Why the Riemann Curvature Tensor needs twenty independent components

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1 Introduction

In General Relativity the Metric is a central object of study. Most commonly used metrics are beautifully symmetric creations describing an idealized version of the world useful for calculations. Researchers approximate the sun or earth as a sphere when calculating the orbits of planets or satellites. Although these symmetric metrics are immensely useful, at some point one must consider an arbitrary metric. Any experimentally measured or derived metric might not be in good coordinates. In this paper I demonstrate how a coordinate transformation of an arbitrary metric can, at least locally, make the metric simpler. While simplifying our arbitrary metric we end up motivating many of the mathematical structures used in General Relativity.

2 Metric at a point

Suppose one is given an arbitrary metric with no symmetries. What is the simplest form a metric can take at a single point? After being evaluated at a point the metric is a 4x4 matrix. For the moment we are ignoring everything but the exact specific real numbers the components of the metric take at a single point. How simple can we make this matrix look? Calling our arbitrary coordinates given by the x^α and our new simpler coordinates y^μ the coordinate transformation takes the form:

$$g_{\mu\nu} = g_{\alpha\beta}\frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu}$$

The analysis of a Symmetric matrix under change of coordinates is well understood. The well known Symmetric matrix theorem allows us to make our metric take the form of the Minkowski metric at a single point.

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} => g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This unsurprising result reiterates that General Relativity takes place on a curved but locally Lorentzian manifold. The interesting result comes from counting the degrees of freedom in transforming the arbitrary metric to a nice metric. We have ten values from the metric that need to be adjusted to the form of the Minkowski metric, however we have sixteen values of the coordinate transformation to diagonalize with. Remember the matrix

$$\frac{dx^\alpha}{dy^\mu} = \frac{dx^\beta}{dy^\nu} = \begin{pmatrix} \frac{dx^0}{dy^0} & \frac{dx^1}{dy^0} & \frac{dx^2}{dy^0} & \frac{dx^3}{dy^0} \\ \frac{dx^0}{dy^1} & \frac{dx^1}{dy^1} & \frac{dx^2}{dy^1} & \frac{dx^3}{dy^1} \\ \frac{dx^0}{dy^2} & \frac{dx^1}{dy^2} & \frac{dx^2}{dy^2} & \frac{dx^3}{dy^2} \\ \frac{dx^0}{dy^3} & \frac{dx^1}{dy^3} & \frac{dx^2}{dy^3} & \frac{dx^3}{dy^3} \end{pmatrix}$$

Already has values for the functions x^α however the functions y^μ are ours to alter as we want. If we want for instance $\frac{dx^1}{dy^2}$ to increase in value by τ at point p. We can redefine $y^2 = y^2 + \frac{-\tau * \frac{dy^2}{dx^1}}{\tau + \frac{dx^1}{dy^2}} * x^1$ At our point this will alter $\frac{dx^1}{dy^2}$ but leave all the other $\frac{dx^\alpha}{dy^\mu}$ unchanged.

If we assume the worst possible situation arises, ten of the independently alterable $\frac{dx^\alpha}{dy^\mu}$ are required to transform the arbitrary metric to the Minkowski metric. After using these ten numbers we still have six remaining degrees of freedom within the coordinate transformation. A little imagination leads one to believe that there might be a six parameter family of local transformations at every point which lead the form of the metric unchanged. Remember we are still only working with a single point so don't go looking for a six parameter family of global transformations for any metric.

The Lorentz group is clearly the six parameter family of transformations left over after diagonalizing the metric. Although the Lorentz transformations predate General Relativity their use can be justified by analyzing the transformations remaining after diagonalizing the metric at a single point.

3 First Order, simplifying derivatives of the metric

After we have simplified the metric at a point we now turn to the derivatives of the metric at the same point. How simple can the derivatives of the metric look. All the derivatives of the coordinate transformations at our point have reduced the metric to the Minkowski metric. To examine how the derivatives of the metric transform at a point start by Taylor expanding our nice metric and the transformation of the arbitrary metric at a point p to first order.

$$g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial y^\sigma} dy^\sigma = g_{\alpha\beta} \frac{dx^\alpha}{dy^\mu} \frac{dx^\beta}{dy^\nu} + \left[\frac{\partial g_{\mu\nu}}{\partial y^\sigma} \frac{dx^\alpha}{dy^\mu} \frac{dx^\beta}{dy^\nu} + 2 g_{\alpha\beta} \frac{dx^\beta}{dy^\nu} \frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\sigma} \right] dy^\sigma$$

We already know the metric equals its coordinate transformation at this point, removing this term then operating both sides on the vector $\frac{\partial}{\partial x^\sigma}$ we have an equation to examine the derivatives of the metric.

$$\frac{\partial g_{\mu\nu}}{\partial y^\sigma} = \frac{\partial g_{\alpha\beta}}{\partial y^\sigma}\frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu} + 2g_{\alpha\beta}\frac{dx^\beta}{dy^\nu}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\sigma}$$

This equation shows the dependence of the derivatives of the metric on the second derivatives of the coordinate change. Since we are free to alter any y^σ, all $\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\sigma}$ are independent and under our control when making a new coordinate system. Since everything else is fixed we can alter the values of the $\frac{\partial g_{\mu\nu}}{\partial y^\sigma}$ as we please. To determine exactly how simple we can make these derivatives of the metric we need to count how many derivatives of the metric we have and how many second derivatives of the coordinate transformations we have to work with.

