METRIC SPACES

Metric Space: How to frame convergency & continuity in a most general framework:

Defn (Limit): $x_n \longrightarrow x_0$ in $\mathbb{R}$ if: $\forall \varepsilon \; \exists N$ st $\forall_{n>N} \; |x_n - x_0| < \varepsilon$

uses the abs value fn to measure distance

Defn (Cont) $f: \mathbb{R} \to \mathbb{R}$ is continuous @ x_0 if:

$$\forall \varepsilon \; \exists \delta \text{ st } \forall_{|x-x_0|<\delta} \; |f(x) - f(x_0)| < \varepsilon$$

Uses abs value to measure distance

The two defn's are the same for <u>any</u> measure of distance. Ie. in $\mathbb{R}^3$ $\|\underset{\sim}{x}\| = \sqrt{x_1^2 + y^2 + z^2}$

$\underset{\sim}{x} = (x, y, z)$ or (x_1, x_2, x_3)

Defn: (Limit) $\underset{\sim}{x}_n \to \underset{\sim}{x}_0$ in $\mathbb{R}^3$ if $\forall \varepsilon \; \exists N$ st $\forall_{n>N} \; \|\underset{\sim}{x} - \underset{\sim}{x}_0\| < \varepsilon$

Defn: (Cont) $f: \mathbb{R}^3 \to \mathbb{R}^3$ is cont @ $\underset{\sim}{x}_0$ if:

$$\forall \varepsilon \; \exists \delta \text{ st } \forall_{\|\underset{\sim}{x} - \underset{\sim}{x}_0\| < \delta} \; \|f(\underset{\sim}{x}) - f(\underset{\sim}{x}_0)\| < \varepsilon$$

General Framework : (d, S)

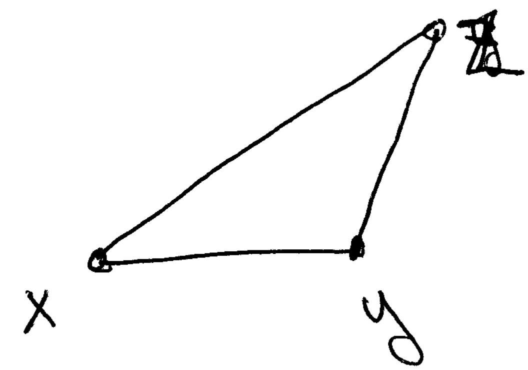
Defn : d is a metric on a set S if:

$$d(x,y) \geq 0 \quad \forall \; x,y \in S \;\&$$

D1. $d(x,\cancel{x}) = 0 \;\forall\, x \in S$, $d(x,y) > 0$ for $x \neq y$ [non-degeneracy]

D2. $d(x,y) = d(y,x)$ [symmetry]

D3. $d(x,z) \leq d(x,y) + d(\cancel{y},z)$ [Δ-inequality]



Defn: (limit) $x_n \longrightarrow x_0$ in S if.

$$\forall \varepsilon \;\exists N \text{ st } \forall_{n>N} \; d(x_n, x_0) < \varepsilon$$

Defn. (Cont) $f: S \to S$ is cont at $x_0 \in S$ if:

$$\forall \varepsilon \;\exists \delta \text{ st } \forall_{d(x, x_0) < \delta} \; d(f(x), f(x_0)) < \varepsilon.$$

<u>Defn</u>: A sequence $x_n \in S$ is Cauchy if:
$\forall \varepsilon\ \exists N$ st $\forall_{m,n>N}\ d(x_n, x_m) < \varepsilon$.

<u>Defn</u>: S is a <u>complete</u> metric space if every Cauchy sequence converges.

