SECTION-7
The Global Solution
of the
Riemann Problem
for the
p-system

Math-280: A Mathematical
Introduction
to

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Riemann Problem for \( p \)-system: (Ch 17 Smoller)

- Lagrangian system for compressible Euler:

\[
\begin{align*}
V_t - U_x &= 0 & v &= \tfrac{1}{3} = \text{spec vol} \\
U_t + P_x &= 0 & u &= \text{velocity} \\
\{ E_t + (Pu)_x \} &= 0 & P &= P(v, s) = \text{pressure} \\
S_t &= 0 & \mathcal{E} &= \tfrac{1}{2} u^2 + e = \text{spec} \\
\end{align*}
\]

- The \( p \)-system: assume \( P = P(v) \) with

\[
P' < 0, \quad P'' > 0
\]

- Recall: polytropic eqn of state:

\[
\begin{align*}
P &= \mathcal{C}v(\gamma - 1) \left( \frac{1}{v} \mathcal{E} \right) \\
P_v &= RT
\end{align*}
\]

Isothermal \( \Rightarrow P = \frac{K}{v} \quad P(v) = \frac{K}{v} \)

Isentropic \( \Rightarrow P = \mathcal{C}v(\gamma - 1) \left( \frac{1}{v} \mathcal{E} \right) \)

\[
P(v) = \kappa \left( \frac{1}{v^\gamma} \right), \quad \gamma > 1
\]

- Note: \( S_t = 0 \) for smooth solution \( \Rightarrow \)

\( s = \text{const. is no approx for smooth soln} \)

Moreover: recall 1,3 - eigenfamilies

\[
\lambda_{1,3} = \pm \sqrt{Pv} \quad R_{1,3} = \left[ \begin{array}{c} 1 \\ \pm \sqrt{Pv} \end{array} \right]
\]

thus integral curves \( R_{1,3} \) have \( s = \text{const.} \)

Since shock-curves have 2nd order contact at \( U_L \), \( s = 0(1) \sigma^2 \) along shock-curves, so \( s = \text{const} \) is a good approx for small amplitude soln or soln` with weak shocks \( \Rightarrow \)

Isentropic is a reasonable assumption.
Riemann Problem for $p$-system:

$p$-system

$$\begin{cases} v_t - u_x = 0 & \text{in} \int_0^\infty p' > 0 \\
U_t + P'(u) = 0 & \text{isentropic: } p = \frac{1}{\gamma} x \\
& \text{isothermal: } p = \frac{1}{\gamma} \end{cases}$$

- Eigenvalue:

$$f(u) = \begin{bmatrix} -u \\ P'(u) \end{bmatrix}, \quad A = df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & 0 \\ \frac{\partial f_2}{\partial x} & 0 \end{bmatrix} = \begin{bmatrix} \gamma & -1 \\ p' & 0 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ p' & -\lambda \end{vmatrix} = \lambda^2 + p' = 0$$

$$\lambda_{\pm} = \pm \sqrt{-p'(u)} \quad \text{"L sound speed"}$$

$$\lambda_1 = \lambda_2 = \lambda^+$$

- Set $U = (v, u)$, $f(u) = (-u, P')(u)$ so $p$-system is $U_t + f(u)_x = 0$

- Eigenvectors: $R_1 = (1, 0)$ satisfies

$$[A - \lambda_1 I] R_1 = 0 \iff \begin{bmatrix} v' - p' \sqrt{1 - p'}/p' \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = 0$$

$$\sqrt{1 - p'} - a = 0 \iff a = \sqrt{1 - p'}$$

$$R_1 = \begin{bmatrix} 1 \\ \sqrt{1 - p'} \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 \\ -\sqrt{1 - p'} \end{bmatrix}$$

(as we got taking s = const in (1))

- Integral curves $R_1$ satisfy

$$\frac{du}{dv} = \sqrt{1 - p'(v)}$$

$$\Rightarrow \quad u = \int \sqrt{1 - p'(v)} \, dv + \text{const}$$

$R_1$ is an integral curve of $R_1$ satisfies

$$u = \int \sqrt{1 - p'(v)} \, dv + \text{const}$$
$\mathbf{R}_2$ are integral curves of $R_2$, satisfy

$$u = -\int \sqrt{-p(v)} \, dv + \text{const}$$

**Conclude:** The integral curves through a state $(v_L, u_L)$ satisfy

$\mathbf{R}_1(u_L) : \quad u = \int_{v_L}^{v} \sqrt{-p(v)} \, dv + u_L = \Phi_1(v)$

