

Shock Wave Interactions in General Relativity and the Emergence of Regularity
Singularities

By

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CONTENTS

Abstract	iii
Acknowledgments	iv
1. Introduction	1
2. Preliminaries	10
3. A Point of Regular Shock Wave Interaction in SSC	20
4. The Obstacles to Coordinate Smoothing at Points of Shock Wave Interaction	26
5. Smoothness Class of the Radial Component $C(t, r)$	29
6. Functions $C^{0,1}$ Across a Hypersurface	34
7. A Necessary and Sufficient Condition for Smoothing Metrics	41
8. Metric Smoothing on Single Shock Surfaces and a Constructive Proof of Israel's Theorem	50
9. Shock Wave Interactions as Regularity Singularities in GR; Transformations in the (t, r) -Plane	61
10. Shock Wave Interactions as Regularity Singularities in GR; the Full Atlas	71
11. The Loss of Locally Inertial Frames	78
12. Construction of a Jacobian on a Deleted Neighborhood and the Gluing Conditions	80
13. Discussion	85
14. Conclusion	89
Appendix A. The Integrability Condition	91
Appendix B. The existence of coordinates from Section 4	92
References	95

ABSTRACT

We show that the regularity of the gravitational metric tensor cannot be lifted from $C^{0,1}$ to $C^{1,1}$ by any $C^{1,1}$ coordinate transformation in a neighborhood of a point of shock wave interaction in General Relativity, without forcing the determinant of the metric tensor to vanish at the point of interaction. This is in contrast to Israel's Theorem [6] which states that such coordinate transformations always exist in a neighborhood of a point on a smooth *single* shock surface. The results thus imply that points of shock wave interaction represent a new kind of singularity in spacetime, singularities that make perfectly good sense physically, that can form from the evolution of smooth initial data, but at which the spacetime is not *locally Minkowskian* under any coordinate transformation. In particular, at such singularities, delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems, but due to cancelation, the curvature tensor remains uniformly bounded.

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1. INTRODUCTION

Albert Einstein introduced his theory of *General Relativity* (GR) in 1915 after eight years of struggle. Einstein's guiding principle in the pursuit of the field equations was the principle that spacetime should be *locally inertial*¹. That is, an observer in freefall through a gravitational field should observe all of the spacetime physics of special relativity, except for the second order acceleration effects due to spacetime curvature (gravity). But the assumption that spacetime is locally inertial is equivalent to assuming the gravitational metric tensor g has a certain level of smoothness around every point. That is, the assumption that spacetime is *locally inertial* at a spacetime point p assumes the gravitational metric tensor g is smooth enough so that one can pursue the construction of Riemann Normal coordinates at p , coordinates in which g is exactly the Minkowski metric at p , and such that all first order derivatives of g vanish at p as well. The nonzero second derivatives at p are then a measure of spacetime curvature. However, the Einstein equations are a system of PDE's for the metric tensor g and the PDE's by themselves determine the smoothness of the gravitational metric tensor by the evolution they impose. Thus the condition on spacetime that it be locally inertial at every point cannot be assumed at the start, but must be determined by regularity theorems for the Einstein equations.

This issue becomes all the more interesting when the sources of matter and energy are modeled by a perfect fluid, and the resulting Einstein-Euler equations form a system of PDE's for the metric tensor g coupled to the density, velocity and pressure of the fluid. It is well known that from the evolution of a perfect fluid governed by the compressible Euler equations, shock wave discontinuities form from smooth

¹Also referred to as *locally Lorentzian* or *locally Minkowskian*, that is, around any point there exist coordinates in which the gravitational metric is Minkowskian at that point with vanishing first derivatives. Therefore, at that point the metric is Minkowskian up to second order errors in distance.

