Theory of the derivative

- f cont @ x_0: $\forall \varepsilon \, \exists \delta$ st $\forall_{|x-x_0|<\delta}$ $|f(x)-f(x_0)|<\varepsilon$

- f diff @ a: $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ ↑ has a limit named

Thm: If f diff @ $x=a$, then f is cont @ $x=a$.

Pf. Assume $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$. We prove:

$\forall \varepsilon \, \exists \delta$ st $\forall_{|x-a|<\delta}$ $|f(x)-f(a)|<\varepsilon$. So fix $\varepsilon>0$

$\boxed{\text{We find the } \delta.}$ But $\exists \delta_1$ st $|x-a|<\delta_1 \Rightarrow$

$$\left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \varepsilon_1 \Rightarrow |f(x)-f(a)-f'(a)(x-a)| < \varepsilon_1 |x-a|$$

Choose δ_2 st $|x-a|<\delta_2 \Rightarrow f'(a)|x-a| <$

Since $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ we know:

$\forall \varepsilon_1, \exists \delta_1$ st $|x-a| < \delta_1 \Rightarrow$

$$\left| \frac{f(x)-f(a)}{x-a} - f'(a) \right| < \varepsilon_1$$

$$|f(x)-f(a)-f'(a)(x-a)| < \varepsilon_1 |x-a|$$

$$|f(x)-f(a)| - |f'(a)(x-a)| < \varepsilon_1 |x-a|$$

$$|f(x)-f(a)| < [\varepsilon_1 + |f'(a)|]\,|x-a|$$

Choose $\delta < \delta_1$ so $|x-a| < \delta \Rightarrow$

$$|\varepsilon_1 + |f'(a)||\,|x-a| < \varepsilon.$$

Note: $\left|\frac{f(x)-f(a)}{x-a}\right| < M$ is enough!

Then $|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon$ ✓

<u>Ex</u>: f cont $\not\Rightarrow$ f diff by $f(x) = |x|$.

<u>Conclude</u>: A diff function is "more regular" than a cont fn.

Outline of proofs we want to prove

<u>Thm</u> ① f & g diff @ $x=a$ $\Rightarrow$

(i) $(cf)'(a) = c f'(a)$

(ii) $(f+g)'(a) = f'(a) + g'(a)$

(iii) $(fg)'(a) = (f'g + fg')(a)$

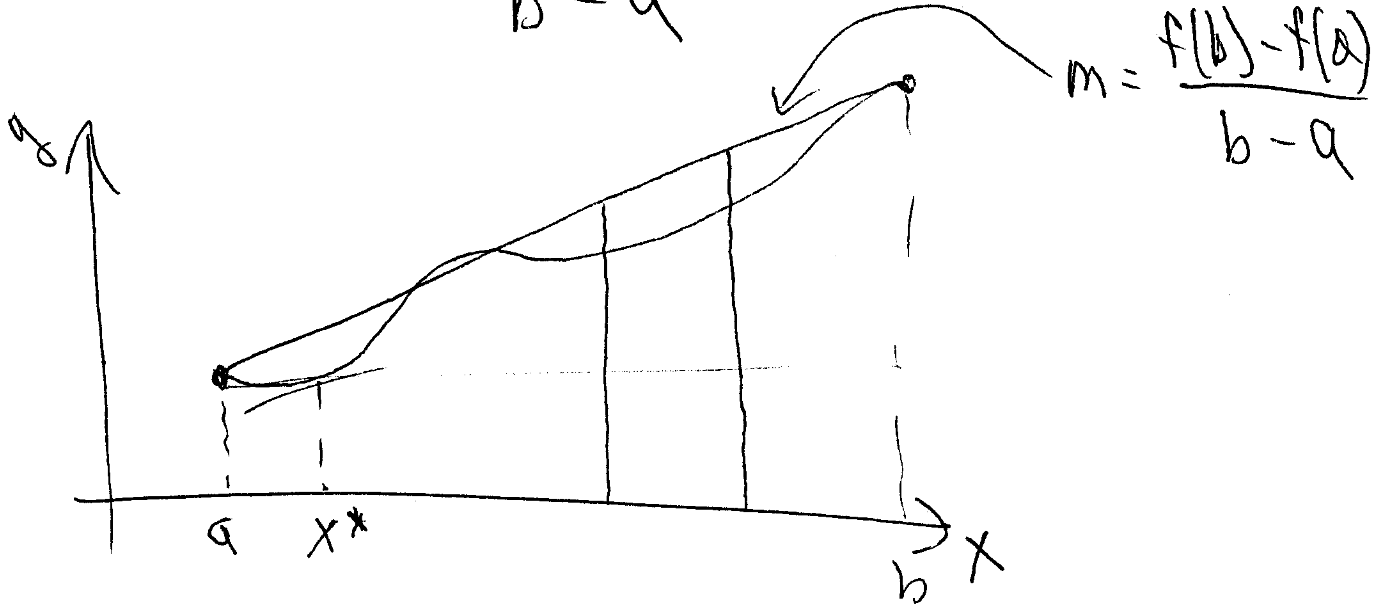
(iv) $\left(\frac{f}{g}\right)'(a) = \left(\frac{gf' - fg'}{g^2}\right)(a)$ if $g \neq 0$