$\mathbf{R}_2(u_L) : \quad u = \int_{v_L}^{v} \sqrt{-p(v)} \, dv + u_L = \Phi_2(v)$

\[ \text{gives param. of } 1,2 \text{-rarefactions} \]

Note: \[ \frac{d\lambda_1}{dv} = \frac{d\sqrt{-p}}{dv} = \frac{p''}{2\sqrt{-p}} > 0 \quad \frac{d\lambda_2}{dv} < 0 \]

\[ \therefore \frac{\nabla \lambda_i \cdot R_i}{\lambda_i} = \frac{d\lambda_i}{dv} > 0 \quad i = 1 \]

\[ < 0 \quad i = 2 \]
Conclude: The p-system is strictly hyperbolic and genuinely nonlinear in physical domain $V > 0$.

\[ \begin{align*}
\mathbb{R}^+_i & \quad \xrightarrow{\lambda_1} \quad \mathbb{R}^+_i \\
\mathbb{R}^-_i & \quad \xleftarrow{-\lambda_1} \quad \mathbb{R}^-_i \\
V_L & \quad \mapsto \quad V \\
\end{align*} \]

Definition: For each $U_L = (V_L, U_L)$, the $i$-rarefaction curve $\mathbb{R}^+_i$ is the portion of $\mathbb{R}^+_i(U_L)$ along which $\lambda_1 > \lambda_1(U_L)$.

We have: for each $U_R \in \mathbb{R}^+_i$ there is a centered $i$-simple wave (rarefaction wave) connecting $U_L$ to $U_R$.

\[ \frac{dx}{dt} = \lambda_1 \]

\[ U = U_L \pm \sqrt{\frac{P'(V) - P'(V_L)}{V - V_L}} (V - V_L) \quad V > V_L \]

\[ U = U_L - \sqrt{(P(V) - P(V_L))(V - V_L)} \quad V < V_L \]

2. Shock waves for p-system:

\[ s[V] = [f] \]

\[ s[V - V_L] = \begin{bmatrix} -u + u_L \\ u - u_L \end{bmatrix} \begin{bmatrix} -P(u) \\ P(u) \end{bmatrix} \]

(1) $s(V - V_L) = -(U - U_L)$

(2) $s(U - U_L) = P(V) - P(V_L)$

\[ \Rightarrow s^2(V - V_L) = -(P(V) - P(V_L)) \]

(3) $s = \pm \sqrt{\frac{P'(V) - P'(V_L)}{V - V_L}}$

Substituting (2) into (1) we obtain:

\[ U = U_L \pm \sqrt{\frac{P'(V) - P'(V_L)}{V - V_L}} (V - V_L) \quad V > V_L \]

\[ U = U_L - \sqrt{(P(V) - P(V_L))(V - V_L)} \quad V < V_L \]
Consider: \( U = u_L \pm \sqrt{(p(v_L) - p(v))(v - v_L)} \)

\[
\frac{du}{dv} = -\frac{p'(v)(v - v_L) + (p(v_L) - p(v))}{2\sqrt{(p(v_L) - p(v))(v - v_L)}} = -\frac{v - v_L}{2\sqrt{(p(v_L) - p(v))(v - v_L)}} \left\{ \frac{p'(v) + \frac{p(v_L) - p(v)}{v_L - v}}{v_L - v} \right\}
\]

\[
= \begin{cases} 
-\frac{1}{2\sqrt{-p(v_L) - p(v)}}, & V > v_L \\
+\frac{1}{2\sqrt{-p(v_L) - p(v)}}, & V < v_L 
\end{cases}
\]

\( (\star) \)

**Note:** (\( \star \)) implies:

as \( v \to v_L^- \), \( \frac{du}{dv} \to -\frac{2p'(v_L)}{2\sqrt{-p(v_L)}} = \pm \sqrt{-p(v_L)} \)

as \( v \to v_L^+ \), \( \frac{du}{dv} \to -\frac{2p'(v_L)}{2\sqrt{-p(v_L)}} = \pm \sqrt{-p(v_L)} \)

Thus define:

(5) \( d_1 (v_L) \): \( U = \begin{cases} 
\sqrt{(p(v_L) - p(v))(v - v_L)} & V < v_L \\
\sqrt{(p(v_L) - p(v))(v - v_L)} & V > v_L 
\end{cases} \)

(6) \( d_2 (v_L) \): \( U = \begin{cases} 
\sqrt{(p(v_L) - p(v))(v - v_L)} & V < v_L \\
\sqrt{(p(v_L) - p(v))(v - v_L)} & V > v_L 
\end{cases} \)

