Metric Space $X, d(x, y)$

$\mathbb{R}^n \quad d(x, y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} = \langle x, y \rangle$

$L^2[a, b] \quad d(f, g) = \sqrt{\int_a^b f(x)g(x)dx} = \langle f, g \rangle$

Metric generated by inner product $\langle f, g \rangle$

Complete $\Rightarrow$ Hilbert Space

Closed function space to $\mathbb{R}^n$

Difficult to prove completeness - need Lebesgue integral

Easiest Way: Consider complex valued function $L^2[-\pi, \pi] = \{ f \rightarrow \mathbb{C} : \int_{-\pi}^{\pi} f^2dx < \infty \}$

$f(x) = u(x) + iv(x) \quad \overline{f(x)} = u(x) - iv(x)$

Real part of $f$ \quad Imaginary part
A slicker way to do this:

\[ L^2 [-\pi, \pi] = \left\{ f: [-\pi, \pi] \to \mathbb{C} \mid \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\} \]

\[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx \]

on basis \( \left\{ e^{inx} = \cos nx + i \sin nx \right\}_{n=-\infty}^{\infty} \)

Easy: \[ \langle e^{inx}, e^{inx} \rangle = \int_{-\pi}^{\pi} e^{inx} \overline{e^{inx}} \, dx = \int_{-\pi}^{\pi} 1 \, dx = 2\pi \]

\[ \langle e^{inx}, e^{inx} \rangle = \int_{-\pi}^{\pi} e^{inx} \overline{e^{inx}} \, dx = \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx = \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx \]

and \[ \frac{1}{i(n-m)} \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx = i(n-m) e^{i(n-m)\pi} \]

\[ = \frac{1}{i(n-m)} e^{i(n-m)\pi} \int_{-\pi}^{\pi} = 0 \quad \text{2\pi-periodic?} \]
Conclude: Any complex \( f \) in \( L^2[-\pi, \pi] \) can be expanded as:

\[
f(x) = \sum_{n=-\infty}^{\infty} \left< f, \frac{e^{inx}}{\sqrt{2\pi}} \right> \frac{e^{inx}}{\sqrt{2\pi}}
\]

or

\[
\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx \cdot e^{inx}
\]

1-F coeff of \( f \) is \( a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \)

"Like a \( T \)-series but based on sines and cosines."

The nicest of all function spaces.

Conclusion: \( L^2[a,b] \) is nicer than \( C^0[a,b] \) because you can work with \( L^2 \)-bases like sines & cosines \( \text{but to get completeness you need a better defn of integral= Lebesgue integral.} \)