- Metric Space X, $d(x,y)$
- $\mathbb{R}^n$ $d(\underset{\sim}{x}, \underset{\sim}{y}) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} = \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle$
- $L^2[a,b]$ $d(f,g) = \sqrt{\int_a^b fg\,dx} = \langle f, g \rangle$

Metric generated by inner product $\langle f, g \rangle$

+ Complete $\Rightarrow$ Hilbert Space –

- Closest function space to $\mathbb{R}^n$
- Difficult to prove complete – need Lebesgue integral
- Easiest Way: Consider complex valued function $L^2[-\pi,\pi] = \{ f\ [-\pi,\pi] \to \mathbb{C} : \int_{-\pi}^{\pi} |f|^2 dx < \infty \}$

$$f(x) = u(x) + i\,v(x) \qquad \bar{f}(x) = u(x) - i\,v(x)$$

real part of f imag part

• A slicker way to do this —

$$L^2[-\pi,\pi] = \left\{ f : [-\pi,\pi] \to \mathbb{C} \;\middle|\; \int_{-\pi}^{\pi} f(x)\,dx < \infty \right\}$$

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx$$

ON basis – $\left\{ \dfrac{e^{inx}}{\sqrt{2\pi}} = \dfrac{\cos nx + i \sin nx}{\sqrt{2\pi}} \right\}_{n=-\infty}^{\infty}$

<u>Easy</u>: $\langle e^{inx}, e^{inx}\rangle = \int_{-\pi}^{\pi} e^{inx} e^{-inx}\,dx = \int_{-\pi}^{\pi} dx = 2\pi$

$$\langle e^{inx}, e^{imx}\rangle = \int_{-\pi}^{\pi} e^{+inx} e^{-imx}\,dx = \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx =$$

$u = i(n-m)$
$du = i(n-m)dx$

$$= \frac{1}{i(n-m)} \int_{x=-\pi}^{x=\pi} e^u\,du = \frac{1}{i(n-m)} e^u \Big]_{x=-\pi}^{x=\pi}$$

$$= \frac{1}{i(n-m)} e^{i(n-m)x} \Big]_{-\pi}^{\pi} = 0 \;!$$

2π-periodic

3

<u>Conclude</u>: Any complex f in $L^2[-\pi,\pi]$ can be expanded as:

$$f(x) = \sum_{n=-\infty}^{\infty} \left\langle f, \frac{e^{inx}}{\sqrt{2\pi}} \right\rangle \frac{e^{inx}}{\sqrt{2\pi}}$$

$$= \boxed{\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx \cdot e^{inx}}$$

n-F-coeff of f is $\boxed{a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx}$

"Like a T-series but based on sines & cosines"

<u>Conclude</u>: $L^2[a,b]$ ← The nicest of all function spaces. is <u>nicer</u> than $C^0[a,b]$ because you can work with ON-bases like sines & cosines <u>BUT</u> to get completeness you need a better defn of integral ⇒ Lebesgue integral.