SECTION-3
The Eulerian and Lagrangian
Equations of Motion
In One Space Dimension

Math-280: A Mathematical Introduction
to
Shock Waves

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Euler equations as a 1-D system of conservation laws —

(4) \( \frac{\partial}{\partial t} s + (su)_x = 0 \)

(5) \( \frac{\partial}{\partial t} (su^2 + p) + (su^2 + p)_x = 0 \)

(6) \( E_t + [(E+p)u]_x = 0 \)

(7) \( s_t + [su]_x = 0 \)

\[ \Rightarrow \quad y_t + f(y)_x = 0 \]

\[ y = \begin{pmatrix} s \\ su \end{pmatrix}, \quad f = \begin{pmatrix} 0 \\ E \end{pmatrix} \]

In general:

\[ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = f(y) \]

Lagrangian Coordinates:

- Assume some reference coordinate \( a \) that names the fluid particle at time \( t=0 \):

\[ a = (a_1, a_2, a_3) \]

Can take either one on smooth solutions.

E.g.: we could choose \( a \) to be \( x \)-coordinate of particle at \( t=0 \), say \( x(a,0) = a \). But we can choose \( a \) to be any smooth coordinate defined at \( t=0 \).
Then we still have
\[ \frac{\partial x}{\partial t} (a, t) = u \]
\[ x(a, 0) = \Phi(a) \]
defines the velocity field, and
\[ \dot{f} = \frac{Df}{Dt} = \frac{\partial}{\partial t} f(x(a, t), t) \bigg|_{a = \text{const}} = \nabla f \cdot u + f_t \]
(same formula).
It follows that if \( J = \frac{\partial x}{\partial a} (a, t) \), \( \dot{J} \) is the same as our old \( \dot{J} \), and we still have
\[ \dot{J} = \text{div} u, \quad \text{indept of } \Phi(a) \]
\[ \Rightarrow \text{all previous derivations go thru unchanged} \]

In particular, we can use \( \text{(MM)} \) to relate the evolution of the density \( \rho \) to the evolution of \( J \):
\[(\text{MM}) \Rightarrow 0 = \rho_t + \text{div} \rho \cdot u = \rho_t + \nabla \rho \cdot u + \rho \text{div} u
\]
\[ = \frac{D\rho}{Dt} + \rho \text{div} u \]
\[ \Rightarrow -\frac{1}{\rho} \frac{D\rho}{Dt} = \text{div} u \]
\[ v = \frac{\rho}{\rho} \]
\[ \Rightarrow -v \frac{D(v)}{Dt} = \frac{1}{v} \frac{Dv}{Dt} = \text{div} u \]
and we conclude that
\[ \frac{1}{v} \frac{Dv}{Dt} = \frac{1}{J} \frac{DJ}{Dt} \]
which we can integrate as follows:
\[ \frac{1}{V} \frac{\partial V}{\partial t} = \frac{\partial}{\partial t} \ln V = \frac{\partial}{\partial t} \ln J(a,t) \]

so

\[ \frac{\partial}{\partial t} \ln J(a,t) = \frac{\partial}{\partial t} \ln \psi(a) \]

\[ \psi(a) = \ln \left( \frac{\ln V}{\ln J(a,t)} \right) \]

In particular,

\[ \psi(a) = \ln \left( \frac{V(a,t)}{J(a,t)} \right) = \ln \left( \frac{\psi(a)}{J(a,t)} \right) \]

\[ V(a,t) = e^{\psi(a)} J(a,t) \]

Lagrangian equations in 1-D:

In 1-space dimension we can define the Lagrangian variables \( a \) by choosing

\[ \psi(a) = 0 \] so that \( \psi(a) = 0 \Rightarrow \psi(a) = J(a,t) \]

and this simplifies the equation when we take \( a \) as the space variable instead of \( x \).

* Restrict to 1-d so \( a, x \in \mathbb{R} \). By (\*), \( \psi(a) = 0 \) if \( V(a,0) = J(a,0) \) or

\[ \frac{1}{p(a,0)} = \frac{\partial x}{\partial a} (a,0) = \phi(a) \]

(\*)

\[ p(a,0) = \frac{\partial a}{\partial x} (x,0) = [\phi^{-1}]'(x) \]

so define

\[ a = \int_0^x p(z,0) dz = [\phi^{-1}]'(x) \]
\[ a = \int_0^T \mathcal{P}(z,t) \, dt \quad \forall t \]

- Taking \( \xi = 0 \) in \( d \xi \), define the Lagrangian coordinate:

\[ \xi = \int_0^t \mathcal{P}(z,t) \, dt \]

This defines a mapping \( (s,t) \mapsto (x,t) \) satisfying

\[ \frac{\partial x}{\partial t}(s,t) = U \quad \frac{\partial x}{\partial s}(s,t) = \frac{1}{s} \]

and

\[ f_x(s,t) = \frac{Df}{Dx} \quad f_x(x,t) = \frac{\partial x}{\partial x} f(s(x,t),t) \]

\[ = f_s \frac{\partial s}{\partial x} = f_s \xi \]
Using \((***)\) in \((Ma)\) \(\Rightarrow\)

\[
(Ma): \frac{\partial P_t}{\partial t} + (Pu)_x = s_t + s_x u + s u_x \\
\Rightarrow \frac{Ds}{Dt}
\]

\[
= \frac{\partial}{\partial t} s(s,t) + s \frac{\partial u}{\partial s}(s,t) \cdot \phi
\]

\[
= \frac{\partial}{\partial t} \frac{1}{\nu} + s^2 u \xi = -s^2 v_t + s^2 u \xi
\]

\(\Rightarrow\)

\[
V_t - U_\xi = 0
\]

Using \((***)\) in \((Bu)\) \(\Rightarrow\)

\[
(Bu): \frac{\partial u}{\partial t} = -\nabla P = -p_x \Leftrightarrow \frac{\partial ^2}{\partial t} u(s,t) + s \frac{\partial P_x}{\partial t} = 0
\]

\[
U_t + P_x = 0
\]
Conclude - Euler Equations in Lagrangian coordinates:

\[(M0)_2: V_t - U_x = 0 \quad \chi \leftrightarrow x\]
\[(M0)_2: U_t + P_x = 0 \quad E = \text{specific total energy}\]
\[(E1)_2: E_t + (Eu)_x = 0 \quad s = \text{specific entropy}\]

\[S_t = 0 \quad \text{take either one on smooth solutions}\]

*Note. We say fluid is barotropic if

\[P = P(V), \quad v = \frac{1}{\gamma}\]

and we assume \(P'(v) < 0, \quad P''(v) > 0\). Then

\[(M0)_2 b (M0)_2 \text{ uncouple from } (E1)_2 \text{ and reduce to the so-called } P\text{-system (name coined by Joel Smoller)}\]

\[V_t - U_x = 0\]
\[U_t + P(V)_x = 0\]