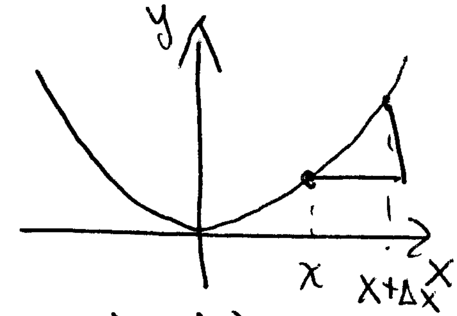
□ Calculus is best expressed with notation of functions

$y = x^2$

$y = f(x) = x^2$

output ↗ ↖ input



$$m = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

General: $f : \mathbb{R} \longrightarrow \mathbb{R}$ "real valued fn of real variable"

outputs $y \in \mathbb{R}$ ↗ ↖ inputs $x \in \mathbb{R}$

if $D \subseteq \mathbb{R}$ we say $D \equiv$ Domain of f

$R \subseteq \mathbb{R}$ we say $R \equiv$ Range of f

$$f : D \longrightarrow R$$

□ When is the graph of f "unbroken"? "continuous"?

Prescientific: "Can draw graph with pencil w/o lifting pencil off paper"

"if the inputs are close to x_0, the outputs should be close to $f(x_0)$"

Two equivalent Defn's: $f: \mathbb{R} \to \mathbb{R}$

Defn 1 f cont @ x_0 if, whenever $x_n \to x_0$, we have $f(x_n) \to f(x_0)$

- We say f cont in D if f cont $\forall\, x_0 \in D$
- If $f: D \to \mathbb{R}$, then f cont @ $x_0 \in D$ if, whenever $D \ni x_n \to x_0$, we have $f(x_n) \to f(x_0)$

Recall: $x_n \to x_0$ if $\forall \varepsilon\, \exists N$ st if $n > N$ then $|x_n - x_0| < \varepsilon$

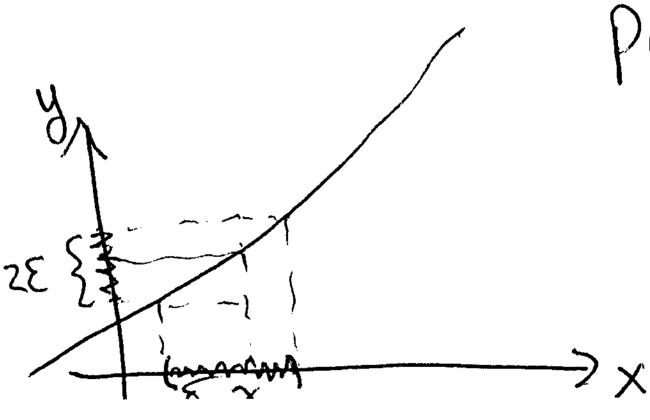
Defn 2 f cont at x_0 if

$$\forall \varepsilon\, \exists \delta \text{ st } [\text{if } |x - x_0| < \delta \text{ then } |f(x) - f(x_0)| < \varepsilon]$$

OR $\forall \varepsilon\, \exists \delta$ st $[\forall x \in I_\delta(x_0)\; P(x)]$

$$P(x) \equiv |f(x) - f(x_0)| < \varepsilon$$

Visualize:



<u>Theorem</u>: f cont by Defn ① iff f cont by Defn ②

- Different books take Defn ① or Defn ② as the starting Defn. Our Book take Defn ①.

☒ Negating Defn ②

Q: How would you say f <u>NOT</u> cont @ x_0?

f cont: $\forall \varepsilon \, \exists \delta$ st $\forall_{x \in I_\delta(x_0)} |f(x) - f(x_0)| < \varepsilon$

If not true: "There must exist some ε st no matter how small δ there must exist $x \in I_\delta(x_0)$ st $|f(x) - f(x_0)| \geq \varepsilon$"

Formally — (Rules of logic)

$$\neg \forall \varepsilon \, \exists \delta \text{ st } \forall_{x \in I_\delta(x_0)} |f(x) - f(x_0)| < \varepsilon \equiv \exists \varepsilon \, \forall \delta \, \exists_{x \in I_\delta(x_0)} \neg P(x)$$

Rule: "The picture keeps track of the logic for you"

3B

Review Logic:

Defn of limit: $x \to x_0$ means:

"$\forall \varepsilon \, \exists N$ st $\underbrace{\text{if } n>N, |x-x_n|<\varepsilon}$"

From a logical point of view, there is an implied "$\forall n > N$"

Better:

"$\forall \varepsilon \, \exists N$ st $\forall_{n>N} P(n)$" $\quad P(n) \equiv |x-x_n|<\varepsilon$

Negation is easy now:

(*)

