(x,t) \) be exact soln to
   \[ u_t + f(u)x = 0 \]
   \( \overline{u}(x,t) \) be approx soln
   \[ u_j^n = u(x_j,t_n) \]
   \[ \overline{u}_j^n = \overline{u}(x_j,t_n) \]

   Really: \( h_x, h_t \to \infty \), \( h_x, h_t \to 0 \).

   \( \overline{u}^n \) = vector of approx values \( (u_j^n)_{j=\infty} \)

2. **Explicit Schemes**
   - \( \overline{u}^{n+1} = \overline{u}^n + u_j^n \)
   - \( \overline{u}_j^{n+1} = \overline{u}_j^n + \frac{\Delta t}{\Delta x} \nabla u \overline{u}_j^n \overline{u}_{j+1}^n \) (very bad)

   Ex: Upwind Scheme for \( u_t + uu_x = 0 \)
   \[ u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} \nabla u (u_j^n - u_{j+1}^n) \]

   Ex: Lax-Friedrichs Scheme for \( u_t + uu_x = 0 \):
   \[ u_j^{n+1} = \frac{u_j^n + u_{j+1}^n}{2} + \frac{f(u_{j+1}^n) - f(u_j^n)}{2\Delta x} \]
   approx \( u_t \)
   approx \( u_x \)
(LaF) Solve for $U_j^{n+1}$:

\[
U_j^{n+1} = \frac{1}{2} (U_{j-1}^n + U_{j+1}^n) - \frac{k}{2h} (f(U_{j+1}^n) - f(U_{j-1}^n))
\]

Stencil

- CFL Condition (1927) (Courant-Friedrichs-Lewy)
  A FDS is unstable if the domain of dependence of FDS does not contain the domain of dep of the PDE.

Ex (LaF)
"Chng i-data 8 get" no chng in approx

Domain of dep of $U_t + f(U)_x = 0$ det's by max $|\lambda_i|$

Need: $\frac{h}{k} > \text{Max } |\lambda_i|$

---

Q: When does a FDS compute the shock discontinuity correctly?

Q1: Do the FDS approximations converge to a weak soln of CL?

Q2: Does the weak soln satisc the entropy cond?

Ex: Not a mute point -

Consider Nonlinear upwind scheme

\[
U_j^{n+1} = U_j^n - k \frac{U_j^n - U_j^{n-1}}{h} (U_j^n - U_j^{n-1})
\]

Compute soln to $U_t + uU_x = 0$ starting from i-data

\[
u(x,0) = \begin{cases} 
1 & x < 0 \\
0 & x > 0 
\end{cases}
\]

Numerical Soln:

\[
U_j^{n+1} = U_j^n = \cdots = U_0^n = 0 \quad j > 0 \\
U_j^{n+1} = U_j^n = \cdots = U_0^n = 1 \quad j < 0
\]
5. True Soln: \( U_t + \left( \frac{1}{2} U^2 \right)_x = 0 \)
\( u_R = 0, u_L = 1 \) \( \Rightarrow \) shock of speed
\[ f(u) = \frac{1}{2} u^2 \]
\[ \frac{f'(u_f) - f'(u_l)}{1} = \frac{1}{2} \]

\[ u \in [0, 1] \Rightarrow 0 \leq \lambda = f'(u) = 0 \leq 1 \]

5. CFL met if \( \frac{h}{k} > 1 \)

5. Numerical Soln:
\[ u_t + f(u)_x = 0 \]
\( \Rightarrow \) wrong shock speed \( \Rightarrow \) does not conv.

6. Consider (LaGr)
\[ \frac{U^{n+1}_j - U^n_j}{h} + \frac{1}{2h} (f(U^n_{j+1}) - f(U^n_{j-1})) \]

Rewrite as:
\[ \frac{U^{n+1}_j - U^n_j}{k} + \frac{1}{2h} (f(U^n_{j+1}) - f(U^n_{j-1})) = \frac{1}{2k} (U^n_{j-1} - 2U^n_j + U^n_{j+1}) \]

\[ (*) \Rightarrow \frac{h}{2k} \left\{ \frac{U^n_{j+1} - U^n_j}{h} - \frac{U^n_{j-1} - U^n_{j+1}}{h} \right\} \]

An approx to \( U_{xx} \)

\( \Rightarrow \) solves \( U_t + f_x = \frac{h}{2k} U_{xx} \cdot h \)

Might expect as \( h \to 0 \) get weak soln's that are limits of viscous method \( \Rightarrow \) expect to solve entropy cond.
- **Claim**: The LaFr scheme results by "solving RPK on a staggered grid."