Since $\frac{\partial g_{\mu\nu}}{\partial y^\sigma}$ is symmetric in μ and ν and σ in four dimensions we have a total of forty derivatives of the metric. Similarly $\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\sigma}$ is symmetric in μ and σ for a total of forty second derivatives of the coordinate transformation. By careful choice of the second derivatives of the coordinate transformations we can make every derivative of the metric equal zero.

In our nice coordinates:

$$\frac{\partial g_{\mu\nu}}{\partial y^\sigma} = 0$$

Any arbitrary metric can locally be made into the Minkowski metric with vanishing first derivatives, consistent with Riemann normal coordinates. To move anywhere in the metric it might not be ideal to simply the metric to local Minkowski space. As previously shown it takes forty functions to describe the first derivative of the metric, any structure used to turn the derivatives of the metric into a useable form then needs to have forty independent components. In GR there are forty independent Christoffel symbols.

4 Second Order, A curvature Tensor.

A similar approach to the derivatives of the metric can be used for the second derivatives. Taylor expanding the metric to second order results in:

$$g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial y^\sigma} dy^\sigma + \frac{1}{2}\frac{\partial^2 g_{\mu\nu}}{\partial y^\sigma \partial y^\rho} dy^\sigma dy^\rho = g_{\alpha\beta}\frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu} + \left[\frac{\partial g_{\mu\nu}}{\partial y^\sigma}\frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu} + 2g_{\alpha\beta}\frac{dx^\beta}{dy^\nu}\frac{\partial^2 x^\alpha}{\partial y^\mu \partial y^\sigma}\right] dy^\sigma +$$

$$\left[\frac{\partial^2 g_{\alpha\beta}}{\partial y^\sigma \partial y^\rho}\frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu} + 4\frac{\partial g_{\alpha\beta}}{\partial y^\sigma}\frac{d^2 x^\alpha}{dy^\mu dy^\rho}\frac{dx^\beta}{dy^\nu} + 2g_{\alpha\beta}\frac{d^2 x^\alpha}{dy^\mu dy^\sigma}\frac{d^2 x^\beta}{dy^\nu dy^\rho} + 2g_{\alpha\beta}\frac{d^3 x^\alpha}{dy^\mu dy^\sigma dy^\rho}\frac{dx^\beta}{dy^\nu}\right] dy^\rho dy^\sigma$$

After equating the second order terms and operating on vectors to remove the one-forms we end up with.

$$\frac{\partial^2 g_{\mu\nu}}{\partial y^\sigma \partial y^\rho} = \frac{\partial^2 g_{\alpha\beta}}{\partial y^\sigma \partial y^\rho}\frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu} + 4\frac{\partial g_{\alpha\beta}}{\partial y^\sigma}\frac{d^2 x^\alpha}{dy^\mu dy^\rho}\frac{dx^\beta}{dy^\nu} + 2g_{\alpha\beta}\frac{d^2 x^\alpha}{dy^\mu dy^\sigma}\frac{d^2 x^\beta}{dy^\nu dy^\rho} + 2g_{\alpha\beta}\frac{d^3 x^\alpha}{dy^\mu dy^\sigma dy^\rho}\frac{dx^\beta}{dy^\nu}$$

We now have an equation for the second derivatives of the metric in terms of the coordinate transformate and there derivatives. We have already fixed the first and second derivatives of the coordinate transformations and are left with the third order derivatives of the coordinate transformations. Again we count the degrees of freedom of both sides of the equation. $\frac{\partial^2 g_{\mu\nu}}{\partial y^\sigma \partial y^\rho}$ is symmetric in μ and ν and by Fubini's theorem we can take the partial derivatives in either order. Combining these symmetries reduces the 256 combinations of μ , ν, σ, and ρ to 100 independent second derivatives of the metric.

The third derivatives of the coordinate transformation, $\frac{d^3 x^\alpha}{dy^\mu dy^\sigma dy^\rho}$ can be analyzed similarly. We have four choices for α and a total of twenty ways to choose the values of μ, σ, and ρ. Multiplying these we end up with eighty independent third derivatives of the coordinate transformation.

An arbitrary metric will have a total of a hundred independent second derivatives of the metric, but only eighty numbers to simplify them. Outside of special cases we end up having twenty second derivatives of the metric be non-zero no matter how cleverly we choose our coordinates. Information describing the essential unsimplifiable nature of the second derivative of the metric is contained in contained in these twenty functions. Any attempt to construct a tensor describing the second derivatives of the metric or any of their properties must have at least twenty functions. Careful analysis shows the Riemann curvature tensor has exactly twenty independent components. We can understand these independent components as conveying the coordinate unsimplifiable nature of the second derivatives of the metric.

5 Conclusion

By Taylor expanding the metric around a point we can determine how mathematical constructions in General Relativity need many of the properties they have. The remaining coordinate transformations of the metric turn out to have the same number of degrees of freedom as the local Lorenz group. Counting degrees of freedom shows how all derivatives of the metric vanish at a point in special coordinates. Meanwhile Counting the number of independent derivatives of the metric prescribes the needed number of independent components of the Christoffel Symbols. Lastly the Riemann curvature tensor can be understood as describing how the second derivatives of the metric cannot be simplifies. All of these contribute to understanding why the necessary mathematical structures in General Relativity have the properties they possess.