Final Exam  
Saturday March 22, 8-10am  
MAT 125A, Temple, Spring 2014

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Your Score</th>
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Problem #1 (20pts): Definitions:

(a) State the definition of the limit of a sequence of real numbers, \( x_n \to x_0 \).

\[
\lim_{n \to \infty} x_n = x_0 \text{ if } \forall \varepsilon > 0 \exists N \forall n > N \ |x_n - x_0| < \varepsilon
\]

(b) State the definition of a Cauchy sequence of real numbers \( x_n \).

\[ x_n \text{ Cauchy if: } \forall \varepsilon > 0 \exists N \forall n, m > N \ |x_n - x_m| < \varepsilon \]

(c) State the \( \varepsilon-\delta \) definition for a function \( f : \mathbb{R} \to \mathbb{R} \) to be continuous at \( x_0 \).

\[ f \text{ cont @ } x_0 \text{ if: } \forall \varepsilon > 0 \exists \delta > 0 \forall x \ |f(x) - f(x_0)| < \varepsilon \]

\[ |x - x_0| < \delta \]

(d) State the \( \varepsilon-\delta \) definition for a function to be uniformly continuous on a set \( S \subset \mathbb{R} \).

\[ f \text{ unif cont on } S \text{ if: } \forall \varepsilon > 0 \exists \delta > 0 \forall x, y \ |f(x) - f(y)| < \varepsilon \]

\[ |x - y| < \delta \]

(e) State the \( \varepsilon-\delta \) definition of a uniformly Cauchy sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \).

\[ f_n \text{ unif Cauchy if: } \forall \varepsilon > 0 \exists N \forall m, n > N \ |f_n(x) - f_m(x)| < \varepsilon \]

\[ x \in \mathbb{R} \]
(f) State the definition for a sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \) to converge pointwise on a set \( S \subset \mathbb{R} \).

\[
f_n \to f \text{ pointwise if: } \forall x \in \mathbb{R}, \quad f_n(x) \to f(x)
\]

(g) State the definition for a sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \) to converge uniformly to a function \( f \) on a set \( S \subset \mathbb{R} \).

\[
f_n \to f \text{ uniformly on } S \text{ if: } \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \forall x \in S, \quad |f_n(x) - f(x)| < \epsilon
\]

(h) State what it means for a sequence of functions \( f_n : \mathbb{R} \to \mathbb{R} \) to NOT converge uniformly to a function \( f \) on a set \( S \subset \mathbb{R} \).

\[
f_n \not\to f \text{ uniformly if: } \exists \epsilon > 0, \forall N \in \mathbb{N}, \exists x \in S \text{ s.t. } \forall n \geq N, \quad |f_n(x) - f(x)| \geq \epsilon
\]

(i) State the Bolzano-Weierstrass Theorem in \( \mathbb{R} \), and state the most general sets to which this theorem applies.

\[
\text{BWTm: If } x_n \text{ lies in a closed and bounded set } E \text{ in } \mathbb{R}, \text{ then } \exists \text{ a convergent subseq } x_{n_k} \to x_0 \text{ and } x_0 \in E.
\]
Problem #2 (30pts):

(a) Assume \( f'(x) \) and \( g'(x) \) exist at \( x = x_0 \). Prove that \( \frac{d}{dx} f(x) g(x) = f'(x) g(x) + f(x) g'(x) \) at \( x = x_0 \).

\[
\frac{d}{dx} (f \cdot g) (x) = \lim_{x \to x_0} \frac{f(x) g(x) - f(x_0) g(x_0)}{x - x_0}
\]

\[
= \lim_{x \to x_0} \frac{f(x) g(x) - f(x_0) g(x) + f(x_0) g(x) - f(x_0) g(x_0)}{x - x_0}
\]

\[
= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x) + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} f(x_0)
\]

\[
= f'(x_0) g(x_0) + f(x_0) g'(x_0)
\]
(b) Prove that if a real valued function is differentiable at a point \( x_0 \), then it is continuous at \( x_0 \).

Assume \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L \) exists.

Fix \( \varepsilon > 0 \). We find \( \delta \) such that \( |x - x_0| < \delta \) implies \( |f(x) - f(x_0)| < \varepsilon \).

