26.2) (a) Start with the geometric series
\[ \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1. \]

Taking the derivative to both sides, we obtain
\[ \sum_{n=1}^{\infty} nx^{(n-1)} = \frac{1}{(1-x)^2} \text{ for } |x| < 1. \]

Then, we multiply both sides by \( x \)
\[ \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \text{ for } |x| < 1 \]
to obtain the desired result.

(b) Notice
\[ \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n. \]

Using the formula derived in a for \( x = \frac{1}{2} \), this evaluates to
\[ \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2}} \right) = 2. \]

(c) Note
\[ \sum_{n=1}^{\infty} \frac{n}{3^n} = \sum_{n=1}^{\infty} n \left( \frac{1}{3} \right)^n \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \sum_{n=1}^{\infty} n \left( \frac{-1}{3} \right)^n. \]

Using the formula derived in a for \( x = \frac{1}{3} \) and \( x = -\frac{1}{3} \), this evaluates to
\[ \sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1}{3} \left( \frac{1}{1 - \frac{1}{3}} \right) = \frac{3}{4} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} = \frac{-3}{4} \left( \frac{1}{1 + \frac{1}{3}} \right) = -\frac{3}{16}. \]

26.4) (a) Start with the power series for \( e^x \)
\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \forall x \in \mathbb{R}. \]

Substitution of \( x \) by \( (-x^2) \) gives us
\[ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \forall x \in \mathbb{R}. \]
(b) Let

\[ F(x) = \int_0^x e^{-t^2} \, dt \]

Using the power series obtained in (a) and integrating term-by-term (Theorem 26.4), we arrive at the power series for \( F(x) \)

\[ F(x) = \int_0^x e^{-t^2} \, dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} \]

26.4) Let

\[ s(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ and } c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}. \]

(a) Differentiating \( s(x) \) term-by-term (Theorem 26.5), we have

\[ s'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = c(x). \]

Notice we keep the sum starting at \( n = 0 \) since the 1st term of \( s(x) \) is not a constant.

Differentiating \( c(x) \) term-by-term (Theorem 26.5), we have

\[ c'(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)!} x^{2k-1} = -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} x^{2k-1} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = -s(x), \]

where we reindexed the sum with \( n = k - 1 \).

(b) We can implicitly differentiate to obtain

\[ (s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0 \quad \forall x \in \mathbb{R}, \]

proving the claim.

(c) Applying the Fundamental Theorem of Calculus to the equation obtained in (a), we have

\[ (s^2 + c^2)' = 0 \quad \forall x \in \mathbb{R} \Rightarrow s^2 + c^2 = C \quad \forall x \in \mathbb{R} \]

where \( C \) is a constant of integration. Since it holds for all \( x \in \mathbb{R} \), we let \( x = 0 \), then

\[ C = [s(0)]^2 + [c(0)]^2 = 0^2 + 1^2 = 1 \]

Hence, \( C = 1 \) and we have

\[ s^2 + c^2 = 1, \]

proving the claim.