Adding Lagrange Multipliers to the Schwarzschild Lagrangian

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1 Introduction

A first approximation of the sun assumes a sphere with mass density dependent on radius. The exact general relativistic solution is the Schwarzschild metric. Orbits of the Schwarzschild solution are found by extremizing path length of the Schwarzschild metric. A system of four second order differential equations results. Exploiting the Symmetry of the system through Killing vectors simplifies the system. Afterwards substituting coordinates in eventually results in an asymptotically solvable differential equation. Many of these steps are quite complicated, but can be verified with Lagrange Multipliers.

2 An idea From Classical Mechanics

Assume a free classical particle given by the Lagrangian.

$$\frac{m}{2} * ((dx/dt)^2 + (dy/dt)^2) \tag{1}$$

Extremizing this Lagrangian unsurprisingly results in.

$$\frac{d^2x}{dt^2} = 0 \frac{d^2y}{dt^2} = 0 \tag{2}$$

Now lets add in a constraint equation to keep our particle in a circle. Using a circle of radius R and a Lagrange Multiplier λ we now have the Lagrangian.

$$\frac{m}{2} * ((dx/dt)^2 + (dy/dt)^2) + \lambda * (x^2 + y^2 - R^2) \tag{3}$$

The first term of the Lagrangian is still the equation of a free particle, the second term requires the particle to travel in a circle. Taking λ to be a variable, standard calculus of variations leads to.

$$m\frac{d^2x}{dt^2} = 2\lambda x \tag{4}$$

$$m\frac{d^2y}{dt^2} = 2\lambda y \tag{5}$$

$$x^2 + y^2 = R^2 \tag{6}$$

Taking two total time derivatives of the equation of the circle and substituting into the first two equations solves for the Lagrange Multiplier.

$$\lambda = -\frac{m}{2}\frac{(dx/dt)^2 + (dy/dt)^2}{x^2 + y^2} \tag{7}$$

The new Lagrangian takes the form.

$$\frac{m}{2} * [(dx/dt)^2 + (dy/dt)^2] * \frac{R^2}{x^2 + y^2} \tag{8}$$

Now lets do an identical calculation for a straight line rather than a circle. Our Lagrangian is now.

$$\frac{m}{2} * ((dx/dt)^2 + (dy/dt)^2) + \lambda * (x - by) \tag{9}$$

The equations of motion become

$$m\frac{d^2x}{dt^2} = \lambda \tag{10}$$

$$m\frac{d^2y}{dt^2} = -b\lambda \tag{11}$$

$$x = by \tag{12}$$

Solving for the Lagrange multiplier reveals.

$$\lambda = 0 \tag{13}$$

When we add the requirement for our particle to travel along a line the Lagrangian is unchanged. Solving the equation of motion for a free particle results in straight lines, implying forcing a free particle to travel in a straight line is redundant. Suppose instead of requiring the particle to travel in a straight line we required constant velocity. Is a constant velocity requirement redundant, or change the form of our Lagrangian. A generalization is called for.

3 Generalizing our Result

Suppose a Lagrangian L describes the motion of a particle. Now suppose a constraint is added to the Lagrangian using a Lagrange Multiplier. To allow the constraint the initial conditions may have fewer degrees of freedom. If the Lagrange multiplier evaluates to zero, then along paths consistent with the reduced initial conditions we are extremizing the original Lagrangian. Along these paths the constraint equation and the Euler-Lagrange equations are both satisfied. An example will clarify this result.

4 Schwarzschild at $\theta = \frac{\pi}{2}$

The Schwarzschild solution orbit is found by extremizing.

$$ds^2 = (1 - \frac{2GM}{r})(\frac{dt}{d\tau})^2 - \frac{1}{(1 - \frac{2GM}{r})}(\frac{dr}{d\tau})^2 - r^2(\frac{d\theta}{d\tau})^2 - r^2 sin^2(\theta)(\frac{d\phi}{d\tau})^2 \quad (14)$$

Now add in a Lagrange multiplier to restrict $\theta = \frac{\pi}{2}$. Physically this corresponds to requiring our motion to take place in a plane. We now have the Lagrangian.

$$L = (1-\frac{2GM}{r})(\frac{dt}{d\tau})^2 - \frac{1}{(1 - \frac{2GM}{r})}(\frac{dr}{d\tau})^2 - r^2(\frac{d\theta}{d\tau})^2 - r^2 sin^2(\theta)(\frac{d\phi}{d\tau})^2 - \lambda(\theta = \frac{\pi}{2}) \quad (15)$$

