Introduction W14

◆ Math 125A
Advanced Calculus
Text: Ross Ch 3, 4, 5
Blake Temple: Distinguished Professor Math
Rm 3148 Hrs MWF 2-3 Appt
Grading: 2 Midterms + 1 Final [Homework tested on Midterms
Disc 6:10-7 Haring 206
TA Evan Smithers
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◆ About me: Philosophy Major UM 1973
Math Logic ⇒ Shock Wave Theory ⇒ GR
4 current research projects —

① Wave Theory for Dark Energy
② Regularity Singularities
③ Relativistic Navier-Stokes Eqn
④ Nonlinear Time-periodic sound waves

Introduction - Advanced Calculus

- 323 Alexander the Great died
- Ptolemy I, distant cousin, one of his generals
  Given lower empire ⇒ Egypt
- 305 BC Ptolemy I became emporer Egypt
- Alexander & Ptolemy thought to have been educated by Aristotle who was taught by Plato

"Highly Educated ~ Alexander the Philosopher King"

- One of the first things he did as emporer was establish 1st modern university in Alexandria

MUSEUM:
- One of 1st 4 permanent scholars was Euclid
- One of 1st things he did was write Elements
- All ships required to turn over books for copying

- Purpose of Elements: To show that all truths of plane geometry could be proven by logic to follow from five Postulates

① A straight line segment can be drawn joining any two points

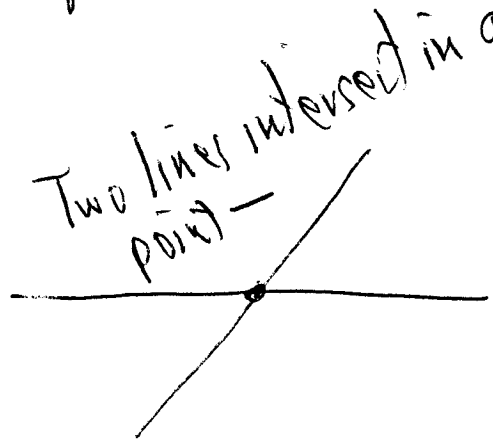
② Any straight line segment can be extended indefinitely into a straight line

③ Given a straight line segment, a circle can be drawn having the line segment as a radius & one endpt the center

④ All right angles are congruent

⑤ ∥-postulate: (restated) Given a line & a pt not on the line ∃! line thru the pt ∥ to the original line

④ Euclid's 1st Postulate raises profound questions:



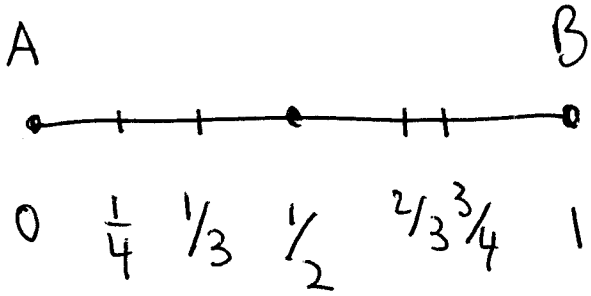
Q1: How do you name all the points on line betn A & B?

Q2: How many are there? Can you count them?

"Euclid's genius was that the notion of a pt as an unambiguous position, and a line as an unambiguous direction, could not be further described – these are starting points"

- The attempts to answer these questions led to the notion of limits & the real line & Advanced Calc.

Q1: How do you name all the pts betw A, B?



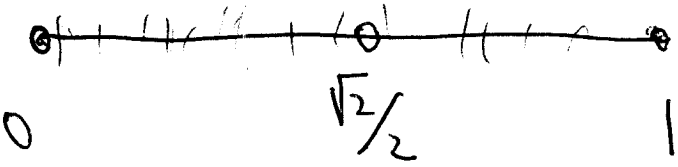
"$\frac{p}{q}$ with $0 < p < q$ names the point p intervals of length $\frac{1}{q}$ from A"

These are called Rational Numbers.