$x \not\to x_0$: $\neg(\forall \varepsilon \, \exists N \text{ st } \forall_{n>N} P(n)) \equiv \exists \varepsilon \, \forall N \, \exists_{n>N} \neg P(n)$

Words: $x \not\to x_0$ if "This is an ε st for all N you can find $n>n$ st $|x-x_n| \not< \varepsilon$"

Logic w/o Quantifiers: $\underbrace{\text{if } P \text{ then } Q}_{P \Rightarrow Q}$; $P \wedge Q$; P iff Q

P, Q statements either T or F

Meaning by truth tables:

P	Q	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$	$P \wedge Q$	P iff Q	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T	T	T
T	F	F	F	F	F	F
F	T	T	T	F	F	F
F	F	T	T	F	T	T

From this we see $P \Rightarrow Q$ claims the same thing as $\neg P \Rightarrow \neg Q$. To prove $P \Rightarrow Q$ is true, you can assume P is true and prove Q is true, is if P is false, $P \Rightarrow Q$ is true since it makes no claim.

Similarly: since $P \Rightarrow Q$ is equivalent logically to $\neg Q \Rightarrow \neg P$ (same truth table!), you can prove $P \Rightarrow Q$ by proving $\neg Q \Rightarrow \neg P$, which requires assuming $\neg Q$ & proving P.

Similarly, P iff Q is equiv to $P \Rightarrow Q$ and $Q \Rightarrow P$, so to prove it, you must both assume P & prove Q, and then assume Q & prove P.

• Application: if you want to prove f is cont, and (f is cont) $\equiv Q$, and you know P is true, you can prove $P \Rightarrow Q$: Often you prove $\neg Q \Rightarrow \neg P$. Ie., assume f not cont, and prove $\neg P$. Then $\neg Q \equiv (*)$!

Note: A Thm in logic states that alternating $\forall$ $\exists$ statements (in a sense) cannot be simplified. Three alt $\forall$-$\exists$ means pretty complicated !

Proof of Theorem

① $\Rightarrow$ ② Assume $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$ & fix $\varepsilon > 0$. We show $\exists\, \delta$ st $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. Assume $\neg$②, that such a δ does not exist. Then $\exists\, \varepsilon > 0$ st $\forall\, \delta > 0$ $\exists\, x \in I_\delta(x_0)$ st $|f(x) - f(x_0)| \geq \varepsilon$.

Choose $\delta = \frac{1}{n}$, and let $x_n \in I_{1/n}(x_0)$ be st $|f(x_n) - f(x_0)| \geq \varepsilon$

Then $x_n \to x_0$, but $f(x_n) \not\to f(x_0)$. So $\neg$①.

We proved $\neg$② $\Rightarrow$ $\neg$① equiv to ① $\Rightarrow$ ② ✓

② $\Rightarrow$ ① Assume Defn ② & let $x_n \to x_0$. We show $f(x_n) \to f(x_0)$. Fix $\varepsilon > 0$. Then $\exists\, \delta$ st $x \in I_\delta(x_0) \Rightarrow$ $|f(x) - f(x_0)| < \varepsilon$. So if $n > N \Rightarrow |x_n - x_0| < \delta$, we have $|f(x_n) - f(x_0)| < \varepsilon$ ✓

Ex. Let $D = [0,1]$, $f(x) = \sqrt{x}$. Prove f is cont at $x = 0$.

(1) Assume $x_n \in D$, $x_n \to 0$. Then $\lim\limits_{x_n \to 0} \sqrt{x_n} = 0$ ✓

That is: fix $\varepsilon > 0$. We show $\exists N$ st $n > N \Rightarrow |\sqrt{x_n}| < \varepsilon$

[Need: $|x_n| < \varepsilon^2$] But $x_n \to 0$, $\Rightarrow \exists N$ st $n \geq N$

$\Rightarrow x_n < \varepsilon^2 \Leftrightarrow \sqrt{x_n} < \varepsilon$ ✓

(2) Fix $\varepsilon > 0$. We find δ st $x \in D$ & $|x| < \delta$

$\Rightarrow \sqrt{x} < \varepsilon$. Choose $\delta = \varepsilon^2 \Rightarrow \sqrt{x} < \varepsilon$ ✓

Ex: $D = \{0\}$. $f : D \to \mathbb{R}$ $f(0) = 1$. Q: is f cont at $x = 0$?

Ans Yes: Choose any $x_n \in D$ st $x_n \to 0$. Then $x_n = 0$ $\forall n$. $f(x_n) = 1 = f(0)$ so $f(x_n) \to f(1)$ ✓

<u>Theorem</u>: Assume f, g cont at x_0. Then

(1) kf, $|f|$, $f+g$, $f \cdot g$, $\mathrm{Max}(f,g)$, $\mathrm{Min}(f,g)$ are cont @ x_0

(2) $g(x_0) \neq 0 \Rightarrow \frac{f}{g}$ cont at x_0

(3) Range $f \subset$ Dom $G \Rightarrow g \circ f$ cont at x_0.