initial data whenever the flow is sufficiently compressive.² At a shock wave, the fluid density, pressure and velocity are discontinuous, and when such discontinuities are assumed to be the sources of spacetime curvature, the Einstein equations imply that the curvature must also become discontinuous at shocks. But discontinuous curvature by itself is not inconsistent with the assumption that spacetime be locally inertial. For example, if the gravitational metric tensor were $C^{1,1}$, (differentiable with Lipschitz continuous first derivatives, [11]), then second derivatives of the metric are at worst discontinuous, and the metric has enough smoothness for there to exist coordinate transformations which transform g to the Minkowski metric at p , with zero derivatives at p as well, [11]. Furthermore, Israel's theorem, [6], (see also [11]) asserts that a spacetime metric need only be $C^{0,1}$, i.e., Lipschitz continuous, across a smooth *single* shock surface³, in order that there exist a $C^{1,1}$ coordinate transformation that lifts the regularity of the gravitational metric one order to $C^{1,1}$ as well, and this again is smooth enough to ensure the existence of locally inertial coordinate frames at each point. In fact, when discontinuities in the fluid are present, $C^{1,1}$ coordinate transformations are the natural atlas of transformations that are capable of lifting the regularity of the metric one order, while still preserving the weak formulation of the Einstein equations, [10]. It is common in GR to assume the gravitational metric tensor is at least $C^{1,1}$, for example, the $C^{1,1}$ regularity of the gravitational metric is assumed at the start in the singularity theorems of Hawking and Penrose, [5]. However, in Standard Schwarzschild Coordinates⁴ (SSC) the gravitational metric will be

²Since the Einstein curvature tensor G satisfies the identity $Div G = 0$, the Einstein equations $G = \kappa T$ imply $Div T = 0$, and so the assumption of a perfect fluid stress tensor T automatically implies the coupling of the Einstein equations to the compressible Euler equations $Div T = 0$.

³We call a hypersurface a shock surface if the Lipschitz continuity of the metric across the hypersurface is such that the jump in the metric derivatives normal to the hypersurface do not vanish ($[g_{\mu\nu,\sigma}]n^\sigma \neq 0$) and such that the Einstein tensor satisfies the Rankine Hugoniot jump condition, that is, $[G^{\mu\nu}]n_\nu = 0$ (c.f. Preliminaries).

⁴It is well known that a general spherically symmetric metric of form $ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + E(t,r)dtdr + C(t,r)^2d\Omega^2$ can be transformed to SSC in a neighborhood of a point where $\frac{\partial C}{\partial r} \neq 0$, c.f. [18]

no smoother than $C^{0,1}$, if a discontinuous energy momentum tensor in the Einstein equations is present.

In this thesis we prove there are no $C^{1,1}$ coordinate transformations that lift the regularity of a gravitational metric tensor from $C^{0,1}$ to $C^{1,1}$ at a point of a shock wave interaction in a spherically symmetric spacetime in GR, without forcing the determinant of the metric tensor to vanish at the point of interaction. This is in contrast to Israel's Theorem [6] which states that such coordinate transformations always exist in a neighborhood of a point on a smooth single shock surface. (Israel's theorem applies to general smooth shock surfaces of arbitrary dimension, [6].) It follows that solutions of the Einstein equations containing single smooth shock surfaces can solve the Einstein equations strongly, (in fact, pointwise almost everywhere in Gaussian normal coordinates), but this fails to be the case at points of shock wave interaction, where the Einstein equations can only hold weakly in the sense of the theory of distributions. The results thus imply that points of shock wave interaction represent a new kind of singularity in General Relativity that can form from the evolution of smooth initial data, that correctly reflects the physics of the equations, but at which the spacetime is not locally Minkowskian under any $C^{1,1}$ coordinate transformation. At such singularities, delta function sources in the second derivatives of the gravitational metric tensor exist in all coordinate systems of the $C^{1,1}$ atlas, but due to cancelation, the curvature tensor remains uniformly bounded.

To state the main result precisely, we consider spherically symmetric spacetime metrics $g_{\mu\nu}$ which solve the Einstein equations in SSC, that is, in coordinates where the metric takes on the form

$$(1.1) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -A(t, r) dt^2 + B(t, r) dr^2 + r^2 d\Omega^2,$$

where either t or r can be taken to be timelike, and $d\Omega^2$ is the line element on the unit 2-sphere, c.f. [12]. In Section 3 we make precise the definition of a regular point

of shock wave interaction in SSC. Essentially, this is a point in (t, r) -space where two distinct shock waves enter or leave the point p at distinct speeds, such that the metric is Lipschitz continuous, the Rankine Hugoniot (RH) jump conditions hold across the shocks [10], and the SSC Einstein equations hold strongly away from the shocks. The main result of the paper is the following theorem, (c.f. Definition 3.1 and Theorem 9.1 below):

Theorem 1.1. *Assume p is a point of regular shock wave interaction in SSC. Then there does not exist a $C^{1,1}$ regular coordinate transformation⁵, defined in a neighborhood of p , such that the metric components are C^1 functions of the new coordinates.*