- Assume pw constant states at time \( t_n \)

- Approx Soln: \[ U(x, t_n) = U^n_{j-1} \text{ for } x_{j-1} < x \leq x_j \]
  \[ U(x, t_n) = U^n_j \text{ for } x_j < x \leq x_{j+1} \]
  \[ U(x, t_{n+1}) = U^{n+1}_j \text{ for } x_j < x \leq x_{j+1} \]

- that is: \( U^n_j \) defined only for \( n+j \) even centered on the \( x \)'s

- Fill in the rectangles by solving the RPK posed at the dot's to get \( U(x, t) \) for \( t < t_n \)

- Assume CFL in form \[ \frac{h}{x} > 2 \text{Max} |\lambda_j| \]
  so waves never hit sides of rectangles @ each time step

- Consider rectangle \( R_{n,j} \equiv \text{bottom center} \ (x_j, t_n) \)

- Now \( U(x, t) \) is a weak soln of (CL) \( u_t + f(u)u_x = 0 \) in \( R_{n,j} \)
• Assume first that $\mathcal{U}(x,t)$ is a smooth soln. in $R_{jn}$

$$0 = \iint_{R_{jn}} \nabla \cdot (\mathcal{U}, f(\mathcal{U})) \, dt \, dx$$

$$= \int_{R_{jn}} \mathcal{U} \cdot \mathcal{U} \, ds$$

$$\Rightarrow \mathcal{U}_{j+1}^{n+1} - \mathcal{U}_{j-1}^{n+1}$$

$$= \int_{x_{j-1}}^{x_{j+1}} \mathcal{U}(x,t_{n+1}) \, dx - \int_{x_{j-1}}^{x_{j+1}} \mathcal{U}(x,t_{n}) \, dx$$

$$\text{top} \quad \text{bottom}$$

$$+ \int_{t_{n}}^{t_{n+1}} f(\mathcal{U}(x_{j+1},t)) \, dt - \int_{t_{n}}^{t_{n+1}} f(\mathcal{U}(x_{j-1},t)) \, dt$$

$$\text{R-side} \quad \text{L-side}$$

But $\mathcal{U}(x,t) = \mathcal{U}_{j+1}^{n}$ on R-side, $\mathcal{U}_{j-1}^{n}$ on L-side.

Assuming $\mathcal{U}_{j}^{n}$ generated by LaFr scheme $\Rightarrow$

$$\mathcal{U}_{j}^{n+1} = \frac{1}{h} \int_{x_{j-1}}^{x_{j+1}} \mathcal{U}(x,t_{n+1}) \, dx$$

$$\Rightarrow \text{"fillin in the LaFr n+1 even with RP's in each strip is consistant."}$$

HW. Show that (8) holds so long as RP inside $R_{jn}$ consists of pm smooth soln. separated by shocks that satisfy RH jump condts. (Assume char's are at lines emanating from center, break up into smooth integrals, apply RH at e's.)
Theorem: Assume $V_j(x,t)$ generated by LaFr with $m_j$ even as above. Assume $V_j(x,t)$ is strictly hyp & gen nonlinear or linearly degen in each char field in a nbhd of $U$ of u-space & assume Lax's soln of RP applies in $U$. Assume that all states $U_j$ in $V_j(x,t)$ lie in $U$. Then (LaFr) approx soln satisfies the entropy inequality

$$\frac{d}{dt} \mathcal{H}(U) + \frac{\partial}{\partial x} \mathcal{H}(U) \leq O(h)$$

in the weak sense.