But \( f'(x_0) \) exists implies \( \exists \delta_1 > 0 \) s.t. \( |x - x_1| < \delta_1 \) implies \( \left| \frac{f(x) - f(x_0)}{x - x_0} \right| < M \) is bounded.

Thus \( |f(x) - f(x_0)| < M |x - x_0| < \varepsilon \) if we choose \( |x - x_0| < \frac{\varepsilon}{M} \) and \( |x - x_0| < \delta_1 \).

Setting \( \delta = \min \left( \frac{\varepsilon}{M}, \delta_1 \right) \) we conclude that

\( |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \).
(c) Give an example of a function \( f : (-1, 1) \to [-1, 1] \) such that \( f \) is differentiable on 
\((-1, 1)\), but \( f' \) is discontinuous at \( x = 0 \). Justify your claims.

\[
f(x) = \begin{cases} 
  x^2 \sin \frac{1}{x} & x \neq 0 \\
  0 & x = 0.
\end{cases}
\]

Since \( \lim_{{x \to 0}} x^2 \sin \frac{1}{x} = 0 \), \( f \) is
continous on \((-1, 1)\) and takes values
in \([-1, 1]\).

\[
\frac{d}{dx} x^2 \sin \frac{1}{x} = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad x \neq 0,
\]
and this has no limit as \( x \to 0 \) because \( \cos \frac{1}{x} \) diverges.

It's enough to say \( f \) is differentiable at \( x = 0 \) because,
being trapped between \( y = x^2 \) and \( y = -x^2 \), \( f'(x) = 0 \) at \( x = 0 \).
Problem #3 (25pts): Assume that $f_n(x) \to f(x)$ for each $x \in [0, 1]$, and assume that each $f_n$ is continuous on $[0, 1]$. Give a careful proof that if $f_n \to f$ uniformly, then $f$ is continuous at each $x_0$.

We give the $\varepsilon/3$ proof:

To prove $f$ cont @ $x_0$: Fix $\varepsilon > 0$. We find $\delta$

$s.t $ $ |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

- Choose $N$ s.t. $n > N \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3} \forall x \in [0, 1]$ Fix such an $n > N$.

- Choose $\delta$ s.t. $ |x - x_0| < \delta \implies |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$

- Then: $ |x - x_0| < \delta \implies$

$|f(x) - f(x_0)| = |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$ as claimed
Problem #4 (20pts): (a) Assume \( f : [0, 1] \to [0, 1] \) is continuous. Prove that \( f \) must have a fixed point \( x \) where \( f(x) = x \). (You may use any of the theorems we proved.)

Let \( g(x) = f(x) - x \) (out). Then \( g(0) = f(0) \geq 0 \) and \( g(1) = f(1) - 1 \leq 0 \). By the EVT \( \exists x \) s.t. \( g(x) = 0 \implies f(x) = x \).

(b) Prove that between any two roots of a polynomial \( p(x) \) there exists a root of \( p'(x) \).

If \( p(x) = 0 = p(y) \), then by MVT/Rolle's Thm.

\[ \exists x^* \in [x, y] \text{ s.t. } p'(x^*) = 0. \]
Problem #5 (25pts): Assume \( \sum_{k=0}^{\infty} |a_k| < \infty \). Use this to prove that the sequence of partial sums \( S_n(x) = \sum_{k=0}^{n} a_k x^k \) is a uniformly convergent sequence of functions on \([-1, 1]\).

(Hint: You may use that a uniformly Cauchy sequence of functions is uniformly convergent.)

We prove \( S_n(x) \) uniformly Cauchy. We need:

\[ \forall \epsilon > 0 \exists N \ni \forall m,n > N \quad |S_n(x) - S_m(x)| < \epsilon, \quad x \in [-1,1] \]

But

\[ |S_n(x) - S_m(x)| \leq \sum_{k=m+1}^{\infty} |a_k| |x|^k \leq \sum_{k=m}^{\infty} |a_k| \]

Since \( \sum_{k=0}^{\infty} |a_k| < \infty \), \( \lim_{n \to \infty} \sum_{k=m}^{\infty} |a_k| = 0 \). So

choose \( N \) s.t. \( \sum_{k=n}^{\infty} |a_k| < \epsilon \) for all \( n > N \).

