Name: ______________________

Student ID#: ______________________

Section: ______________________

Final Exam

Saturday March 22, 8-10am

MAT 125A, Temple, Winter 2014

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1 (20pts): Definitions:

(a) State the definition of the *limit* of a sequence of real numbers, $x_n \to x_0$.

(b) State the definition of a *Cauchy sequence* of real numbers x_n.

(c) State the ϵ-δ definition for a function $f : \mathcal{R} \to \mathcal{R}$ to be *continuous* at x_0.

(d) State the ϵ-δ definition for a function to be *uniformly continuous* on a set $S \subset \mathcal{R}$.

(e) State the ϵ-δ definition of a *uniformly Cauchy* sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$.

(f) State the definition for a sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$ to converge *pointwise* on a set $S \subset \mathcal{R}$.

(g) State the definition for a sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$ to converge *uniformly* to a function f on a set $S \subset \mathcal{R}$.

(h) State what it means for a sequence of functions $f_n : \mathcal{R} \to \mathcal{R}$ to NOT converge *uniformly* to a function f on a set $S \subset \mathcal{R}$.

(i) State the Bolzano-Weierstrass Theorem in $\mathcal{R}$, and state the most general sets to which this theorem applies.

Problem #2 (25pts):
(**a**) Assume $f'(x)$ and $g'(x)$ exist at $x = x_0$. Prove that $\frac{d}{dx}(fg)(x) = f'(x)g(x) + f(x)g'(x)$ at $x = x_0$.

(**b**) Prove that if a real valued function f is differentiable at a point x_0, then it is continuous at x_0.

(c) Give an example of a function $f : (-1,1) \to [-1,1]$ such that f is differentiable on $(-1,1)$, but f' is discontinuous at $x = 0$. Justify your claims.

Problem #3 (25pts): Assume that $f_n(x) \to f(x)$ for each $x \in [0, 1]$, and assume that each f_n is continuous on $[0, 1]$. Give a careful proof that if $f_n \to f$ uniformly, then f is continuous at each x_0.

Problem #4 (25pts): (a) Assume $f : [0, 1] \to [0, 1]$ is continuous. Prove that f must have a fixed point x where $f(x) = x$. (You may use any of the theorems we proved.)

(b) Prove that between any two roots of a polynomial $p(x)$ there exists a root of $p'(x)$.

Problem #5 (25pts): Assume $\sum_{k=0}^{\infty} |a_k| < \infty$. Use this to prove that the sequence of partial sums $S_n(x) = \sum_{k=0}^{n} a_k x^k$ is a uniformly Cauchy sequence of functions for $x \in [-1, 1]$.

Problem #6 (25pts): Let $(\mathcal{S}, d)$ be a metric space.

(a) State the three conditions on d for it to be a metric.

(b) Give the definition of *Cauchy sequence* $x_n \in \mathcal{S}$.

(c) State the additional condition required of a metric space (S, d) to make it *complete*.

(d) Define an *open set* in $(\mathcal{S}, d)$.

(e) Define a *closed set* in $(\mathcal{S}, d)$.

Problem #7 (25pts): Let $(\mathcal{S}, d)$ be a metric space.
(**a**) Give the ϵ-δ definition for a function $f : \mathcal{S} \to \mathcal{S}$ to be continuous at a point $x_0 \in \mathcal{S}$.

(**b**) Prove that if f satisfies the condition that the pre-image of open sets are open, then f is continuous at each $x_0 \in \mathcal{S}$ in the ϵ-δ sense.

Problem #8 (25pts): Let $(\mathcal{S}, d)$ be a metric space. Recall that a compact set $E \subset \mathcal{S}$ is one such that every open covering admits a finite subcover. Prove that if $E \subset S$ is compact, then E satisfies the Bolzano-Weierstrass property: Every subsequence x_n in E contains a convergent subsequence $x_{n_k} \to x_0$, where $x_0 \in E$.