. But Pythagorus proved $\sqrt{2}$ is not rational

We know: π, e, $\sqrt{2}/2$, ... not rational

Q1: How do you name <u>all</u> the points betw 0 & 1?



- Name real numbers by sequences ≡ decimals

$x = .2517\cdots$ name!:

$$x = \frac{2}{10} + \frac{5}{100} + \frac{1}{1000} + \frac{7}{10^4} + \cdots$$

$x = .a_1 a_2 a_3 000$ means

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} \qquad x = \lim_{N\to\infty} S_N \,,\; S_N = \sum_{n=1}^{N} \frac{a_n}{10^n}$$

Q1: does this name <u>all</u> the points betw 0 & 1?

<u>Intuition</u>: Yes !

Q: What assumptions are we making when we say decimals name <u>all</u> the points?

Defn ①: We say a sequence converges, $S_N \to L$, if $\forall \varepsilon \; \exists N$ such that $n > N$ implies $|S_N - L| < \varepsilon$.

But you need to know L exists to apply Defn, & for a decimal $x = \lim_{N \to \infty} S_N$ we're trying to use the sequence to name the limit.

② Completeness Axiom: "Every set of real numbers bounded from above contains a LUB"

This Says: "The line (Segment) contains no holes"

• Theorem: A sequence of real numbers converges to a unique limit iff it is Cauchy. (1789-1857)

Defn. A sequence $\{S_N\}$ is Cauchy if:

$\forall \varepsilon > 0 \ \exists N$ st if $n, m > N$, then

$$|S_n - S_m| < \varepsilon.$$

Cor: Decimals Name a unique real number:

P.f. We show $\{S_n\}$ Cauchy. Fix $\varepsilon > 0$. Then

$$|S_n - S_m| = \left| \sum_{k=m+1}^{n} \frac{a_k}{10^k} \right| \leq 10 \sum_{k=m+1}^{n} \left(\frac{1}{10^k} \right)$$

Set $S_m^n = \sum_{k=m+1}^{n} r^k$, $r = \left(\frac{1}{10}\right)$

$$rS_m^n = \sum_{k=m+2}^{n+1} r^k$$

$$(1-r)S_m^n = S_m^n - rS_m^n = r^{m+1} - r^{n+1}$$

$$0 \le S_m^n = \frac{r^{m+1} - r^{n+1}}{1-r} \le \frac{10}{9} \cdot \left(\frac{1}{10}\right)^{m+1}$$

$$\therefore \quad |S_n - S_M| = |S_m^n| \le \frac{10}{9}\left(\frac{1}{10}\right)^{m+1}$$

Choose N large so $\frac{10}{9}\left(\frac{1}{10}\right)^{N+1} < \varepsilon$.

Then $m, n > N \Rightarrow |S_n - S_m| < \varepsilon \Rightarrow$ Cauchy

Conclude: Decimals are Cauchy $\Rightarrow$ name unique real numbers $\equiv$ points on line.

Q: Are there any other points on line?

Ans: We can assume the real numbers ≡ set of decimals name all points on the line or we can pull it all back to Dedikind Cuts (Richard Dedikind 1831-1916)

Defn: Any Partition of the rational numbers into two disjoint nonempty sets

$$\mathbb{Q} = S_1 \cup S_2$$

such that $s_1 < s_2 \ \forall \ s_1 \in S_1, s_2 \in S_2$ names a unique real number.

Theorem: (+ a bunch of obvious axioms for $+, \cdot, <, >$) implies both are equivalent.

☑ Conclude: The real numbers consist of rational numbers & Cauchy Sequences of rational numbers.

- The notion of limit and convergence of sequences is the starting point of the study of real number line
- Q2: Are there more real numbers than rational numbers?

Ans: Cantor (1845–1918) YES

I.e. There are more sequences of ~~rational~~ ~~numbers~~ $\{0,1\}$ than there are rational numbers

Pf (Cantor diagonalization)

Just as the notion of the limit of a sequence of numbers is the starting point for the study of the real number line, the notion of the limit of a function is the starting point for Calculus. This is the topic of MAT125A.

Defn: A function is continuous at $x = x_0$ if

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.