17.6) Homework assignment 17.5b gives us the following:

**Proposition 1** Every polynomial is continuous on \( \mathbb{R} \).

Suppose \( f(x) = \frac{p(x)}{q(x)} \) on the domain \( \{ x \in \mathbb{R} | q(x) \neq 0 \} \) where \( p(x) \) and \( q(x) \) are polynomials. By Proposition 1, both \( p(x) \) and \( q(x) \) are continuous on \( \mathbb{R} \). As long as \( q(x) \neq 0 \), the ratio \( \frac{p(x)}{q(x)} \) is continuous (Theorem 17.4). But all \( x \)'s where \( q(x) = 0 \) are not in the domain of \( f(x) \). Hence, \( f(x) \) is continuous on \( \text{dom}(f) \).

17.8) (a) If \( f(x) \leq g(x) \) for a given \( x \), then \( f(x) - g(x) \leq 0 \). So we have

\[
\min(f, g)(x) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}(-f(x) - g(x)) = f(x)
\]

and we obtain the minimum function output \( f(x) \).

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\]

and we obtain the minimum function output \( g(x) \).

Hence, \( \min(f, g) \) function can be correctly defined as

\[
\min(f, g) := \frac{1}{2}(f + g) - \frac{1}{2}|f - g|
\]

(b) A quick calculation using formula in Section 17 Example 5 shows

\[
-\max(-f, -g) = -\left( \frac{1}{2}(-f - g) - \frac{1}{2} - f + g \right) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g| = \min(f, g)
\]

(c) Looking at the formula obtained in (a), suppose \( f \) and \( g \) are continuous. Then \( f + g \) and \( f - g \) are continuous because of the addition and subtraction laws of continuity (Theorem 17.4), respectively. The function \( |f - g| \) is continuous because the absolute value function preserves continuity (Theorem 17.3). Also, the functions \( \frac{1}{2}(f + g) \) and \( \frac{1}{2}|f - g| \) are continuous through the scalar multiplication law (Theorem 17.3). Finally, \( \min(f, g) \) is continuous by another application of the addition law of continuity.

17.10) (a) The given function is

\[
f(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x \leq 0 
\end{cases}
\]

Construct the sequence \( \{x_n\} \) by \( x_n = \frac{1}{n} \ \forall n \in \mathbb{N} \). For this sequence we have

\[
\{x_n\} \to 0 \quad \text{and} \quad f(x_n) = 1 \ \forall n
\]

Consequently, \( \lim_{n \to \infty} f(x_n) = 1 \neq f(0) = 0 \). Hence, \( f \) is not continuous at \( x = 0 \).

(b) The given function is

\[
g(x) = \begin{cases} 
0 & \text{if } x = 0 \\
\sin(\frac{1}{x}) & \text{if } x \neq 0 
\end{cases}
\]
Construct the sequence \( \{x_n\} \) by
\[
x_n = \frac{1}{\frac{x}{2} + 2\pi n} \quad \forall n \in \mathbb{N}
\]
For this sequence we have
\[
\{x_n\} \to 0 \quad \text{and} \quad g(x_n) = 1 \quad \forall n
\]
Consequently, \( \lim_{n \to \infty} g(x_n) = 1 \neq g(0) = 0 \). Hence, \( g \) is not continuous at \( x = 0 \).

(d) The given function \( P(x) \) with domain \([0, \infty)\) is
\[
P(x) = \begin{cases} 
15 & \text{if } 0 \leq x < 1 \\
15 + 13n & \text{if } n \leq x < n + 1 
\end{cases}
\]
Choose \( x_0 \in \mathbb{N} \), and let \( m = x_0 - 1 \). Construct the sequence \( \{x_n\} \) by \( x_n = x_0 - \frac{1}{n} \) \( \forall n \in \mathbb{N} \). For this sequence we have
\[
\{x_n\} \to x_0 \quad \text{and} \quad P(x_n) = 15 + 13m \quad \forall n
\]
since \( m \leq x_n < m + 1 \).

Consequently,
\[
\lim_{n \to \infty} P(x_n) = 15 + 13m \neq P(x_0) = 15 + 13(m + 1)
\]
since \( m + 1 \leq x_0 < m + 2 \). Hence, \( P \) is not continuous at \( x_0 \). Since \( x_0 \) was an arbitrary positive integer, \( P(x) \) is discontinuous on the positive integers.

Proposition 2 Let \( f \) be a continuous function with domain \((a, b)\). If \( f(r) = 0 \) for each rational number \( r \) in \((a, b)\), then \( f(x) = 0 \) for all \( x \in (a, b) \).

Suppose \( f \) and \( g \) are continuous on \((a, b)\) and \( f(x) = g(x) \) \( \forall x \in \mathbb{Q} \). Consider the difference function \((f - g)(x)\) on \((a, b)\), which is continuous by the subtraction law of continuity (Theorem 17.4). By our assumption, \((f - g)(x) = f(x) - g(x) = 0 \) \( \forall x \in \mathbb{Q} \). By the above Proposition, \((f - g)(x) = 0 \) \( \forall x \in (a, b) \). This implies \( f(x) - g(x) = 0 \) \( \forall x \in (a, b) \). Thus, \( f(x) = g(x) \) \( \forall x \in (a, b) \).

