1. (pts) Let \( f(x) = \sin \frac{1}{x} \). Use the definition of the limit to prove that \( \lim_{x \to 0} f(x) \) does not exist.

Consider sequence \( \{x_n\} \to 0 \) with \( x_n = \frac{1}{2\pi n} \), so \( f(x_n) = \sin(2\pi n) = 0 \forall n \).

Then \( \lim_{n \to 0} f(x_n) = 0 \).

Consider another sequence \( \{x_k\} \to 0 \) with \( x_k = \frac{1}{2\pi k + \frac{\pi}{2}} \).

So \( f(x_k) = \sin \left( 2\pi k + \frac{\pi}{2} \right) = 1 \forall k \).

Then \( \lim_{k \to 0} f(x_k) = 1 \).

Since \( \{x_n\}, \{x_k\} \subseteq (-3,0) \cup (0,2) \) where \( \{x_n\} \to 0 \) and \( \{x_k\} \to 0 \),

\[ \lim_{k \to 0} f(x_k) \neq \lim_{n \to 0} f(x_n), \quad x \to 0 f(x) \text{ D.N.E. by definition.} \]

2. (pts) Find the interval of convergence for the following power series.

\[ \sum_{n=0}^{\infty} \frac{2^n}{n5^{n+1}}x^n \]

\[ B = \lim_{n \to \infty} \sup \left| a_n \right|^{\frac{1}{n}} = \lim_{n \to \infty} \sup \left| \frac{2^n}{n5^{n+1}} \right|^{\frac{1}{n}} = \lim_{n \to \infty} \frac{2}{n^5} \frac{1}{5^{n+1}} = \frac{2}{5}. \]

Thus, radius of convergence \( R = \frac{1}{B} = \frac{5}{2} \).

Check endpoints:
- Let \( x = \frac{5}{2} \Rightarrow \sum \frac{2^n}{n5^{n+1}} \left( \frac{5}{2} \right)^n = \sum \frac{1}{5} \frac{1}{n} = \frac{1}{5} \sum \frac{1}{n} \) which is a divergent \( p \)-series.
- Let \( x = -\frac{5}{2} \Rightarrow \sum \frac{2^n}{n5^{n+1}} \left( -\frac{5}{2} \right)^n = \sum \frac{1}{5} \frac{1}{n} = \frac{1}{5} \sum \frac{1}{n} \).

Let \( a_n = \frac{1}{n} \) which is \( \geq 1 \), and \( \lim_{n \to \infty} a_n = 0 \).

By Alternating Series Test, this is a convergent series.

Therefore, the interval of convergence is \( \left[ -\frac{5}{2}, \frac{5}{2} \right] \).
3. (pts) Let the sequence of functions \( \{f_n\} \) be \( f_n(x) = x - x^n \) for \( x \in [0, 1] \).

(a) Find \( f(x) \) such that \( \{f_n\} \to f \) on \([0, 1]\).

\[
\text{For } x = 1, \quad f_n(1) = 0 \quad \forall n \implies \lim_{n \to \infty} f_n(1) = 0
\]

\[
\text{For } x \in [0, 1), \quad \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x - x^n = x
\]

Define \( f(x) = \begin{cases} x & x \in [0, 1) \quad \text{where } \{f_n\} \to f \text{ on } [0, 1] \\ 0 & x = 1 \end{cases} \)

(b) Using the definition, prove \( \{f_n\} \) does not converge uniformly to \( f \) (found in part a) on \([0, 1]\).

Choose \( \varepsilon = \frac{1}{2} \). Let \( N \) be given and let \( N^* = \lceil N \rceil \in \mathbb{N} \).

Choose \( n = N^* + 1 \). For \( x \neq 1 \),

\[
|f_n(x) - f(x)| = |x - x^n - x| = x^n \geq \frac{1}{2} \quad (\Rightarrow) \quad x \geq \sqrt[2]{\frac{1}{2}}.
\]

Choose \( x = \sqrt[2]{\frac{1}{2}} \) where we chose \( n = \lceil N \rceil + 1 > N \), and we have \( |f_n(x) - f(x)| \geq \frac{1}{2} = \varepsilon \).

Thus, \( \{f_n\} \) does not converge uniformly to \( f \) on \([0, 1]\).
4. (pts) Let the sequence of functions \( \{f_n\} \) be \( f_n(x) = \frac{1}{1 + nx} \) for \( x \in [2, \infty) \). Let \( f(x) = 0 \) for \( x \in [2, \infty) \). Using the definition, prove \( \{f_n\} \) converges uniformly to \( f \) on \( x \in [2, \infty) \).

Let \( \varepsilon > 0 \) be given.

Notice \( |f_n(x) - f(x)| = \left| \frac{1}{1 + nx} - 0 \right| = \frac{1}{1 + nx} < \frac{1}{1 + 2n} \)

since \( x \geq 2 \).

Then, \( |f_n(x) - f(x)| < \varepsilon \iff \frac{1}{1 + 2n} < \varepsilon \iff n > \frac{1}{2}(\frac{1}{\varepsilon} - 1) \).

