2 Topology of metric spaces —

- Idea of topology — everything about convergence & continuity can be stated/expressed in terms of open sets.

__Defn__: The topology of a metric space is the collection of all __open sets__

We define the open sets in (S, d)

- __Defn__: $B_\varepsilon(x_0) = \{x \in S : d(x, x_0) < \varepsilon\} \subseteq$ "open ball center x_0 radius ϵ"

- __Defn__: O is an open set $O \subseteq S$ if $\forall$ $x \in S$ $\exists$ ε st $B_\varepsilon(x) \subseteq S$.

"Every element of S is surrounded by elements of S" ~ "No boundary pts"

- **Thm**: (1) infinite unions of open sets are open
  (2) Every open set is an infinite union of open Balls
  (3) Finite intersections of open sets are open

Pf. (1) & (2) clear (3) Suffices to prove $\forall x \in \bigcap O_i$ $\exists \varepsilon$ st $B_\varepsilon(x) \subseteq \bigcap O_i$

(3) $x \in O_1 \cap \cdots \cap O_n \Rightarrow \forall O_i \; \exists \varepsilon_i$ st $B_{\varepsilon_i}(x) \subseteq O_i$. Take $\varepsilon = \min\{\varepsilon_1 \cdots \varepsilon_n\}$.

Then $B_\varepsilon(x) \subseteq O_i \; \forall i \Rightarrow B_\varepsilon(x) \subseteq O_1 \cap \cdots \cap O_n$.

✓

<u>Defn</u>: A set is closed if it is the complement of an open set.

<u>Thm</u>: If E is <u>closed</u> and $x_n \to x_0$, then $x_0 \in E$.

Pf. assume $x_n \to x_0$ & $x_0 \notin E_{closed}$. Then $x_0 \in E^c_{closed} = \mathcal{O}_{open}$. But $\nexists\ \varepsilon$ st $B_\varepsilon(x_0) \subseteq \mathcal{O}$ because $x_n \to x_0$ & $x_n \notin \mathcal{O}$. ※ $\mathcal{O}$ open.

- <u>Thm</u>: $x_n \to x_0$ iff x_n is eventually within every open set containing x_0. Ie iff $\forall\ \mathcal{O}_{\ni x_0}\ \cancel{\exists}\ \exists N$ st $\forall_{n>N}\ x_n \in \mathcal{O}$.

Pf. ~~G~~ Given $\mathcal{O}$, choose $B_\varepsilon(x_0) \subseteq \mathcal{O}$, & choose N st $n > N \Rightarrow x_n \in B_\varepsilon(x_0) \subseteq \mathcal{O}$ ✓

• Q: What sets E in (S,d) have the BW-property? i.e. $\{x_n\} \subseteq E \Rightarrow \exists$ convergent subsequen $x_n \to x_0 \in E$.

In $\mathbb{R}^n$: Closed bded sets satisfy BW property.
Not so in general, eg not so in $C^0[a,b]$.

<u>Defn</u>: A set $E \subseteq S$ is <u>compact</u> if every covery of E by open sets admits a finite subcover. Ie, $E \subseteq \bigcup_{\lambda \in \Lambda} O_\lambda \Rightarrow \exists$ $O_1 \cdots O_N$ st $E \subseteq \bigcup_{i=1}^{N} O_i$.

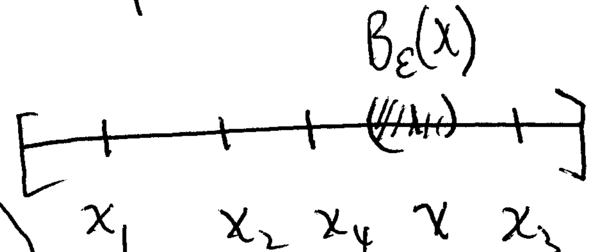
<u>Thm</u>: If E is <u>compact</u>, then E has the BW property.

Pf. Assume E compact & that $\{x_n\} \subseteq E$.

We prove $\exists$ convergent subseq^n $x_{n_k} \to x_0 \in E$.

Assume for ※ no such subsequence exists.

Then $\forall\, x \in E\ \exists\ B_{\varepsilon_x}(x)$



such that for only finitely many n ~~many~~ x_n lies in $B_{\varepsilon_x}(x)$. I.e., if not, then each $B_{\frac{1}{k}}(x)$ contains some x_{n_k} $\Rightarrow$ $x_{n_k} \to x$.

Thus $\bigcup_{x \in E} B_{\varepsilon_x}(x) \supseteq E$, and hence admits

a finite subcover $B_{\varepsilon_1}(\bar{x}_1) \cup \cdots \cup B_{\varepsilon_N}(\bar{x}_N) \supseteq E$.