<u>Thm</u> ② Chain Rule

$\frac{d}{dx} f \circ g(a) = f'(g(a))\, g'(a)$ so long as f is diff @ $g(a)$.

Thm 3 (MVT) Assume f cont on $[a,b]$ & diff on (a,b). Then $\exists\, x^* \in (a,b)$ st

$$f'(x^*) = \frac{f(b)-f(a)}{b-a}$$



Main Application (Preview)

Needed to prove FTC

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \underbrace{\sum_{k=1}^{N} f(x_k^*)\Delta x} = \lim_{N\to\infty} \sum_{k=1}^{N} \left[F(x_{k+1}) - F(x_k)\right]$$
$$= F(b) - F(a)$$

$$f(x_k^*)\Delta x = F(x_{k+1}) - F(x_k)$$

f cont $\Rightarrow$ $|f(x_k) - f(x_k^*)| < \varepsilon$ for $|\Delta x| < \delta$

Thus:

$$\sum_{k=1}^{N} f(x_k)\Delta x = \underbrace{\sum_{k=1}^{N} f(x_k^*)\Delta x}_{F(b)-F(a)} + \underbrace{\sum_{k=1}^{N} f(x_k)-f(x_k^*)\Delta x}_{\mathrm{II}}$$

$$|\mathrm{II}| \leq \sum_{k=1}^{N} |f(x_k)-f(x_k^*)|\Delta x \leq \varepsilon \sum_{k=1}^{N} \Delta x$$

$$\leq \varepsilon |b-a|$$ (Uniform Continuity of f on [a,b])

Thus: $\sum_{k=1}^{N} f(x_k)\Delta x = F(b)-F(a) + \varepsilon|b-a|$

$\forall \varepsilon\ \exists \delta$ st $\Rightarrow$

$|\Delta x| < \delta$

$$\therefore \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x = F(b)-F(a)$$

- Product Rule: $\frac{d}{dx}(fg) = f'g + fg'$

Pf. We prove $\lim\limits_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} = f'(a)g(a) + f(a)g'(a)$

<u>But</u>: $\frac{f(x)g(x) - f(a)g(a)}{x-a} = \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x-a}$

$$= f(x)\left[\frac{g(x)-g(a)}{x-a}\right] + g(a)\left[\frac{f(x)-f(a)}{x-a}\right]$$

$\therefore$

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} = f(a) \lim [\ ] + g(a) \lim [\ ]$$

$$= f(a)\,g'(a) + g(a)\,f'(a)$$

- Chain Rule: $(f \circ g)'(a) = f'(g(a))\, g'(a)$

Pf. Assume $g'(a)$ & $f'(g(a))$ exist.

$$(f\circ g)'(a) = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a}$$

$$= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \, \frac{g(x)-g(a)}{x-a} \qquad (*)$$

Now: if $g(x)-g(a)$ takes value zero in every $B_\delta(g(a))$, then $\lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a} = 0$

If not, $(*)$ holds and so

$$(f\circ g)'(a) = \underbrace{\lim_{x\to a} \frac{f(g(x))-f(g(a))}{g(x)-g(a)}}_{} \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

$\lim_{x\to a} g(x) - g(a) = 0 \Rightarrow \quad f'(g(a)) \qquad \searrow g'(a)$