Then \( m,n > N \Rightarrow |S_n(x) - S_m(x)| < \epsilon \). √
Problem #6 (25pts): Let \((S, d)\) be a metric space.

(a) State the three conditions on \(d\) for it to be a metric.

\[
\begin{align*}
d(x, y) & \geq 0 \quad \text{and} \quad d(x, y) = 0 \quad \text{iff} \quad x = y \\
d(x, y) & \leq d(y, z) + d(z, x) \\
d(x, z) & \leq d(x, y) + d(y, z)
\end{align*}
\]

(b) Give the definition of Cauchy sequence \(x_n \in S\).

\(x_n\) is Cauchy if: \(\forall \varepsilon \in \mathbb{N} \exists N \in \mathbb{N} : \forall m, n > N \quad d(x_n, x_m) < \varepsilon\)

(c) State the additional condition required of a metric space to make it complete.

Every Cauchy sequence has a limit in \(S\).

(d) Define an open set in \((S, d)\).

\(\emptyset\) is open if \(\forall x \in \emptyset, \exists \varepsilon > 0 \quad \beta_\varepsilon(x) \subseteq \emptyset\)

(e) Define a closed set in \((S, d)\).

\(E\) is closed if \(E = \emptyset^c \) for some \(\emptyset\) open
**Problem #7 (25pts):** Let \( (S, d) \) be a metric space.

(a) Give the \( \epsilon \)-\( \delta \) definition for a function \( f : S \to S \) to be continuous at a point \( x_0 \in S \).

\[
\forall \epsilon > 0 \exists \delta > 0 \forall x \left( d\left(f(x), f(x_0)\right) < \epsilon \right) \delta(x, x_0) < \delta
\]

(b) Prove that if \( f \) satisfies the condition that the pre-image of open sets are open, then \( f \) is continuous at each \( x_0 \in S \) in the \( \epsilon \)-\( \delta \) sense.

Assume \( f^{-1}(\emptyset) \) open \( \forall \emptyset \) open in \( S \).

Fix \( \epsilon > 0 \). We find \( \delta > 0 \) such that \( d(x, x_0) < \delta \) implies \( d\left(f(x), f(x_0)\right) < \epsilon \). But \( B_{\epsilon}(f(x_0)) \) open \( \Rightarrow f^{-1}\left(B_{\epsilon}(f(x_0))\right) \) open \( \Rightarrow \exists \delta > 0 \) such that \( B_{\delta}(x_0) \subseteq f^{-1}\left(B_{\epsilon}(f(x_0))\right) \). Clearly \( f(B_{\delta}(x_0)) \subseteq B_{\epsilon}(f(x_0)) \) which means \( d(x, x_0) < \delta \Rightarrow d\left(f(x), f(x_0)\right) < \epsilon \) as claimed.
Problem #8 (25pts): Let $(S, d)$ be a metric space. Recall that a compact set $E \subset S$ is one such that every open covering admits a finite subcover. Prove that if $E \subset S$ is compact, then $E$ satisfies the Bolzano-Weierstrass property: Every subsequence $x_n$ in $E$ contains a convergent subsequence $x_{n_k} \to x_0$, where $x_0 \in E$.

Assume $E$ compact and $\{x_n\} \subseteq E$. Assume for contradiction that $x_n$ has no convergent subseq.

Then $\forall x \in E \exists \delta_x > 0 \forall x_n \in B_{\delta_x}(x)$ for only finitely many $n$. But $\bigcup_{x \in E} B_{\delta_x}(x)$ covers $E \Rightarrow x \in E$.

$\exists$ finite subcover $B_{\delta_1}(x_1) \cup \ldots \cup B_{\delta_n}(x_n)$.

Since $x_n \in E \forall n$, an $\infty$-number of them must lie in one of $B_{\delta_1}(x_i)$, i.e., $x_n \in B_{\delta_i}(x_i)$ for $\infty$-many $n$. [low, $\exists$ finitely # of $n \not{\in}$]

This contradicts $B_{\delta_i}(x_i)$ contains $x_n$ for only finitely many $n$. √