Name: Solutions

Student ID#: ____________

Section: ____________

Final Exam

Saturday March 22, 8-10am

MAT 125A, Temple, ~~Spring~~ Winter 2014

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (20pts): Definitions:

(a) State the definition of the *limit* of a sequence of real numbers, $x_n \to x_0$.

$\lim_{n\to\infty} x_n = x_0$ if $\forall \varepsilon\ \exists N$ st $\forall n > N$ $|x_n - x_0| < \varepsilon$

(b) State the definition of a *Cauchy sequence* of real numbers x_n.

x_n Cauchy if: $\forall \varepsilon\ \exists N$ st $\forall_{n,m>N}$ $|x_n - x_m| < \varepsilon$

(c) State the ϵ-δ definition for a function $f : \mathcal{R} \to \mathcal{R}$ to be *continuous* at x_0.

f cont @ x_0 if: $\forall \varepsilon\ \exists \delta$ st $\forall_{\substack{x \\ |x-x_0|<\delta}}$ $|f(x) - f(x_0)| < \varepsilon$

(d) State the ϵ-δ definition for a function to be *uniformly continuous* on a set $S \subset \mathcal{R}$.

f unif cont on S if: $\forall \varepsilon\ \exists \delta$ st $\forall_{\substack{x,y \\ |x-y|<\delta}}$ $|f(x) - f(y)| < \varepsilon$

(e) State the ϵ-δ definition of a *uniformly Cauchy* sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$.

f_n unif Cauchy if: $\forall \varepsilon\ \exists N$ st $\forall_{\substack{m,n>N \\ x \in \mathbb{R}}}$ $|f_n(x) - f_m(x)| < \varepsilon$

(f) State the definition for a sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$ to converge *pointwise* on a set $S \subset \mathcal{R}$.

$f_n \to f$ pointwise if: $\forall x \in \mathbb{R}$, $f_n(x) \to f(x)$

(g) State the definition for a sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$ to converge *uniformly* to a function f on a set $S \subset \mathcal{R}$.

$f_n \to f$ uniformly on S if: $\forall \varepsilon\ \exists N$ st $\forall_{\substack{n>N \\ x \in S}} |f_n(x) - f(x)| < \varepsilon$

(h) State what it means for a sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$ to NOT converge *uniformly* to a function f on a set $S \subset \mathcal{R}$.

$f_n \not\to f$ unf if: $\exists \varepsilon$ st $\forall N\ \exists_{\substack{n>N \\ x \in S}} |f_n(x) - f(x)| \geq \varepsilon$

(i) State the Bolzano-Weierstrass Theorem in $\mathcal{R}$, and state the most general sets to which this theorem applies.

BW Thm: If x_n lies in a closed & bounded set E in $\mathbb{R}$, then $\exists$ a convergent subseq $x_{n_k} \to x_0$ and $x_0 \in E$.

Problem #2 (30pts):
(**a**) Assume $f'(x)$ and $g'(x)$ exist at $x = x_0$. Prove that $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$ at $x = x_0$.

$$\frac{d}{dx}(f \cdot g)(x) = \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x) + \lim_{x \to x_0} f(x_0) \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

(**b**) Prove that if a real valued function is differentiable at a point x_0, then it is continuous at x_0.

Assume $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = L$ exists.

Fix $\varepsilon > 0$. We find δ such that $|x-x_0|<\delta$ implies $|f(x)-f(x_0)|<\varepsilon$.

But $f'(x_0)$ exists implies $\exists\ \delta_1$ st $|x-x_1|<\delta_1$ implies $\left|\frac{f(x)-f(x_0)}{x-x_0}\right| < M$ is bounded.

Thus $|f(x)-f(x_0)| < M|x-x_0| < \varepsilon$

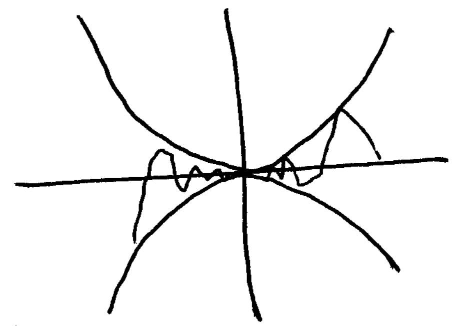
if we choose $|x-x_0| < \frac{\varepsilon}{M}$ and $|x-x_0|<\delta_1$.

Setting $\delta = \mathrm{Min}\left(\frac{\varepsilon}{M}, \delta_1\right)$ we conclude that

$$|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon.$$

(c) Give an example of a function $f : (-1, 1) \to [-1, 1]$ such that f is differentiable on $(-1, 1)$, but f' is discontinuous at $x = 0$. Justify your claims.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$



Since $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$, f is cont on $(-1,1)$ & takes values in $[-1,1]$. $\frac{d}{dx} x^2 \sin\frac{1}{x} = 2x \sin\frac{1}{x} - \cos\frac{1}{x}$ $x \neq 0$, and this has no limit as $x \to 0$ because $\cos\frac{1}{x}$ discont @ $x=0$.

Its enough to say f is diff @ $x=0$ because, being trapped betw $y=x^2$ & $y=-x^2$, $f'(x)=0$ @ $x=0$.

Problem #3 (25pts): Assume that $f_n(x) \to f(x)$ for each $x \in [0,1]$, and assume that each f_n is continuous on $[0,1]$. Give a careful proof that if $f_n \to f$ uniformly, then f is continuous at each x_0.

We give the $\varepsilon/3$ proof:

To prove f cont @ x_0: Fix $\varepsilon > 0$. We find δ st $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

- Choose N st $n > N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3}$ $\forall x \in [0,1]$
  Fix such an $n > N$.

