19.2) (a) Let $\epsilon > 0$ be given. Notice
\[ |f(x) - f(y)| = |3x + 11 - (3y + 11)| = 3|x - y|. \]
Then
\[ |f(x) - f(y)| < \epsilon \iff 3|x - y| < \epsilon \iff |x - y| < \frac{\epsilon}{3}. \]
Choose $\delta = \frac{\epsilon}{3}$. Thus, if
\[ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \]
and $f$ is uniformly continuous on $\mathbb{R}$.

(b) Let $\epsilon > 0$ be given. Notice
\[ |f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y|. \]
So
\[ |f(x) - f(y)| < \epsilon \iff |x - y||x + y| < \epsilon. \]
But $x, y \in [0, 3]$ which gives us the bound $|x + y| \leq 6$. Then
\[ |x - y||x + y| \leq 6|x - y| < \epsilon \text{ if and only if } |x - y| < \frac{\epsilon}{6}. \]
Choose $\delta = \frac{\epsilon}{6}$. Thus, if
\[ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \]
and $f$ is uniformly continuous on $[0, 3]$.

19.4) (a) Suppose $f$ is uniformly continuous on a bounded set $S$, but (for a contradiction) $f$ is not bounded. So for any $n \in \mathbb{N}$ there exists an $x_n \in S$ where $|f(x_n)| > n$. In particular, we can use this fact
to construct a sequence $\{x_n\}$ with $\lim_{n \to \infty} |f(x_n)| = \infty$. By the Bolzano-Weierstrass theorem, \{x_n\} has a
convergent subsequence \{x_{n_k}\}. Since \{x_{n_k}\} converges, it’s a Cauchy sequence. By the uniform continuity
of $f$, \{f(x_{n_k})\} and consequently \{|f(x_{n_k})|\} are both Cauchy sequences (Theorem 19.4). On $\mathbb{R}$, Cauchy
sequences are convergent which means \{|f(x_{n_k})|\} is bounded, but $\lim_{k \to \infty} |f(x_{n_k})| = \infty$. Contradiction.
Hence, $f$ is uniformly continuous.

(b) $f(x) = \frac{1}{x^2}$ is not bounded because of the division by zero at $x = 0$. By homework 19.4a), since
interval $(0, 1)$ is a bounded set, $f$ is not uniformly continuous on $(0, 1)$.

19.6) (a) $f'(x) = \frac{1}{2\sqrt{x}}$ is unbounded on $(0, 1]$ because of the division by zero that occurs at $x = 0$.
We can build a (trivial) continuous extension of $f(x) = \sqrt{x}$ on $(0, 1]$ by $\tilde{f}(x) = \sqrt{x}$ on $[0, 1]$, which is
continuous since $x^p$ is continuous for $p = 1/2$ and $x \geq 0$. Since there exists a continuous extension of $f(x)$
on $(0, 1]$, $f$ is uniformly continuous on $(0, 1]$ (Theorem 19.5).

(b) Notice that $|f'(x)| < \frac{1}{2}$ for all $x \in (1, \infty)$. Hence, $f$ is differentiable with $f'$ bounded on the interval
$(1, \infty)$, which implies $f$ is uniformly continuous on $[1, \infty)$ (Theorem 19.6).
19.8) Let \( f(x) = \sin x \) which implies \( f'(x) = \cos x \), so \( f \) is differentiable on \( \mathbb{R} \). Let \( x, y \in \mathbb{R} \). By the Mean Value Theorem, there exists \( c \in \mathbb{R} \) such that

\[
    f'(c) = \frac{\sin x - \sin y}{x - y} \Rightarrow |f'(c)| = \frac{|\sin x - \sin y|}{|x - y|}.
\]

But \( |f'(x)| = |\cos x| \leq 1 \ \forall x \in \mathbb{R} \). Thus,

\[
    \frac{|\sin x - \sin y|}{|x - y|} \leq 1 \Rightarrow |\sin x - \sin y| \leq |x - y|
\]

(b) Let \( \epsilon > 0 \) be given. Choose \( \delta = \epsilon \). If

\[
    |x - y| < \delta \Rightarrow |f(x) - f(y)| = |\sin x - \sin y| \leq |x - y| < \epsilon,
\]

using inequality proved in homework 8a). Hence, \( f \) is uniformly continuous on \( \mathbb{R} \).

18.10) (a) Yes. I observe

\[
    g(x) = \begin{cases} 
    0 & \text{if } x = 0 \\
    x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0
    \end{cases}
\]

is continuous on \( \mathbb{R} \).

(b) \( g \) is uniformly continuous on any bounded subset \( \mathbb{R} \) because \( g \) is continuous on any closed and bounded subset of \( \mathbb{R} \). So for any bounded set \( S \subseteq \mathbb{R} \), we can easily build a continuous extension on the closure of \( S \) (i.e. \( S^\prime \) in Definition 13.8).

(c) Yes. \( g \) is uniformly continuous on \( \mathbb{R} \) because \( g' \) is bounded away from \( x = 0 \) (See the book solution for homework 19.9 for detailed discussion of this).