So that

\( d_1 (v_L) \) is tangent to \( R_1 \) at \( U = v_L \)

\( d_2 (v_L) \) is tangent to \( R_2 \) at \( U = v_L \)
Lemma: The soln of \((R-H)\) \(S[u]=[f]\) for \([u]=u-u_L\) consists of \(u \in \mathcal{A}_1(u) \cup \mathcal{A}_2(u)\). Moreover, for \(u \in \mathcal{A}_1(u)\) we must take

\[ S = -\sqrt{-\frac{p'(u)-p'(u_L)}{V-V_L}} \]

and for \(u \in \mathcal{A}_2(u)\) take

\[ S = +\sqrt{-\frac{p'(u)-p'(u_L)}{V-V_L}} \]

Pf. This follows directly from (3), (4), (5), (6).

- As a consequence of Lax's general theory, we know \(\mathcal{S}_2\) has \(C^2\)-contact with \(R_2\).
- Define the wave curves:

\[
\mathcal{W}_1 = \mathcal{W}_1(\mathcal{U}_L) = \begin{cases} \mathcal{R}_1^+(\mathcal{U}_L) & V > V_L \\ \mathcal{S}_1^{-}(\mathcal{U}_L) & V < V_L \end{cases}
\]

\[
\mathcal{S}_1^{-}(\mathcal{U}_L): u = u_L - \sqrt{(p'(u)-p(u))(V-V_L)} \quad V < V_L
\]

\[
\mathcal{W}_2 = \mathcal{W}_2(\mathcal{U}_L) = \begin{cases} \mathcal{R}_2^+(\mathcal{U}_L) & V < V_L \\ \mathcal{S}_2^{-}(\mathcal{U}_L) & V > V_L \end{cases}
\]

\[
\mathcal{S}_2^{-}(\mathcal{U}_L): u = u_L - \sqrt{(p'(u)-p(u))(V-V_L)} \quad V > V_L
\]

Note: On \(\mathcal{W}_1 \cap \mathcal{W}_2\) \(u\) is a fn of \(V\), so wave curves never "hit" \(V=0 \Rightarrow V \geq 0\) is "invariant region."
Principle: Pick out the shock curves $S_2^-$ that move in the directions of $R_2$ in which $\lambda$ decreases, since there are directions in which we cannot obtain simple wave solutions.

I.e. “put in the shocks only when $\lambda$ a smoother solution ...”

Lemma 1: $s$ decreases on $S_2^-(U_L)$ going away from $U_L$, and moreover

$$\lambda_i(U) < S(U) < \lambda_i(U_L)$$

for $U \in S_1^-(U_L)$, and (4) fails on $S_1^+(U_L)$.

Homework: Prove this.

Cor: shock waves $[U_L, U]$ for $U \in S_1^-(U_L)$ give the admissibly $\epsilon$-shocks with left state $U_L$.

Homework: How many characteristics impinge on an $\epsilon$-shock? for $p$-system? in general?
\textbf{Soln: From (3), \( i = 1 \)}

\[ S = -\sqrt{-\frac{p(V)-p(V_L)}{V-V_L}} \]

\( p \) convex up \( \iff \) \( V < V_L \) on \( \mathcal{A} \)

\[ p'(V) < \frac{p(V)-p(V_L)}{V-V_L} < p'(V_L) \]

and put in minus signs...

\textbf{From (4) we see that 2-shocks satisfy the condition that characteristic in the family at the shock imping on the shock, \( B \) characteristic in the opposite family cross the shock. Note: this is global (shock & rarefaction waves can be arbitrarily strong) while Lax's thm is local.}
To solve the R.P., given $U_L$, for any $U_R$:

1. For each $U_m \in W_1(U_L)$, draw inzwave curve $W_2(U_m)$.
2. Given $U_R$, find $U_m$ s.t. $U_R \in W_2(U_m)$. The R.P. is solved by the (neg speed) 1-wave taking $U_L \to U_m$ followed by the (positive speed) 2-wave taking $U_m \to U_R$.

**Picture:**

$U_R \in W_2(U_m)$

$U_m \in W_1(U_L)$

$W_1(U_m)$

$W_2(U_L)$

$B_i^+(U_L)$

$R_i^+(U_L)$

$W_1(U_R)$

$W_2(U_R)$

**Missing Steps:**

A. We need that the curves $W_2(U_m)$ for $U_m \in W_1(U_L)$ fill up all of $-\infty < u < +\infty$, $v \geq 0$ in order that each $U_R$ has a $U_m$ s.t. $U_R \in W_2(U_m)$.