Choose \( N = \frac{1}{2}(\frac{1}{\varepsilon} - 1) \). Then, we have

\( \forall x \in [2, \infty), \forall n > N \implies |f_n(x) - f(x)| < \varepsilon. \)

Therefore, \( \{f_n\} \Rightarrow f \) on \( [2, \infty) \).
5. (pts) For \( x \in [0, 1] \), we have the following power series

\[
\sqrt{1 + x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)} x^n.
\]

Use this fact to build a power series for \( \frac{1}{\sqrt{1-x^2}} \).

Start with

\[
\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)} x^n
\]

Differentiating both sides, we get

\[
\frac{1}{2\sqrt{1+x}} = \sum_{n=1}^{\infty} \frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)} x^{n-1}
\]

Substituting \( -x^2 \) for \( x \) & multiplying by 2, we have

\[
\frac{1}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)} (-x^2)^{n-1}
\]

6. (pts) Prove the following series converges uniformly on \( \mathbb{R} \) to a continuous function

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx
\]

Consider the sequence \( \{a_n\} \) where \( a_n = \frac{1}{n^2} \). All the terms are nonnegative and \( \sum \frac{1}{n^2} < \infty \) since it's a convergent p-series. Notice

\[
\left| \frac{1}{n^2} \cos nx \right| \leq \frac{1}{n^2} \quad \forall x \in \mathbb{R}
\]

Thus, the series \( \sum \frac{1}{n^2} \cos nx \) converges uniformly on \( \mathbb{R} \). Also, the limit is continuous since each partial sum is continuous (addition preserves continuity).
7. (pts) Use the definition of the derivative to prove the Quotient Rule.

Let \( f, g \) be differentiable at \( a \) where \( g(a) \neq 0 \).

Since \( g(a) \neq 0 \) and \( g \) is continuous at \( a \) (from differentiability of \( g \)), there exist an open interval \( I \) with \( a \in I \) such that \( g(x) \neq 0 \) \( \forall x \in I \). For \( x \in I \), we have

\[
(f/g)'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x)g(a) - f(a)g(x)}{(x-a) \cdot g(x)g(a)}
\]

\[
= \lim_{x \to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{(x-a) \cdot g(x)g(a)}
\]

\[
= \lim_{x \to a} \left[ g(a) \frac{f(x) - f(a)}{x-a} - f(a) \cdot \frac{g(x) - g(a)}{x-a} \right] \frac{1}{g(x)g(a)} = \frac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}
\]

Therefore, \( f/g \) is differentiable at \( a \).

8. (pts) Use the definition of the derivative to show \( f(x) = |x| + |x+1| \) is not differentiable at \( x = -1 \).

Notice by definition of \( |x| \),

\( f(x) = \begin{cases} 
-2x-1 & x \leq -1 \\
1 & -1 < x \leq 0 \\
2x+1 & x > 0 
\end{cases} \).

So

\[
\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x + 1} = \lim_{x \to -1^-} \frac{-2x-1-1}{x + 1} = \lim_{x \to -1^-} \frac{-2(x+1)}{x+1} = -2
\]

and

\[
\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x + 1} = \lim_{x \to -1^+} \frac{1-1}{x + 1} = 0
\]

Since

\[
\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x + 1} \neq \lim_{x \to -1^-} \frac{f(x) - f(-1)}{x + 1}
\]

\( f'(-1) := \lim_{x \to -1} \frac{f(x) - f(-1)}{x + 1} \) does not exist.

Thus, \( f \) is not differentiable at \( x = -1 \).
9. (pts) Let the sequence of functions \( \{f_n\} \) be \( f_n(x) = \frac{nx}{1+n^2x^2} \) for \( x \in [0, 1] \). Let \( f(x) = 0 \) for \( x \in [0, 1] \). Prove \( \{f_n\} \) does not converge uniformly to \( f \) on \( x \in [0, 1] \).

For fixed \( n \in \mathbb{N} \), consider
\[
\sup \left\{ |f_n(x) - f(x)| : x \in [0, 1] \right\} = \sup \left\{ \frac{nx}{1+n^2x^2} : x \in [0, 1] \right\}
\]

We need to find max value for \( g(x) = \frac{nx}{1+n^2x^2} \),

\[
\frac{d}{dx} g(x) = \frac{n(1+n^2x^2)-nx(2n^2x)}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = 0
\]

\[\Rightarrow x = \pm \frac{1}{n}\]

\[
\begin{array}{c|c|c}
& 0 & 0 \\
\hline
x = -\frac{1}{n} & + & 0\
x = \frac{1}{n} & - & \rightarrow
\end{array}
\]

Thus, max occurs at \( x = \frac{1}{n} \in [0, 1] \) and we have
\[
\sup \left\{ |f_n(x) - f(x)| : x \in [0, 1] \right\} = \frac{n}{1+n^2\left(\frac{1}{n}\right)^2} = \frac{1}{2}.
\]

So \( \lim_{n \to \infty} \sup \left\{ |f_n(x) - f(x)| : x \in [0, 1] \right\} = \frac{1}{2} \neq 0 \).

Therefore, \( \{f_n\} \) does not converge uniformly to \( f \) on \( [0, 1] \).