Pf (3): Assume Rang $f \subset$ Dom G so that $g(f(x))$ is defined. Assume f, g cont at x_0. We show that $\forall\, x_n \in \mathrm{Dom}\, f$, $x_n \to x_0 \Rightarrow g(f(x_n)) \to g(f(x_0))$.

But $x_n \to x_0 \underset{\substack{\uparrow \\ \text{cont of } f}}{\Rightarrow} f(x_n) \to f(x_0) \underset{\substack{\uparrow \\ \text{cont of } g}}{\Rightarrow} g(f(x_n)) \to g(f(x_0))$

<u>Example</u>: A function that is <u>continuous</u> at every irrational # but <u>discontinuous</u> at every rational #

$f: [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$

Claim ①: f is cont at every irrational #

Pf $\overset{(*)}{\wedge}$ fix $\varepsilon>0$. Choose $n \in \mathbb{N}$ st $\frac{1}{n}<\varepsilon$. Write out all rational #'s with denom less than or equal to n.

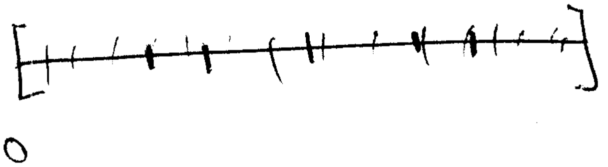
$p=1$ $\frac{0}{1}, \frac{1}{1}$

$p=2$ $\frac{0}{2}, \frac{1}{2}, \frac{2}{2}$

$p=3$ $\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$

$\vdots$

$p=n$ $\frac{0}{n}$ $\frac{1}{n}$ $\frac{2}{n}$ $\cdots$ $\frac{n}{n}$



Let $S_n = \{ \frac{q}{p} : 0 \le q \le p \ \& \ p \le n \}$. finite set

* [Now assume x_0 is irratl. ~~Then $x_0 \notin S$~~. We find δ st $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)| = |f(x)| < \varepsilon$.

But $x_0 \notin S_n$ so let $\delta = \min_{x \in S} |x-x_0| < \frac{1}{n}$. Then $|x-x_0|<\delta$

$\Rightarrow$ either x is irratl & $f(x_0)=0$ or

x is ratl & $x=\frac{q}{p}$ where $p>n$ & $|f(x)|<\frac{1}{n}<$

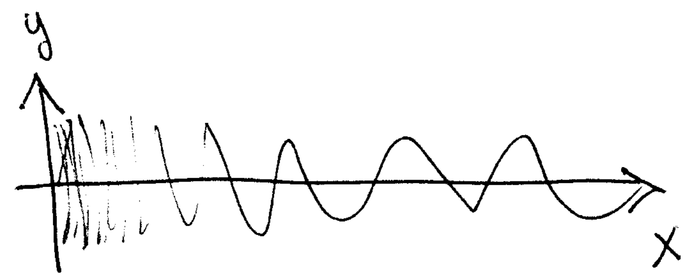
Either way, $|f(x)|<\varepsilon$ for $|x-x_0|<\delta$ ✓

Claim ②: f is discont at every rational.

Assume $x_0 = \frac{q}{p}$ rational, lowest terms. Then $\exists$ irrat'l # $x_n \to x_0$. But $f(x_n) = 0$ & $f(x_0) = \frac{1}{p}$
$\Rightarrow \lim f(x_n) \neq f(x_0)$ ✓

Example: A function that is discontinuous by oscillation

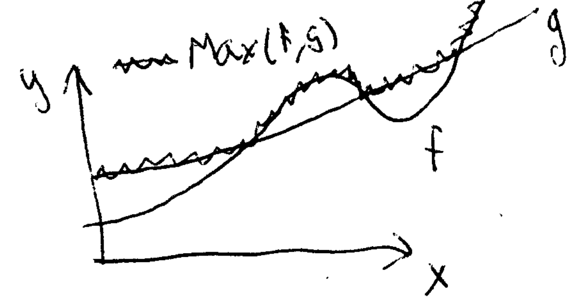
$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



Claim: f discont at $x = 0$.

Choose x_n st $\frac{1}{x_n} = 2\pi n + \frac{\pi}{2}$ Then $x_n \to 0$

but $\sin\left(\frac{1}{x_n}\right) = \sin\left(2\pi n + \frac{\pi}{2}\right) = 1 \not\to \sin(0)$ ✓



- $\mathrm{Max}(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$

$\big(f(x) \geq g(x)$, $\frac{1}{2}(f+g) - \frac{1}{2}(f-g) = g$

$f(x) \leq g(x)$, $\frac{1}{2}(f+g) + \frac{1}{2}(f-g) = f\big)$

$\therefore$ cont of max follows from cont $f+g$, $f-g$, $|f|$