Name: Solutions

Student ID#: ____________

Section: ____________

Midterm Exam 1

Wednesday, Jan 29

MAT 125A, Temple, Winter 2014

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): **(a)** State the $\epsilon - \delta$ condition for a function to be continuous at a point x_0, and state its negation. (You may assume that f is defined for all real numbers.)

f is continuous at x_0 if: $\forall \varepsilon \exists \delta$ st $\forall_{x \in I_\delta(x_0)} |f(x) - f(x_0)| < \varepsilon$

f is not cont @ x_0 if: $\exists \varepsilon$ st $\forall \delta \exists_{x \in I_\delta(x_0)} |f(x) - f(x_0)| \geq \varepsilon$

(b) Use the negation to prove directly that if for every sequence $x_n \to x_0$ we have $f(x_n) \to f(x_0)$, then f is continuous at x_0 by the $\epsilon - \delta$ condition.

Assume f does not meet the $\varepsilon\delta$-criterion. We construct a sequence $x_n \to x_0$ s.t. $f(x_n) \not\to f(x_0)$. For this, f not cont by $\varepsilon\delta$ means there exists $\varepsilon > 0$ st st $\forall \delta > 0$ we can find x such that $|x - x_0| < \varepsilon$ but $|f(x) - f(x_0)| \geq \varepsilon$. In particular, such an x_n exists for each $\delta = \frac{1}{n}$. Thus $|x_n - x_0| < \frac{1}{n}$, but $|f(x_n) - f(x_0)| \geq \varepsilon$. Conclude that $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$. Thus our assumption that f does not meet the $\varepsilon\delta$-criterion is false, and hence it does. $\square$

Problem #2 (20pts): **(a)** Recall that a continuous function on a closed interval takes on its max and min values m and M, respectively. Prove the case that it takes on its minimum value m.

Assume f cont on $[a,b]$, and let $m = \inf\{f(x) : x \in [a,b]\}$. Then by defn $\exists$ x_n such that $f(x_n) \to m$. By BW x_n contains a convergent subsequence $x_n \to x_0 \in [a,b]$. Since f is cont at x_0, $\lim_{n\to\infty} f(x_n) = f(x_0) = m$. ✓

(b) State the Intermediate Value Theorem.

Let f be continuous on $[a,b]$. Then for any value y between $f(a)$ & $f(b)$, $\exists$ $x_0 \in [a,b]$ such that $f(x_0) = y$.

(c) Using only parts (a) and (b), prove that if $f : [a,b] \to \mathcal{R}$ is continuous, then its range is exactly $[m, M]$.

By defn of m, M, f takes no values outside the interval $[m,M]$. By (a), $\exists$ $\bar{a}, \bar{b} \in [a,b]$ such that $f(\bar{a}) = m$ & $f(\bar{b}) = M$. WLOG, assume $\bar{a} \le \bar{b}$. By IVT, for every y betw m & M, there exist $x_0 \in [\bar{a}, \bar{b}]$ such that $f(x_0) = y$. Since $x_0 \in [\bar{a}, \bar{b}] \subseteq [a,b]$, we conclude that every value in $[m,M]$ is taken on by f, so Range $f = [m,M]$.

Problem #3 (20pts): Assume that $f(x)$ is a function defined and continuous on the closed interval $[a, b]$. Let y be given, and define $x_0 = Inf\mathcal{S}$ where $\mathcal{S} = \{x : f(x) > y\}$. Assume that $a < x_0 < b$, and prove that $f(x_0) = y$.

Since x_0 is a limit x_n of points in S where $f(x_n) > y$, cont of f @ x_0 implies $f(x_0) \geq y$. Assume for contradiction that $f(x_0) < y$, say $f(x_0) = y - \varepsilon$. But since $x_0 \neq a, b$, and f cont at x_0, we know $\exists\, \delta > 0$ such that if $|x_0 - x| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$, so

$$f(x) < y, \text{ all } 0 \leq |x - x_0| < \delta.$$

But this means $x_0 - \frac{\delta}{2} \in S$, contradicting our assumption that $x_0 = \text{Inf}\, S$. $\therefore f(x_0) = y$.

Problem #4 (20pts): Let $f : \mathcal{R} \to \mathcal{R}$ be a uniformly continuous function, and let x_n denote a sequence of real numbers.

(a) State the Cauchy criterion for convergence of x_n.

x_n Cauchy if $\forall \varepsilon\ \exists N$ st $m,n > N$ implies $|x_n - x_m| < \varepsilon$

(b) State the definition of uniform continuity of f.

f unif continuous if $\forall \varepsilon\ \exists \delta$ st $\forall_{\substack{x,y \\ |x-y|<\varepsilon}}$ $|f(y) - f(x)| < \varepsilon$.

(c) Use (a) and (b) to prove directly that if x_n is Cauchy, then so is $f(x_n)$.

Assume f unif cont and x_n Cauchy. Let $\varepsilon > 0$ be given. It suffices to find N st if $n,m > N$ then $|f(x_n) - f(x_m)| < \varepsilon$. But by unif cont $\exists\ \delta$ st if $|x_n - x_m| < \delta$, then $|f(x_n) - f(x_m)| < \varepsilon$. And since x_n Cauchy, $\exists$ N st $m,n > N$ implies $|x_n - x_m| < \delta$. Conclude, $m,n > N$ implies $|f(x_n) - f(x_m)| < \varepsilon$, & $f(x_n)$ is Cauchy $\square$

Problem #5 (20pts): Let $f : \mathcal{R} \to \mathcal{R}$ and $g : \mathcal{R} \to \mathcal{R}$ be uniformly continuous functions. Prove that $f \circ g$ is uniformly continuous. (Recall, $(f \circ g)(x) = f(g(x))$.)

f is unif cont if: $\forall \varepsilon\ \exists \delta$ st $\forall_{\substack{x,y \\ |x-y|<\delta}}\ |f(y)-f(x)| < \varepsilon$

Thus to prove $f \circ g$ unif cont, fix $\varepsilon > 0$. We find δ st if $|x-y|<\delta$ then $|f\circ g(y) - f \circ g(x)| < \varepsilon$.

~~But~~ Since f is unif cont., $\exists\ \bar{\delta}$ st if $|g(y)-g(x)| < \bar{\delta}$, then $|f(g(y)) - f(g(x))| < \varepsilon$.

But since g is also unif cont, $\exists\ \delta$ st if $|y-x|<\delta$, then $|g(y)-g(x)| < \bar{\delta}$. Thus, if $|y-x|<\delta$, then $|f(g(y)) - f(g(x))| < \varepsilon$, as needed $\square$