Solving the Euler-Lagrange equation for theta in addition to the constraint gives.

$$-2r^2\frac{d^2\theta}{d\tau^2} - 4r\frac{dr}{dtau}\frac{d\theta}{d\tau} = -2r^2 \sin\theta\cos\theta\frac{d^2\phi}{d\tau^2} - \lambda \quad (16)$$

$$\theta = \frac{\pi}{2} \quad (17)$$

We can differentiate the constraint to get.

$$\frac{d\theta}{d\tau} = 0 \quad (18)$$

$$\frac{d^2\theta}{d\tau^2} = 0 \quad (19)$$

Now the Euler-Lagrange equation for theta shows $\lambda = 0$. Note this calculation doesn't work if θ is not $\frac{\pi}{2}$. With much less Machinery than the Killing Vector approach we have proven motion in the Shwarzschild metric takes place in a plane. In any hypothetical orbit we can adjust our coordinates to make our object in a plane. Once in this plane the object will never leave the plane and is subject to the same Lagrangian a free particle would experience.

5 Energy and Angular Momentum

Usually the conserved quantities associated with Schwarzschild orbits are found using Killing vectors. Lagrange multipliers only allows one to check a conserved quantity. To that end we now show Energy and Angular Momentum are conserved.

$$L = (1-\frac{2GM}{r})(\frac{dt}{d\tau})^2 - \frac{1}{(1 - \frac{2GM}{r})}(\frac{dr}{d\tau})^2 - r^2(\frac{d\theta}{d\tau})^2 - r^2 sin^2(\theta)(\frac{d\phi}{d\tau})^2 - \lambda((1-\frac{2GM}{r})\frac{dt}{d\tau} - E) \quad (20)$$

Varying with respect to time and our Lagrange multiplier yields

$$\frac{d}{d\tau}(2(1-\frac{2GM}{r})\frac{dt}{d\tau}-\lambda(1-\frac{2GM}{r})) \tag{21}$$

$$(1-\frac{2GM}{r})\frac{dt}{d\tau}=E \tag{22}$$

Substituting the constraint equation into the variational equation and taking the derivative with respect to proper time yields.

$$\lambda\frac{2GM}{r^2}\frac{dr}{d\tau}=0 \tag{23}$$

Which is satisfied in General by $\lambda = 0$.

Similarly with Angular momentum

$$L=(1-\frac{2GM}{r})(\frac{dt}{d\tau})^2-\frac{1}{(1-\frac{2GM}{r})}(\frac{dr}{d\tau})^2-r^2(\frac{d\theta}{d\tau})^2-r^2sin^2(\theta)(\frac{d\phi}{d\tau})^2-\lambda(r^2\frac{d\phi}{d\tau}-A) \tag{24}$$

Varying with respect to ϕ and our Lagrange multiplier yields

$$\frac{d}{d\tau}[-2r^2sin^2(\theta)(\frac{d\phi}{d\tau})-\lambda r^2]=0 \tag{25}$$

$$r^2\frac{d\phi}{d\tau}=A \tag{26}$$

Substituting the constraint equation into the variational equation and taking the derivative with respect to proper time yields.

$$\lambda 2r\frac{dr}{d\tau}=0 \tag{27}$$

Again in General λ is zero.

Using the method of applying constraint equations We see that energy, $(1-\frac{2GM}{r})\frac{dt}{d\tau}$ and Angular momentum $r^2\frac{dr}{d\tau}$ are both conserved along orbits of the Schwarzschild metric. A Philosophical point is the slight difference between conservation laws derived from killing vectors and those derived above. If we derive the conserved quantities from the Schwarzschild solution we end are first solving for orbits, and then finding conserved quantities based on those orbits. Lagrange multipliers instead assume a specific conserved quantity then demonstrate the conservation law has no effect on the solution of the orbit.

6 Conclusion

If two Lagrangians yield the same equation a mathematician can use either equation. Adding in a constraint equation using Lagrange multipliers restricts the possible solutions, if we haven't actually changed the Lagrangian, the constraint was already satisfied. In this way we check and verify the solutions to the Schwarzschild metric can indeed be regarded as taking place in a plane and with fixed energy and angular momentum. These results are derived more directly than the traditional method.