Proposition 2 (b) First we show \( h \) is continuous at \( x = 0 \). Suppose \( \{x_n\} \) is any sequence converging to \( 0 \). If \( x_n \) is rational, then \( h(x_n) = x_n \). If \( x_n \) is irrational, then \( h(x_n) = 0 \). Either way, \( |h(x_n)| \leq |x_n| \).

Now, we will show that \( \{h(x_n)\} \) converges to \( h(0) \). Let \( \epsilon > 0 \) be given. Since \( \{x_n\} \) converges to \( 0 \), for this \( \epsilon \), there exists an \( N \) such that whenever \( n \geq N \) we have \( |x_n| < \epsilon \). But then for this same \( N \), we have that \( |h(x_n)| \leq |x_n| < \epsilon \) whenever \( n \geq N \). Since \( \epsilon > 0 \) was arbitrary, we conclude that \( h(x_n) \to h(0) \), i.e. \( h \) is continuous at \( x = 0 \).

Before showing that \( h \) is discontinuous at any \( x \neq 0 \), we state and prove the reverse triangle inequality: for any \( a, b \in \mathbb{R} \) we have
\[
|a| - |b| \leq |a - b|
\]
To prove this, one uses the regular triangle inequality, which says that for any \( x, y \in \mathbb{R} \) we have
\[
|x + y| \leq |x| + |y|
\]
Let \( x = a - b, \ y = b \) in the triangle inequality. Then
\[
|a| \leq |a - b| + |b|
\]
Subtracting \(|b|\) from both sides of the equation, we see that
\[
|a| - |b| \leq |a - b| \tag{1}
\]
Now let \( x = b - a, \ y = a \) in the triangle inequality. Then
\[
|b| \leq |b - a| + |a| = |a - b| + |a|
\]
Subtracting \(|a|\) from both sides of this equation gives
\[
|b| - |a| \leq |a - b| \tag{2}
\]
Combining (1) and (2) yields the reverse triangle inequality.

Now, we will prove that \( h \) is discontinuous at every nonzero \( x \) using the reverse triangle inequality. Suppose \( x \neq 0 \). Then there exists some \( \epsilon > 0 \) such that \( |x| > 2\epsilon \). Fix this \( \epsilon \). (A side note: we use \( 2\epsilon \) rather than \( \epsilon \) to make the end result neater, but the process is entirely the same either way, up to dividing all \( \epsilon \) terms in the proof by 2.) We now break the situation up into two separate cases.

First, suppose \( x \in \mathbb{R} \setminus \mathbb{Q} \). Then there exists a sequence \( \{x_n\} \subset \mathbb{Q} \) such that \( \{x_n\} \to x \). This means that for our particular \( \epsilon \), there exists some \( N \) such that whenever \( n \geq N \) we have \(|x_n - x| < \epsilon\). Fix this \( N \). Since \( \{x_n\} \subset \mathbb{Q} \), \( h(x_n) = x_n \) for all \( n \). Since \( x \in \mathbb{R} \setminus \mathbb{Q} \), \( h(x) = 0 \). Thus
\[
|h(x_n) - h(x)| = |x_n| = |x - (x - x_n)|
\]
By applying the reverse triangle inequality to \( a = x, \ b = x - x_n \), we see that
\[
|h(x_n) - h(x)| \geq |x - |x - x_n|| = |x - |x_n - x||
\]
For our fixed \( N \), we have \(|x_n - x| < \epsilon\) whenever \( n \geq N \), so that
\[
|h(x_n) - h(x)| > 2\epsilon - \epsilon = \epsilon
\]
whenever \( n \geq N \). The fact that this above inequality holds for a particular \( \epsilon \) and for any \( n \geq N \) means that \( h(x_n) \) cannot converge to \( h(x) \), i.e. \( h \) is not continuous at \( x \).

The second case can be proved without resorting to the reverse triangle inequality. Suppose \( x \in \mathbb{Q} \) and let \( \{x_n\} \subset \mathbb{R} \setminus \mathbb{Q} \) be a sequence converging to \( x \). Since \( x \neq 0 \), we will continue to operate under the assumption that \(|x| > 2\epsilon\). In this case, we have \( h(x_n) = 0 \) for all \( n \), while \( h(x) = x \). Thus
\[
|h(x_n) - h(x)| = |0 - x| = |x| > 2\epsilon
\]
for our particular choice of $\epsilon$ and for any $n \in \mathbb{N}$. Thus $h(x_n)$ does not converge to $h(x)$, i.e. $h$ is not continuous at $x$.

Since we have shown $h$ is discontinuous at any nonzero $x \in \mathbb{Q}$ as well as any nonzero $x \in \mathbb{R} \setminus \mathbb{Q}$, we conclude that $h$ is discontinuous at any nonzero $x \in \mathbb{R}$. (Thanks to Evan Smothers)