B. We need that $W_2(U_m) \cap W_2(U_m') = \emptyset$ for each $U_m \neq U_m'$. 

(A) $\equiv$ required for existence of R.P. soln $[U_L, U_R]$ for every $U_R$.

(B) $\equiv$ required for uniqueness of R.P. soln $[U_L, U_R]$ for every $U_R$.
- We verify (A) in region I, solving in II, III, built in IV.
- We check region I.
- Choose $\Omega_R$ in $I$, $\Omega_R = (v_R, u_R)$.
- Recall $\delta_2$ given by (A) $U = u^0 - \sqrt{(P(v_0) - P(u)) (v - v_0)}$

$$\delta_2^2 = (v - v_0)^2 + (u - u_0)^2$$

We need that for some $P \in \Omega_A$,

$$P(u_0, v_0, v_R) = u_R = f(P), P = (v_0, u_0)$$

But $f$ is continous on $[u, \bar{u}]$,

$$B = f(\bar{u}) \leq u_L \leq f(A) = A$$

$\Rightarrow$ there exists $P \in \Omega_A$ such that $f(P) = u_L$.

Intermediate Value Theorem

- we verify (B)
- Assume region (I).

$$u = u_0 - \sqrt{(P(v_0) - P(u)) (v - v_0)}$$

$$= T(u_0, v_0, v_R)$$

$$u_0 = u_L + \int_{v_L}^{v_R} \sqrt{P'(s)} \, ds$$

Fix $v_L, v_R$:

$$u = u_L + \int_{v_L}^{v_R} \sqrt{P'(s)} \, ds - \sqrt{(P(v_0) - P(v_0)) (v_R - v_0)} = \varphi(v_0)$$

For uniqueness, need only $\frac{du}{dv_0} = \varphi'(v_0) \neq 0$.

$$\varphi'(v_0) = \sqrt{P'(v_0)} - \frac{P'(v_0) (v_R - v_0) - (P(v_0) - P(v_0))}{\sqrt{(P(v_0) - P(v_0)) (v_R - v_0)}}$$

$$= \sqrt{-P_0} - \frac{(P_0 - P) (V_R - V_0)}{\sqrt{-P_0 - P}} \frac{V_R - V_0}{V_R - V_0}$$

$$> 0 \text{ as } P_0 < 0 \vee$$
In region I, ambiguous
is clear since \( \mathcal{R}_2^+(v) \)
are integral and of on
(autonomous) vector field \( \mathcal{A} \)
hence are nonintersecting:
Problem: \( \mathcal{R}_1^+ \) may not reach all values of \( u \! \)
\( \mathcal{R}_1^+ : u = u_L + \int_{v}^{v} \sqrt{-p'(s)} \, ds \)
if \( \sqrt{-p'(s)} \) is integrable, then \( u \to \hat{u} \) as \( v \to \infty \)
- In this case, \( \mathcal{R}_2^- (v) \) does not reach all values of \( u \).
\( \mathcal{R}_2^- : u = u_L - \int_{v}^{v} \sqrt{-p'(s)} \, ds \)
- For Example:
\( \mathcal{R}_2^- (v) : 
\begin{align*}
u &= u_L - \int_{v}^{v} \sqrt{-p'(s)} \, ds \\
u &= u_L + \sum_{i=0}^{n} (u_i - u_{i-1}) \\
&= u_L + \sum_{i=0}^{n} \int_{v}^{v} \sqrt{-p'(s)} \, ds + \int_{v}^{v} \sqrt{-p'(s)} \, ds \\
&= u_L + \sum_{i=0}^{n} \int_{v}^{v} \sqrt{-p'(s)} \, ds + \int_{v}^{v} \sqrt{-p'(s)} \, ds
\end{align*} \)
ie. the solution of the RPE is given as follows:
defined everywhere except  
\( t=0 \) where \( V = \frac{\beta}{\gamma} = \infty \)  
and \( u \) is undefined 
\( \Rightarrow \) vacuum

\[ p' < 0, \quad p'' > 0 \Rightarrow p' \text{ increasing} \]
\[ \Rightarrow -p' \text{ decreasing} \Rightarrow \sqrt{-p'} \text{ decreasing} \]

Thus \( \sqrt{-p'(v)} \) integrated, \( 0 \Rightarrow \sqrt{-p'(v)} \rightarrow 0 \text{ as } V \rightarrow \infty \)
\( \Rightarrow \lambda(v), \lambda'(v) \rightarrow 0 \text{ as } V \rightarrow \infty \)

*** plot is justified

HW: Recall polytropic \( p = C v^{(n-1)} \) \( \frac{1}{\gamma} \); \( \gamma < 1 \)
If isentropic \( \Rightarrow \rho = \text{const} \Rightarrow p = \frac{\alpha^2}{v^2} \)
Also \( PV = kT \) so isothermal \( \Rightarrow p = \frac{\alpha^2}{v} \)
Show: Isentropic \( \Rightarrow \) vacuum, isothermal \( \Rightarrow \) no vacuum

Wave Interactions:
2-2 shocks interact:
what waves will come out?
Ans (real physics!) a 2-shock & a 1-rarefaction

We prove this:

We need: if
\( \Omega_m \in \Omega_2(D_c) \),
then \( \Omega_2(\Omega_m) \in \text{region I for } D_c \).

