24.2) For $x \in [0, \infty)$, let the sequence of functions $\{f_n\}$ be defined by $f_n(x) = \frac{x}{n}$ $\forall n$.

(a) For $x = 0$, we have

$$f_n(0) = 0 \quad \forall n \Rightarrow \{f_n(0)\} \to 0.$$ 

For $x \in (0, \infty)$, we have

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x}{n} = 0.$$ 

Thus, we choose to define $f(x) = 0 \quad \forall x \in [0, \infty)$ so that $\{f_n\} \to f$ on $[0, \infty)$.

(b) For a given $n \in \mathbb{N}$, we have

$$\sup\{|f(x) - f_n(x)| : x \in [0,1]\} = \sup\{\frac{x}{n} : x \in [0,1]\} = \frac{1}{n}.$$ 

Then

$$\lim_{n \to \infty} \sup\{|f(x) - f_n(x)| : x \in [0,1]\} = \lim_{n \to \infty} \frac{1}{n} = 0.$$ 

Thus, $\{f_n\} \Rightarrow f$ on $[0,1]$ by Proposition in Remark 24.4.

(c) For a given $n \in \mathbb{N}$, we have

$$\sup\{|f(x) - f_n(x)| : x \in [0,\infty)\} = \sup\{\frac{x}{n} : x \in [0,\infty)\} = \infty.$$ 

Then

$$\lim_{n \to \infty} \sup\{|f(x) - f_n(x)| : x \in [0,\infty)\} = \infty.$$ 

Thus, $\{f_n\}$ does not uniformly converge to $f$ on $[0, \infty)$ by Proposition in Remark 24.4.

24.6) Let the sequence of functions $\{f_n\}$ be $f_n(x) = \left(x - \frac{1}{n}\right)^2$ be defined on $x \in [0,1]$.

(a) Let $x \in [0,1]$. Then

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \left(x - \frac{1}{n}\right)^2 = x^2$$

Thus, we choose to define $f(x) = x^2 \forall x \in [0,1]$ so that $\{f_n\} \to f$ on $x \in [0,1]$.

(b) Yes. Let $\epsilon > 0$ be given and $x \in [0,1]$. Then we have

$$|f_n(x) - f(x)| = \left|x - \frac{1}{n}\right|^2 - x^2 = \left|\frac{1}{n^2} - \frac{2x}{n}\right| \leq \frac{1}{n^2} - \frac{2}{n}$$

from the Triangle Inequality and $x \in [0,1]$. Since $n^2 > n \quad \forall n \in \mathbb{N}$, we have the following bound

$$|f_n(x) - f(x)| \leq \frac{1}{n^2} - \frac{2}{n} < \frac{3}{n}.$$ 

So

$$|f_n(x) - f(x)| < \epsilon \iff \frac{3}{n} < \epsilon \iff n > \frac{3}{\epsilon}.$$ 

Choose $N = \frac{3}{\epsilon}$. Thus,

$$\forall n > N \Rightarrow |f_n(x) - f(x)| < \epsilon.$$
Since \( x \in [0, 1] \) was arbitrary, it holds for all \( x \in [0, 1] \). Therefore, \( \{f_n\} \Rightarrow f \) on \([0, 1] \) by definition.

24.8) Let the sequence of functions \( \{f_n\} \) be \( f_n(x) = \sum_{k=0}^{n} x^k \) be defined on \( x \in [0, 1] \).

(a) For \( x = 1 \), \( f_n(1) = n \forall n \). Clearly,

\[
\lim_{n \to \infty} f_n(1) = \infty,
\]

so the limit does not exist. Therefore, the sequence \( \{f_n\} \) does not converge pointwise on \([0, 1] \).

(b) No. Since \( \{f_n\} \) does not converge pointwise on \([0, 1] \) (see part a), \( \{f_n\} \) cannot converge uniformly \([0, 1] \).

24.10) Suppose \( \{f_n\} \Rightarrow f \) and \( \{g_n\} \Rightarrow g \) on a set \( S \). Consider the sequence of functions \( \{f_n + g_n\} \) on \( S \). Let \( \epsilon > 0 \) be given and \( x \in S \). Notice

\[
|(f_n + g_n)(x) - (f + g)(x)| = |f_n(x) - f(x) + g_n(x) - g(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)|
\]

by the Triangle Inequality.

Consider the number \( \frac{\epsilon}{2} > 0 \). Since \( \{f_n\} \Rightarrow f \), there exists \( N_1 \) such that

\[
\forall n > N_1 \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}.
\]

Also, since \( \{g_n\} \Rightarrow g \), there exists \( N_2 \) such that

\[
\forall n > N_2 \Rightarrow |g_n(x) - g(x)| < \frac{\epsilon}{2}.
\]

Choose \( N = \max\{N_1, N_2\} \). Then

\[
\forall n > N \Rightarrow |(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

by (1) and (2). Since \( x \in S \) was arbitrary, it holds for all \( x \in S \). Therefore, \( \{f_n + g_n\} \Rightarrow f + g \).