<u>Ex</u> 1 $\mathbb{R}$ is a complete metric space with
$$d(x,y) = |x-y|$$

<u>Ex</u> 2 Complex #'s $\mathbb{C} = \{a+ib : a,b \in \mathbb{R}\}$ is a complete metric space with
$$d(z,\tilde{z}) = \|(a-\tilde{a}) + i(b-\tilde{b})\| = \sqrt{(a-\tilde{a})^2 + (b-\tilde{b})^2}$$

<u>Ex</u> 3 $\mathbb{R}^2$ with $d(\underset{\sim}{x},\underset{\sim}{y}) = \|y-x\|$
$$= \sqrt{(y_1-x_1)^2 + (y_2-x_2)^2}$$
$\underset{\sim}{x} = (x_1, x_2)$
$\underset{\sim}{y} = (y_1, y_2)$

<u>Ex</u> (4) $\mathbb{R}^3$ with $\overset{d(x,y)}{\|\underset{\sim}{x}-\underset{\sim}{y}\|} = \sqrt{\sum_{i=1}^{3}(x_i-y_i)^2}$

<u>Ex</u> (4) $\mathbb{R}^n$ $d(\underset{\sim}{x}-\underset{\sim}{y}) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$

called the Euclidean Norm

<u>Note</u>: Normed spaces are special metric Spaces with $d(x,y) = \|\underbrace{x-y}\|$ where $\|c\underset{\sim}{x}\| = |c|\|\underset{\sim}{x}\|$

distance is measured by the norm of difference

<u>Defn</u>: A complete normed linear space is called a ~~Hilbert~~ Banach Space.

Ex: Let $S = \{f : [a,b] \to \mathbb{R} \ \& \ f \text{ continuous}\}$.

Define $d(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| = \max_{x\in[a,b]} |f(x)-g(x)|$

cont fn's take max/min values on closed intervals

$= \|f-g\|_{sup}$

$\|f\|_{sup}$ is called the supnorm.

Theorem: S, d is a complete metric space.
(In fact, it's a Banach Space not a Hilbert Space, usually denoted $C^0[a,b]$)

Proof. (i) Check d is a metric

D1) $d(f,g) = \sup|f-g| = 0$ iff $f(x) = g(x) \ \forall x$
iff $f = g$.

D2) $d(f,g) = d(g,f)$

$\uparrow$ $\sup|f-g| = \sup|g-f|$

D3) Δ-inequality

$$d(f,h) = \sup_{x\in[a,b]} |f(x)-h(x)| = \sup_{x\in[a,b]} |f(x)-g(x)+g(x)-h(x)|$$

$$\le \sup_{x\in[a,b]} |f(x)-g(x)| + \sup_{x\in[a,b]} |g(x)-h(x)|$$

- Completeness: Assume f_n is Cauchy. Then

$\forall \varepsilon\ \exists N$ st $\forall_{n,m>N}\ \|f_m(x)-f_n(x)\|_{sup} < \varepsilon.$

But that means at each x, $f_n(x)$ is a Cauchy sequence of real #'s $\Rightarrow$ at each x $f_n(x) \longrightarrow f(x)$. But f_n must converge uniformly to f because $|f_n(x)-f_m(x)| < \varepsilon \Rightarrow \lim_{m\to\infty} |f_n(x)-f(x)| \le \varepsilon$.

Thus: $\forall \varepsilon\ \exists N$ st $\forall_{\substack{m,n>N\\ x\in[a,b]}}\ \underbrace{|f_m(x)-f_n(x)| < \frac{\varepsilon}{2}}_{|f_n(x)-f(x)| < \varepsilon}$

$\forall \varepsilon\ \exists N$ st $\forall_{m,n>N}\ \|f_n-f\|_{sup} < \varepsilon.$ ✓ $f_n \to f$ in $\|\ \|_s$

Since the unif limit of cont fn's is cont.,
we have $f_n \to f$ in $\underset{\sim}{S}$ & $f \in \underset{\sim}{S}$ $\Rightarrow$
(S,d) complete. $\square$

$\square$ $L^2[a,b] = \{ f : [a,b] \to \mathbb{R} : \int_a^b |f(x)|^2 dx < \infty \}$

This is a Hilbert Space $\Rightarrow$ Banach Space $\Rightarrow$ Metric Space.

