Name: Solutions

Student ID#: ______________________

Section: ______________________

Midterm Exam 2

Monday, March 3

MAT 125A, Temple, Winter 2014

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		15
2		15
3		20
4		15
5		20
6		15
Total		100

Problem #1 (15pts):

(a) State the $\epsilon - \delta$ definition of a uniformly continuous function f defined on the Domain set $\mathcal{S}$. Then state the negation of this statement. That is, give the definition of *not* uniformly continuous.

Defn: f is uniformly cont on $\mathcal{S}$ if: $\forall \varepsilon \ \exists \delta$ st

$\forall_{|x-y|<\delta} \ |f(y)-f(x)|<\varepsilon$

Defn: f is not uniformly cont on $\mathcal{S}$ if: $\exists \varepsilon$ st $\forall \delta$

$\exists x, y \ |x-y|<\delta$ st $|f(y)-f(x)| \geq \varepsilon$

(b) Let $f : \mathcal{R} \to \mathcal{R}$ and $g : \mathcal{R} \to \mathcal{R}$ be uniformly continuous functions. Prove that $f \circ g$ is uniformly continuous. (Recall, $(f \circ g)(x) = f(g(x))$.)

Fix $\varepsilon > 0$. We find δ st $|x-y|<\delta$ implies $|f(y)-f(x)|<\varepsilon$.

Choose δ_1 so $|\bar{x}-\bar{y}|<\delta_1$ implies $|f(\bar{x})-f(\bar{y})|<\varepsilon$.

Choose δ so $|x-y|<\delta$ implies $|g(x)-g(y)|<\delta_1$.

Then $|x-y|<\delta$ implies $|g(y)-g(x)|<\delta_1$ so

$|f(g(y))-f(g(x))|<\varepsilon$ as claimed.

$g(x_0) \neq 0$

Problem #2 (15pts): Assume that f and g are differentiable at $x = x_0$, and assume that $\cancel{g'(x_0) \neq 0}$. Give a careful proof that

$$\frac{d}{dx}\left(\frac{f}{g}\right)(x_0) = \frac{g(x_0)f'(x_0) - g'(x_0)f(x_0)}{g(x_0)^2}.$$

$$\frac{d}{dx}\left(\frac{f}{g}\right)(x_0) = \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{(x - x_0)\,(g(x)g(x_0))}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{(x - x_0)\,g(x)g(x_0)}$$

$$= \lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0} \frac{g(x_0)}{g(x)g(x_0)} - \frac{f(x_0)}{g(x)g(x_0)} \frac{g(x) - g(x_0)}{x - x_0} \right\}$$

$$= \frac{f'(x_0)\,g(x_0)}{g(x)g(x_0)} - \frac{f(x_0)\,g'(x_0)}{g(x_0)g(x_0)} = \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Problem #3 (20pts): Consider the geometric series starting at $k = 2$,

$$f(x) = \sum_{k=2}^{\infty} x^k.$$

(a) Derive a formula for the partial sum $S_n(x) = \sum_{k=2}^{N} x^k$.

$$S_n(x) - xS_n(x) = \sum_{k=2}^{N} x^k - \sum_{k=2}^{N} x^{k+1} = x^2 - x^{N+1}$$

$$S_n(x) = \frac{x^2 - x^{N+1}}{1 - x} = \frac{x^2}{1-x}(1 - x^{N-1})$$

(b) Define what it would mean for $S_n(x)$ to be a uniformly Cauchy sequence of functions. on $(-r, r)$.

