25.2) Let the sequence of functions \( \{f_n\} \) be defined by \( f_n(x) = \frac{x^n}{n} \) on \([-1, 1]\). First we find the pointwise limit \( f \). Let \( x \in [-1, 1] \). Then we have

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{x^n}{n}.
\]

Since \( |x| \leq 1 \), the following inequality holds

\[
-\frac{1}{n} \leq \frac{x^n}{n} \leq \frac{1}{n} \quad \forall n.
\]

Clearly, the following limits are true

\[
\lim_{x \to \infty} \frac{1}{n} = \lim_{x \to \infty} -\frac{1}{n} = 0.
\]

Then,

\[
\lim_{n \to \infty} \frac{x^n}{n} = 0
\]

by the Squeeze Theorem (Exercise 8.5). Thus, we choose to define \( f(x) = 0 \) \( \forall x \in [-1, 1] \) so that \( \{f_n\} \to f \) on \([-1, 1]\).

Now we show the uniform convergence. Let \( \epsilon > 0 \) be given and \( x \in [-1, 1] \). We have

\[
|f_n(x) - f(x)| = \left| \frac{x^n}{n} - 0 \right| = \left| \frac{x^n}{n} \right| \leq \frac{1}{n}
\]

So

\[
|f_n(x) - f(x)| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}
\]

Choose \( N = \frac{1}{\epsilon} \). Thus,

\[
\forall n > N \Rightarrow |f_n(x) - f(x)| < \epsilon
\]

Since \( x \in [-1, 1] \) was arbitrary, it holds for all \( x \in [-1, 1] \). Therefore, \( \{f_n\} \Rightarrow f \) on \([-1, 1]\) by definition.

25.4) Let the sequence of functions \( \{f_n\} \) on \( S \subseteq \mathbb{R} \). Suppose \( \{f_n\} \Rightarrow f \) on \( S \). Let \( \epsilon > 0 \) be given. Note

\[
|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \quad \forall x \in S
\]

by the Triangle Inequality.

Consider the number \( \frac{\epsilon}{2} > 0 \). Since \( \{f_n\} \Rightarrow f \), there exists \( N \) such that

\[
\forall n > N \quad \forall x \in S \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}.
\]

Moreover, we have

\[
\forall m > N \quad \forall x \in S \Rightarrow |f_m(x) - f(x)| < \frac{\epsilon}{2}.
\]

Thus,

\[
\forall m, n > N \quad \forall x \in S \Rightarrow |f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
by (1) and (2). Therefore, \( \{f_n\} \) is uniformly Cauchy on \( S \) by definition.

25.6) (a) Suppose \( \sum |a_k| < \infty \) (i.e. a convergent series of numbers). Here we have a sequence \( \{|a_k|\} \) of nonnegative numbers with \( \sum |a_k| < \infty \). Consider the power series \( \sum a_k x^k \) on \([-1, 1] \). Since \( |x| \leq 1 \), we have

\[
|a_k x^k| = |a_k| |x|^k \leq |a_k| \forall k \text{ and } \forall x \in [-1, 1].
\]

Thus, the power series \( \sum a_k x^k \) converges uniformly on \([-1, 1] \) by the Weierstrass M-Test. Clearly, a power series is a series of continuous functions (since they are just polynomials). Therefore, \( \sum a_k x^k \) converges uniformly to a continuous functions by Theorem 25.5.

(b) Yes. Since \( a_k = \frac{1}{k^2} > 0 \forall k \), and \( \sum \frac{1}{k^2} \) is a convergent \( p \)-series, the power series \( \sum \frac{1}{k^2} x^k \) converges uniformly to a continuous function on \([-1, 1] \) by the assertion proved in part (a).

25.12) Suppose \( \sum g_k \) is a series of continuous functions \( g_k \) on \([a, b] \) that converges uniformly to \( g \) on \([a, b] \). Define the corresponding sequence of partial sums \( \{f_n\} \) defined by \( f_n(x) = \sum_{k=1}^{n} g_k(x) \) for all \( n \) and \( x \in [a, b] \). Notice for all \( n \) that \( f_n \) is continuous (since addition preserves continuity), and we have \( \{f_n\} \Rightarrow g \) on \([a, b] \) by definition of uniform convergence on a series of functions. From Theorem 25.2, we have

\[
\int_{a}^{b} g(x) \, dx = \lim_{n \to \infty} \int_{a}^{b} f_n(x) \, dx = \lim_{n \to \infty} \int_{a}^{b} \sum_{k=1}^{n} g_k(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} \int_{a}^{b} g_k(x) \, dx = \sum_{k=1}^{\infty} \int_{a}^{b} g_k(x) \, dx
\]

since the integral ‘distributes’ over addition (Theorem 35.8), proving the claim.

25.14) Suppose \( \sum g_k \) is a series of functions that converges uniformly to \( g \) on \( S \), and \( h \) is a bounded function on \( S \). \( h \) is bounded on \( S \) means there exists an \( M \in \mathbb{R} \) such that \( |h(x)| \leq M \forall x \in S \). Notice

\[
|\sum_{k=1}^{n} h(x)g_k(x) - h(x)g(x)| = |h(x)||\sum_{k=1}^{n} g_k(x) - g(x)| \leq M|\sum_{k=1}^{n} g_k(x) - g(x)|
\]

Let \( \epsilon > 0 \) be given, and consider the value \( \frac{\epsilon}{M} > 0 \). Since the series of functions converges uniformly to \( g \) on \( S \), there exists an \( N \) such that

\[
\forall n > N \implies \forall x \in S \implies |\sum_{k=1}^{n} g_k(x) - g(x)| < \frac{\epsilon}{M}.
\]

For this \( N \), we have

\[
\forall n > N \implies |\sum_{k=1}^{n} h(x)g_k(x) - h(x)g(x)| \leq M|\sum_{k=1}^{n} g_k(x) - g(x)| < M\frac{\epsilon}{M} = \epsilon.
\]

Therefore, the series of functions \( \sum h g_k \) converges uniformly to \( h g \) on \( S \) by definition.