29.2) Let \( f(x) = \cos x \) which is continuous and differentiable on \( \mathbb{R} \) from known facts. Consider \( x, y \in \mathbb{R} \).

By the Mean Value Theorem, there exists a \( c \) between \( x \) and \( y \) such that

\[
\frac{f(x) - f(y)}{x - y} = f'(c) \iff \frac{\cos x - \cos y}{x - y} = -\sin c
\]

Taking the absolute values of both sides, we obtain

\[
\left| \frac{\cos x - \cos y}{x - y} \right| = |\sin c| \leq 1.
\]

Rearranging, gives us the final result of

\[
|\cos x - \cos y| \leq |x - y|.
\]

Since \( x, y \in \mathbb{R} \) were arbitrary, this inequality holds for all \( x, y \in \mathbb{R} \), proving the claim.

28.8) Let \( f \) be differentiable on \((a, b)\).

(ii) Suppose \( f'(x) < 0 \ \forall x \in (a, b) \). Consider \( x_1 \) and \( x_2 \) with \( a < x_1 < x_2 < b \). Since \( f \) is differentiable on \((a, b)\), it is continuous and differentiable on \([x_1, x_2]\) by Theorem 28.2. By the Mean Value Theorem, there exists a \( c \in (x_1, x_2) \) such that

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) < 0,
\]

where the inequality comes from the assumption. Since \( x_2 - x_1 > 0 \), we have

\[
f(x_2) - f(x_1) < 0 \Rightarrow f(x_2) < f(x_1).
\]

Thus, \( f \) is strictly decreasing.

(iii) Suppose \( f'(x) \geq 0 \ \forall x \in (a, b) \). Consider \( x_1 \) and \( x_2 \) with \( a < x_1 < x_2 < b \). Since \( f \) is differentiable on \((a, b)\), it is continuous and differentiable on \([x_1, x_2]\) by Theorem 28.2. By the Mean Value Theorem, there exists a \( c \in (x_1, x_2) \) such that

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0,
\]

where the inequality comes from the assumption. Since \( x_2 - x_1 > 0 \), we have

\[
f(x_2) - f(x_1) \geq 0 \Rightarrow f(x_1) \leq f(x_2).
\]

Thus, \( f \) is increasing.

(iv) Suppose \( f'(x) \leq 0 \ \forall x \in (a, b) \). Consider \( x_1 \) and \( x_2 \) with \( a < x_1 < x_2 < b \). Since \( f \) is differentiable on \((a, b)\), it is continuous and differentiable on \([x_1, x_2]\) by Theorem 28.2. By the Mean Value Theorem, there exists a \( c \in (x_1, x_2) \) such that

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \leq 0,
\]

where the inequality comes from the assumption. Since \( x_2 - x_1 > 0 \), we have

\[
f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_2) \leq f(x_1).
\]
Thus, $f$ is decreasing.

28.14) Suppose $f$ is differentiable on $\mathbb{R}$, $1 \leq f'(x) \leq 2 \forall x \in \mathbb{R}$, and $f(0) = 0$. For $x = 0$, the inequality $x \leq f(x) \leq 2x$ hold trivially.

Let $x > 0$. Since $f$ is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$ by Theorem 28.2. By the Mean Value Theorem, there exists a $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c) \iff f'(c) = \frac{f(x)}{x}$$

Since $1 \leq f'(x) \leq 2 \forall x \in \mathbb{R}$, we have

$$1 \leq \frac{f(x)}{x} \leq 2 \iff x \leq f(x) \leq 2x$$

Since $x > 0$ was arbitrary, the inequality holds for all $x > 0$. Combining this with the $x = 0$ case, we obtain

$$x \leq f(x) \leq 2x \forall x \geq 0,$$

proving the claim.

28.18) Let $f$ be differentiable on $\mathbb{R}$ with $a := \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Choose $s_0 \in \mathbb{R}$ and define sequence $\{s_n\}$ by $s_n = f(s_{n-1})$ for $n \geq 1$. Since $f$ is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$ by Theorem 28.2. Consider $n \in \mathbb{N}$. By the Mean Value Theorem, there exists a $c$ between $s_n$ and $s_{n-1}$ such that

$$\frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}} = f'(c) \iff \frac{s_{n+1} - s_n}{s_n - s_{n-1}} = f'(c) \iff \frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = |f'(c)| \leq a,$$

by the assumption. Rearranging this inequality and the fact that $n \in \mathbb{N}$ was arbitrary, gives us

$$|s_{n+1} - s_n| \leq a|s_n - s_{n-1}| \text{ for } n \geq 1.$$

Notice by repeated use of the above inequality, we obtain

$$|s_n - s_{n-1}| \leq a|s_{n-1} - s_{n-2}| \leq a|s_{n-2} - s_{n-3}| \leq \ldots \leq a^{n-1}|s_1 - s_0| \forall n \in \mathbb{N}$$

Consider $m, n \in \mathbb{N}$ where without loss of generality $n > m$, with the above inequality, we have

$$|s_n - s_m| = |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \ldots + s_{m+1} - s_m|$$

$$\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \ldots + |s_{m+1} - s_m|$$

$$\leq a^{n-1}|s_1 - s_0| + a^{n-2}|s_1 - s_0| + \ldots + a^m|s_1 - s_0|$$

$$\leq a^m \left( \sum_{k=0}^{n-m-1} a^k \right) |s_1 - s_0| \leq a^m \left( \sum_{k=0}^{\infty} a^k \right) |s_1 - s_0| = \frac{a^m}{1-a}|s_1 - s_0|,$$

since its a geometric series with $a < 1$. Then, we have the following

$$|s_n - s_m| \leq \frac{a^m}{1-a}|s_1 - s_0|. \quad (1)$$
Now we are going to prove that \( \{s_n\} \) is a Cauchy sequence. Let \( \epsilon > 0 \) be given. From (1), we have the following

\[
|s_n - s_m| < \epsilon \quad \text{if} \quad |s_n - s_m| \leq \frac{a^m}{1 - a}|s_1 - s_0| < \epsilon
\]

for \( m, n \in \mathbb{N} \) where \( n > m \). But

\[
\frac{a^m}{1 - a}|s_1 - s_0| < \epsilon \quad \text{if and only if} \quad m > \log_a \left( \frac{(1 - a)\epsilon}{|s_1 - s_0|} \right).
\]

Choose

\[
N = \log_a \left( \frac{(1 - a)\epsilon}{|s_1 - s_0|} \right).
\]

If \( m, n > N \) with \( n > m \), then

\[
|s_n - s_m| < \epsilon
\]

Thus, the sequence \( \{s_n\} \) is Cauchy. Since \( \mathbb{R} \) is complete, the sequence \( \{s_n\} \) converges.