- Choose δ st $|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$

- Then: $|x - x_0| < \delta \Rightarrow$

$$|f(x) - f(x_0)| = |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{as claimed}$$

Problem #4 (20pts): (a) Assume $f : [0,1] \to [0,1]$ is continuous. Prove that f must have a fixed point x where $f(x) = x$. (You may use any of the theorems we proved.)

Let $g(x) = f(x) - x$ cont. Then $g(0) = f(0) \geq 0$

and $g(1) = f(x) - 1 \leq 0$. By the IVT $\exists x$ st

$g(x) = 0 \Rightarrow f(x) = x$.

(b) Prove that between any two roots of a polynomial $p(x)$ there exists a root of $p'(x)$.

If $P(x) = 0 = P(y)$, then by MVT/Rolles Thm

$\exists\ x^* \in [x, y]$ st $P'(x^*) = 0$.

Problem #5 (25pts): Assume $\sum_{k=0}^{\infty} |a_k| < \infty$. Use this to prove that the sequence of partial sums $S_n(x) = \sum_{k=0}^{n} a_k x^k$ is a uniformly convergent sequence of functions on $[-1, 1]$. (Hint: You may use that a uniformly Cauchy sequence of functions is uniformly convergent.)

We prove $S_n(x)$ uniformly Cauchy. We need:

$\forall \varepsilon \; \exists N$ st $\forall_{\substack{m,n>N \\ x\in[-1,1]}} \; |S_n(x) - S_m(x)| < \varepsilon$.

But $|S_n(x) - S_m(x)| \leq \sum_{k=m}^{\infty} |a_k| |x|^k \leq \sum_{k=m}^{\infty} |a_k|$

Since $\sum_{k=0}^{\infty} |a_k| < \infty$, $\lim_{n\to\infty} \sum_{k=m}^{\infty} |a_k| = 0$. So

choose N st $\sum_{k=n}^{\infty} |a_k| < \varepsilon$ for all $n > N$.

Then $m, n > N \Rightarrow |S_n(x) - S_m(x)| < \varepsilon$. ✓

Problem #6 (25pts): Let $(\mathcal{S}, d)$ be a metric space.

(a) State the three conditions on d for it to be a metric.

$d(x,y) \geq 0$ & $d(x,y) = 0$ iff $x = y$

$d(x,y) = d(y,x)$

$d(x,z) \leq d(x,y) + d(y,z)$

(b) Give the definition of *Cauchy sequence* $x_n \in \mathcal{S}$.

x_n ~~Ca~~ Cauchy if: $\forall \varepsilon\ \exists N$ st $\forall m,n > N$ $d(x_n, x_m) < \varepsilon$

(c) State the additional condition required of a metric space to make it *complete*.

Every Cauchy sequ has a limit in $\mathcal{S}$.

(d) Define an *open set* in $(\mathcal{S}, d)$.

$\mathcal{O}$ open if $\forall x \in \mathcal{O}$, $\exists \varepsilon$ st $B_\varepsilon(x) \subseteq \mathcal{O}$

(e) Define a *closed set* in $(\mathcal{S}, d)$.

E closed if $E = \mathcal{O}^c$ for some $\mathcal{O}$ open

Problem #7 (25pts): Let $(\mathcal{S}, d)$ be a metric space.
(**a**) Give the ϵ-δ definition for a function $f : \mathcal{S} \to \mathcal{S}$ to be continuous at a point $x_0 \in \mathcal{S}$.

$\forall \varepsilon \ \exists \delta$ st $\underset{d(x,x_0)<\delta}{\forall x}$ $d(f(x), f(x_0)) < \varepsilon$

(**b**) Prove that if f satisfies the condition that the pre-image of open sets are open, then f is continuous at each $x_0 \in \mathcal{S}$ in the ϵ-δ sense.

Assume $f^{-1}(\mathcal{O})$ open $\forall$ $\mathcal{O}_{open}$ in $\mathcal{S}$.
Fix $\varepsilon > 0$. We find δ st $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \varepsilon$. But $B_\varepsilon(f(x_0))$ open $\Rightarrow$ $f^{-1}(B_\varepsilon(f(x_0)))$ open $\Rightarrow \exists \delta$ st $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$. Cleary $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$ which means $d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \varepsilon$ as claimed.

Problem #8 (25pts): Let $(\mathcal{S}, d)$ be a metric space. Recall that a compact set $E \subset \mathcal{S}$ is one such that every open covering admits a finite subcover. Prove that if $E \subset S$ is compact, then E satisfies the Bolzano-Weierstrass property: Every subsequence x_n in E contains a convergent subsequence $x_{n_k} \to x_0$, where $x_0 \in E$.

Assume E compact and $\{x_n\} \in E$. Assume for contradiction that x_n has no convergent subsequ.

Then $\forall x \in E$ $\exists B_{\delta_x}(x)$ s.t. $x_n \in B_{\delta_x}(x)$ for only finitely many n. But $\bigcup_{x \in E} B_{\delta_x}(x)$ covers $E \Rightarrow$ $\exists$ finite subcover $B_{\delta_1}(x_1) \cup \cdots \cup B_{\delta_n}(x_n)$.

Since $x_n \in E$ $\forall n$, an ∞-number of them must lie in one of $B_{\delta_i}(x_i)$ i.e., $x_n \in B_i(x_i)$ for ∞-many n. [o.w., $\exists$ finite # of n !].

This contradicts $B_{\delta_i}(x_i)$ contains x_n for only finitely many n. ✓