$S_n(x)$ uniformly Cauchy if $\forall \varepsilon$ $\exists N$ st

$\forall$ $m, n > N$ $\Rightarrow$ $|S_n(x) - S_m(x)| < \varepsilon$
$x \in (-r, r)$

(c) Prove that $S_n(x)$ is a uniformly Cauchy sequence of functions on Domain $|x| < r$ for any $r < 1$.

$$|S_n(x) - S_m(x)| = \left| \sum_{k=2}^{n} x^k - \sum_{k=2}^{m} x^k \right|$$

$$= \left| \sum_{k=m+1}^{n} x^k \right| \leq \sum_{k=m+1}^{n} r^k = |S_n(r) - S_m(r)|$$

But $S_n(r) = \frac{r^2}{1-r}(1 - r^{N-1}) \longrightarrow \frac{r^2}{1-r} < 1$

So $S_n(r)$ is a Cauchy Sequence. Thus $\forall \varepsilon$ $\exists N$ st $\forall\, m,n > N$ we have $|S_n(r) - S_m(r)| < \varepsilon$.

Thus $m, n > N$ implies

$$|S_n(x) - S_m(x)| \leq |S_n(r) - S_m(r)| < \varepsilon$$

hence $S_n(x)$ is a Uniformly Cauchy Sequ.

Problem #4 (15pts): Let f be a function continuous on the closed interval $[a, b]$, assume f is differentiable on the open interval (a, b), and assume $f(a) = f(b)$. Using only theorems about continuous functions that we proved before, give a careful proof of Rolle's Theorem: There exists a point $x^* \in (a, b)$ at which $f'(x^*) = 0$. [You may use properties of deriv like $f'(x)=0$ at max/min]

- f cont on $[a,b] \Rightarrow f$ takes a max and min value on $[a,b]$.
- $f'(x)=0$ at max/min values.
- If $f(x) = \text{const} = f(a) = f(b)$ for all $x \in [a,b]$, then $f'(x)=0$ for all $x \in (a,b)$ because every x is a max/min.
- Assume $f(x)$ not constant. Then $\exists\, x \in (a,b)$ such that either (i) $f(x) > f(a) = f(b)$ or (ii) $f(x) < f(a) = f(b)$. In Case (i) $\exists$ max pt $f(x^*) \neq f(a)$ & in (ii) $\exists$ min pt $f(x^*) \neq f(a)$. In either case, $f'(x^*)=0$ & $x^* \neq a$ or b. ✓

Problem #5 (20pts): Assume that $f_n(x) \to f(x)$ for each $x \in [0,1]$, and assume that each f_n is continuous on $[0,1]$.

(a) Give an example of a sequence f_n such that the limit f is discontinuous.

$f_n(x) = x^n$. Then $f_n(x) \to 0 \quad x \neq 1,$

$f_n(1) \to 1 \quad x = 1.$

(b) Define what it means for the sequence of functions f_n to converge *uniformly* to f.

$f_n \to f$ uniformly on $[0,1]$ if $\forall \varepsilon \; \exists N$ st

$\forall \underset{x \in [0,1]}{n > N} \quad |f_n(x) - f(x)| < \varepsilon.$

(c) Give a careful proof that if $f_n \to f$ uniformly, then f is continuous.

Assume $f_n \to f$ uniformly. We prove f is continuous at each $x_0 \in [0,1]$. For this, fix $\varepsilon > 0$. We find δ st if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. First choose N st $n > N$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in [0,1]$. Next choose $n > N$, & use cont. of f_n to fix $\delta > 0$ st $|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$. Then for $|x - x_0| < \delta$ we have:

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

as claimed.

Problem #6 (15pts): Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = \infty$. Use our theorems about power series to derive a formula for the coefficients a_n in terms of derivatives of f at $x = 0$. Then marvel at the idea that the entire function is determined by what is happening at $x = 0$!.

We know you can diff a power seris TxT within its radius of convergence – Thus

$$f'(x) = \sum_{n=1}^{\infty} a_n \cdot n x^{n-1} \Rightarrow f'(0) = a_1$$

$$f''(x) = \sum_{n=2}^{\infty} a_n \cdot n(n-1) x^{n-2} \Rightarrow f''(0) = 2a_2$$

$\vdots$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n \, n(n-1)\cdots(n-k+1) \Rightarrow f^{(k)}(0) = a_k k!$$

Conclude: $f^{(k)}(0) = a_k k! \Rightarrow \boxed{a_k = \frac{f^{(k)}(0)}{k!}}$