Need following assumptions about stability of approx soln's:

1. $|V_j(x_j,t_{n+1}) - V_j(x_j,t_{n-1})| \leq C \|U_j - U_j^{n-1}\| \quad \forall \ x_{j-1} < x < x_{j+1}$

   (This follows from stability of Lax RP)

2. Total Variation Bound (not yet proven)

   $$\sum_{j=-\infty}^{\infty} |U_j^{n+1} - U_j^{n-1}| < V < \infty \quad \forall \ n > 0$$

   $\dot{u} = -\infty$ for $n+j$ even.
Proof: Let \( \phi \) be a smooth pos test fn. of compact supp in \( t>0, x \in \mathbb{R} \). We prove
\[
- \int_{-\infty}^{\infty} \int_{t_0}^t (\partial u/\partial t + \partial u/\partial x) \phi \, dx \, dt \leq 0.
\]
- \( u(x,t) \) is an exact weak soln in each strip \( t_{n-1} < t < t_n \), thus
\[
- \sum_{n=0}^{\infty} \int_{t_{n-1}}^{t_n} (\partial u/\partial t + \partial u/\partial x) \phi \, dx \, dt
\]

\[
= \sum_{n=0}^{\infty} \int_{t_{n-1}}^{t_n} (\partial u/\partial x) \phi \, dx \, dt
\]

\[
= \sum_{n=0}^{\infty} \int_{t_{n-1}}^{t_n} \frac{d}{dt} \left( u(x,t) \phi(x) \right) \, dx
\]

\[
= \sum_{n=0}^{\infty} \int_{t_{n-1}}^{t_n} \frac{d}{dt} (u(x,t) \phi(x)) \, dx
\]

\[
= \sum_{n=0}^{\infty} \int_{t_{n-1}}^{t_n} \frac{d}{dt} (u(x,t) \phi(x)) \, dx
\]

\[
= \sum_{n=0}^{\infty} \int_{t_{n-1}}^{t_n} \frac{d}{dt} (u(x,t) \phi(x)) \, dx
\]

Now:
\[
\int_{X_{j-1}}^{X_{j+1}} \gamma(\sigma(x,t_{n+1})) \, dx = 2h \gamma \left( \frac{1}{2h} \int_{X_{j-1}}^{X_{j+1}} \gamma(\sigma(x,t_{n+1})) \, dx \right)
\]

\[
\leq 2h \left( \frac{1}{2h} \right) \int_{X_{j-1}}^{X_{j+1}} \gamma(\sigma(x,t_{n+1})) \, dx
\]
Conclude: If $\Phi(x,t)$ were constant on $x_j$, we'd have:

$$\int_{x_{j-1}}^{x_{j+1}} \left[ \delta (U(x,t_{m_i}))-\eta (U(x,t_{m+})) \right] \Phi(x,t) \, dx \leq 0$$

To take advantage, write

$$\Phi(x,t) = \Phi(x_j,t_n) + O(h)$$

Conclude:

$$\int_{-\infty}^{\infty} \left[ \int_{t_0}^{t_{n+1}} \sum_{j=1}^{n} \sum_{m=1}^{M} \left[ \delta (U(x_j,t_{m+}))-\eta (U(x_j,t_{m+})) \right] \Phi(x_j,t_n) + O(h) \right] \, dx \, dt$$

$$\leq \frac{h}{2} \sum_{j=1}^{n} \sum_{m=1}^{M} \left[ \max_{x_j} |U^n_j - U^n_{j-1}| \right] \leq O(h) \cdot \frac{h}{2} \cdot V \leq V = O(h)$$
Numerical methods based on Riemann Problems

Lax-Friedrichs:
\[ U_{j}^{n+1} = \frac{1}{2} (U_{j-1}^{n} + U_{j+1}^{n}) + \frac{h}{2} (F(U_{j+1}^{n}) - F(U_{j-1}^{n})) \]

Can view \( U_{j}^{n+1} \) as average of soln obtained by RK on a staggered grid. No need to solve RK!

Godunov: \( U_{j}^{n+1} \) as average of soln obtained by RK on an unstagged grid

\[ U_{j}^{n} = \text{value of numerical approx } U(x_{j}t_{n}), x_{j+\frac{1}{2}} \leq x < x_{j+1} \]

Extend to \( t_{n} < t < t_{n+1} \) by solving RK's \( \Rightarrow U(x,t) \)
*Note: (God) requires soln of RP, (LaFl) does not!

* Modified (God) use some approx to soln of RP to approx \( U^* \)

* Glenn's Method: Sample instead of average (staggered or unstagged)
  
  - Not conservative
  - Only method proven to converge