But then one of $B_{\varepsilon_k}(\bar{x}_k)$ contains x_n for ∞ many n, ※.

→ <u>Note</u>: In general, eg $C^0[a,b]$, closed bounded sets are not compact!

- Continuity: Let $f: S \to S^*$

<u>Theorem</u>: f is continuous iff the inverse image of open sets are open. I.e. $\forall\, O \subseteq S$, $f^{-1}(O) = \{x \in S : f(x) \in O\}$ is open.

Pf. ($\Rightarrow$) Assume f cont & let $O \subseteq S$ be open. We prove $f^{-1}(O)$ open. If $f^{-1}(O) = \emptyset$ then done, because $\emptyset$ & S are always assumed open. If $f^{-1}(O) \neq \emptyset$ then $\exists\ x \in S$ st $f(x) = y \in O$. But ~~~~

O open $\Rightarrow \exists\ \varepsilon$ st $B_\varepsilon(y) \subseteq O$. But f cont $\Rightarrow \exists$ δ st if $x \in B_\delta(x)$, then $f(x) \subseteq B_\varepsilon(y) \subseteq O$.

Thus $\forall\ x \in f^{-1}(O)$, $B_\delta(x) \subseteq f^{-1}(O)$ & $f^{-1}(O)$ is open ✓

($\Leftarrow$) Assume $f^{-1}(O)$ open $\forall\, O \subseteq S'$ open. We prove f cont. @ each $x \in S$. But $f(x) = y$ means $\forall\, \varepsilon > 0$ we must find δ st $x \in B_\delta(x)$ implies $y \in B_\varepsilon(y)$. But $f^{-1}(B_\varepsilon(y))$ open $\Rightarrow \exists$ δ st $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(y))$, so $f(B_\delta(x)) \subseteq B_\varepsilon(y)$ as needed for cont ✓

All in a metric space

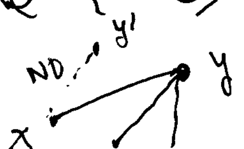
Theorems about cont fn's on compact sets:

<u>Thm</u> ① A cont fn maps compact sets to compact sets

<u>Thm</u> ② A cont fn on a compact set is uniformly continuous

<u>Thm</u> ③ A real valued fn on a compact set is bounded, & takes on its max/min values

<u>Pf. of Thm ①</u>: Assume E compact & $f: S \to S^*$ we prove $f(E)$ compact. Let $\bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$ be a covering of $f(E)$ by open sets. Then $\bigcup_{\lambda\in\Lambda} f^{-1}(\mathcal{O}_\lambda)$ covers E. Since $f^{-1}(\mathcal{O}_\lambda)$ open $\exists$ finite subcov $f^{-1}(\mathcal{O}_1)\cup\cdots\cup f^{-1}(\mathcal{O}_N) \supseteq E$. Thus $\mathcal{O}_1\cup\cdots\cup\mathcal{O}_N$ covers $f(E)$. [$y \in f(E) \Rightarrow y=f(x)$ $x\in E \Rightarrow$ $x \in f^{-1}(\mathcal{O}_i)$ some i, $\Rightarrow y\in\mathcal{O}_i$ because <u>only</u> <u>y</u> has preimage x



Pf. of Thm ②: Assume f cont on E compact & $\varepsilon > 0$. We find δ st $d(s,t) < \delta \Rightarrow d^*(f(s), f(t)) < \varepsilon$. $(s,t \in E)$

- $\forall s \in E \ \exists \ \delta_s$ st $t \in B_{\delta_s}(s) \Rightarrow d^*(f(s), f(t)) < \frac{\varepsilon}{2}$
- E compact $\Rightarrow B_{\delta_{s_1}/2}(s_1) \cup \cdots \cup B_{\delta_{s_n}/2}(s_n) \supseteq E$
- Idea: choose δ small enough so that $d(s,t) < \delta$ places **both** s & t in one of these balls.

Then $d(s,t) \leq d(s, s_i) + d(s_i, t) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ✓

How small? Let $\delta = \frac{1}{2} \operatorname{Min}\{\delta_{s_i}\}$. Then

$s \in B_{\delta_{s_i}/2}(s_i) \quad d(s,t) < \delta_{s_i}/2$

$\therefore d(s_i, t) \leq d(s_i, s) + d(s, t) \leq \frac{\delta_{s_i}}{2} + \frac{\delta_{s_i}}{2} = \delta_{s_i}$

$\Rightarrow s, t$ both in $B_{\delta_{s_i}}(s_i)$ ✓