Hilbert Space ~~has~~ is a B-space in which norm generated by dot/inner product.

I.e. $d(f,g) = \sqrt{\int_a^b |f(x)-g(x)|^2 dx} = \langle f, f \rangle$

$$\langle f, g \rangle = \underbrace{\int_a^b f(x) g(x) dx}$$

Like the dot product in $\mathbb{R}^n$

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \underset{\sim}{x} \cdot \underset{\sim}{y} = \sum_{k=1}^{n} x_i y_i$$

Norm is generated by inner product $\Rightarrow$ Hilbert

- Very important space ~ setting for Fourier Series
- The problem - f_n could be Cauchy in $L^2[a,b]$, $f_n \to f$ but the Riemann Integral for f^2 is undefined - $\int_a^b f_n^2\,dx \to$ value

But $\int_a^b f^2\,dx$ undefined as a Riemann Integral

The Fix - Define the Lebesgue Integral

Then - $\lim_{n\to\infty} \int_a^b f_n(x)^2\,dx = \int_a^b \lim_{n\to\infty} f_n(x)^2\,dx$

$= \int_a^b f(x)^2\,dx$

Answer - With the Lebesgue Integral, $L^2[a,b]$ is a complete space $\Rightarrow$ Hilbert Space

- Why is $L^2[a,b]$ so important?

In $\mathbb{R}^3$ you have ON bases — Eg i, j, k

Then: $x = \langle x, i\rangle i + \langle x, j\rangle j + \langle x, k\rangle k$

In fact: any ON basis e_1, e_2, e_3 satisfying

$\langle e_i, e_j\rangle = \delta_{ij}$. Then

$$x = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \langle x, e_3\rangle e_3$$

I.e. $x = a_1 e_1 + a_2 e_2 + a_3 e_3$ because $\{e_i\}$ basis

$$x \cdot e_i = (a_1 e_1 + \cdots + a_3 e_3) \cdot e_i = a_i = \langle x, e_i\rangle \checkmark$$

~~Then this in $L^2[a,b]$~~

<u>Conclude</u>: The expansion of x in terms of an ON basis is <u>easy</u>

• Similarly: $L^2[-\pi,\pi]$ has ON basis

$$\left\{ \frac{\sin nx}{a_n}, \frac{\cos nx}{b_n} \right\}_{n=-\infty}^{\infty}$$

$$a_n^2 = \int_{-\pi}^{\pi} \sin^2 nx\, dx = \pi$$

$$b_n^2 = \int_{-\pi}^{\pi} \cos^2 nx\, dx = \pi \quad n \neq 0$$

$$b_0^2 = 2\pi$$

$$\int_{-\pi}^{\pi} \sin^2 n\pi x\, dx = \int_{-\pi}^{\pi} \frac{1-\cos 2n\pi x}{2}\, dx$$

$$\cos^2 n\pi x - \sin^2 n\pi x = \cos 2n\pi x$$

$$1 - 2\sin^2 n\pi x = \cos 2n\pi x$$

$$\sin^2 n\pi x = \frac{1-\cos 2n\pi x}{2}$$

$$= \frac{1}{2}\cdot 2\pi - \int_{-\pi}^{\pi} \frac{\cos 2n\pi x}{2}\, dx = \pi$$

Thus: $$f(x) = \sum_{n=-\infty}^{\infty} \left\langle f, \frac{\sin nx}{a_n} \right\rangle \frac{\sin nx}{a_n} + \sum_{n=-\infty}^{\infty} \left\langle f, \frac{\cos nx}{b_n} \right\rangle \frac{\cos nx}{b_n}$$

$$= \sum_{n=-\infty}^{\infty} A_n \sin nx + \sum_{n=\infty} B_n \cos nx$$

$$A_n = \langle f, \sin nx \rangle \frac{1}{a_n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$n \neq 0$ $$B_n = \langle f, \cos nx \rangle \frac{1}{b_n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$