Theorem. The interaction of two 2-shocks produces a 1-rarefaction wave and a
2-shock wave.
Proof. Assume that at \( t=0 \) we have

\[
\begin{align*}
U_L &= \frac{u_n}{u_R} \quad x
\end{align*}
\]

Then \( U_n \in \mathcal{E}_2^-(U_L) \), \( U_R \in \mathcal{E}_2^-(U_M) \)

\[
S(u_L, u_n) = \sqrt{- \frac{P(u_L) - P(u_n)}{v_n - v_L}}
\]

\[
> S(u_n, u_R) = \sqrt{- \frac{P(u_R) - P(u_n)}{v_n - v_R}}
\]

because \( P \) is convex up

Because the two shocks must interact at some location, the absolute magnitude of the slope gives speed of the pos. speed

\[
\begin{align*}
U_R &= u_m - \sqrt{(P(u_m) - P(u_L))(v_m - v_L)} \\
U_m &= u_L - \sqrt{(P(u_L) - P(u_m))(v_m - v_L)} \\
U_n &= u_n - \sqrt{(P(u_n) - P(u_L))(v_n - v_L)} \\
U_m &= u_m - \sqrt{(P(u_m) - P(u_L))(v_m - v_L)} \\
U_n &= u_n - \sqrt{(P(u_n) - P(u_L))(v_n - v_L)}
\end{align*}
\]

\[
\begin{align*}
\overline{u}_n &= u_n - u_m + u_m - u_L + u_L - \overline{u}_n = -H(R,L) - H(u_m) + H(u_L)
\end{align*}
\]
\[ u_R - \bar{u}_R = \sqrt{(P(u_A) - P(u)))(P(u) - P(u_B))} \]
\[ - \sqrt{(P(v_R) - P(v_A))(P(v_A) - P(v_B))} \]

**Homework**

**Let** \[ H(a, b) = \sqrt{(P(a) - P(b))(b - a)} = H(b, a) \]

**Need:** \( H(a, c) - H(a, b) - H(b, c) \geq 0 \) for all \( 0 < a < b < c \)

Show that \( H(x, z) > H(x, y) + H(y, z) \) for \( x > y > z \).

For proof, reverse the following steps:

\[ \sqrt{(x - y)(P(x) - P(y))} > \sqrt{(y - z)(P(y) - P(z))} + \sqrt{(x - z)(P(x) - P(z))} \]

\[ (x - z)(P(x) - P(z)) > (x - y)(P(x) - P(y)) + (y - z)(P(y) - P(z)) \]

\[ \frac{P(x) - P(y)}{x - y} + \frac{P(y) - P(z)}{y - z} > 2 \sqrt{\frac{P(x) - P(z)}{x - y} \cdot \frac{P(y) - P(z)}{y - z}} \]

\[ \frac{P(z) - P(y)}{y - z} + \frac{P(y) - P(x)}{x - y} > 2 \sqrt{\frac{P(z) - P(x)}{y - z} \cdot \frac{P(y) - P(x)}{x - y}} \]

\[ \alpha + \beta > 2 \sqrt{\alpha \beta} \]

\[ (\sqrt{\alpha} - \sqrt{\beta})^2 > 0 \]
Note: the nonelectrolyte cases are integrated curves: $R_1^+(U_M) < R_2^+(U_M)$ if $U_M \in \mathcal{R}_i^-(U_C)$

The shock cases are most integrated curves of vector fields: $S_i^-(U_M) \neq S_i^-(U_C)$ for

$U_M < S_i^-(U_C)$

This is what causes most of the theoretical problems:
open question: it is not known that 1-st order problem is well-posed for $p$-shock, even for bounded shock:
Existence $\checkmark$ (GLMM)
Uniqueness (open) $\checkmark$ Bressan Sol./Conv. of Glaz-Schen
Cont. Dep. (open) $\checkmark$ Global Existence (Diperna/Compensated Compactness)