The proof of Theorem 1.1 is constructive, providing Jacobians that, if they smooth the metric on a deleted neighborhood of p , must have a vanishing determinant at p itself. In this sense the metric becomes singular at p , if a C^1 regularity is forced upon it, and having this effect, of a vanishing metric determinant opposed by a lack of C^1 regularity, in mind we refer to p as a *regularity singularity*. We expect this type of singularity to form out of smooth initial data within a finite time, in correspondence with fluids governed by the special relativistic Euler equations.⁶ Our assumptions in Theorem 1.1 apply to the upper half ($t \geq 0$) and the lower half ($t \leq 0$) of a shock

⁵The atlas of $C^{1,1}$ coordinate transformations is a generic choice to address shock waves in General Relativity, since C^2 coordinate transformations cannot lift the metric regularity in the first place (c.f. Section 7), while a $C^{1,\alpha}$ atlas seems to be appropriate only for metric tensors in $C^{0,\alpha}$. Furthermore, due to the quasilinear structure of the Einstein equations a $C^{1,1}$ atlas is natural as it preserves the weak formalism, while for any atlas regularity below C^1 (e.g., a $C^{0,1}$ atlas with resulting discontinuous metric components) we expect that a weak formulation of the Einstein equations fails to exist. Given this, points of regular shock wave interaction in SSC represent *regularity singularities* in the sense that they are points where the gravitational metric is less regular than C^1 in any coordinate system that can be reached within the $C^{1,1}$ atlas.

⁶It is commonly expected that general relativistic shock waves form out of smooth initial, for example, the breakdown of classical solutions of the coupled Einstein Euler Equations has been shown in plane symmetric spacetimes, with strong indications that this is due to a formation of

wave interaction (at $t = 0$) separately, general enough to include the case of two timelike (or spacelike⁷) interacting shock waves of opposite families that cross at the point p , (two such shock waves typically change their speeds discontinuously at the point of interaction), but also general enough to include the cases of two outgoing shock waves created by the focusing of compressive rarefaction waves, or two incoming shock waves of the same family that interact at p to create an outgoing shock wave of the same family and an outgoing rarefaction wave of the opposite family, c.f. [10]. In particular, our framework is general enough to incorporate the shock wave interaction which was numerically simulated in [17]. We want to point out that even though our research was motivated by shock wave solutions of the Einstein-Euler equations constructed by Temple and Groah [3], we do not explicitly assume a perfect fluid as the matter source, we only require $T^{\mu\nu}$ to be bounded and to satisfy the Rankine Hugoniot jump conditions $[T^{\mu\nu}]n_\mu = 0$ on each of the shock curves.

Historically, the issue of the smoothness of the gravitational metric tensor across interfaces began with the matching of the interior Schwarzschild solution to the vacuum across an interface, followed by the celebrated work of Oppenheimer and Snyder who gave the first dynamical model of gravitational collapse by matching a pressureless fluid sphere to the Schwarzschild vacuum spacetime across a dynamical interface. In [11], Smoller and Temple extended the Oppenheimer-Snyder model to nonzero pressure by matching the Friedmann metric to a static fluid sphere across a shock wave interface that modeled a blast wave in GR. In his celebrated 1966 paper [6], Israel gave the definitive conditions for regular matching of gravitational metrics at smooth interfaces, by showing that if the second fundamental form is continuous across a single smooth interface, then the RH jump conditions also hold, and Gaussian normal coordinates provide a locally inertial coordinate system at each point on the surface.

shock waves [9], however, as far as we know the formation of shock waves in spherically symmetric spacetimes has not been shown yet.

⁷In fact the theorem applies to non-null surfaces that can be regularly parameterized by the SSC time or radial variable, c.f. Theorem 9.1 below.

In [3] Groah and Temple addressed these issues rigorously in the first general existence theory for shock wave solutions of the Einstein-Euler equations in spherically symmetric spacetimes. In coordinates where their analysis is feasible, SSC, it turned out that the gravitational metric was Lipschitz continuous at shock waves, but no smoother, and it has remained an open problem whether the weak solutions constructed by Groah and Temple could be smoothed to $C^{1,1}$ by coordinate transformation, like the single shock surfaces addressed by Israel. The results in this paper resolve this issue by proving definitively that the weak solutions constructed by Temple and Groah *cannot* be smoothed within the class of $C^{1,1}$ coordinate transformations when they contain points of shock wave interaction.

Points of shock wave interaction are straightforward to construct for the relativistic compressible Euler equations in flat spacetime, but to our knowledge no one has yet constructed an exact solution of the Einstein equations containing a point of shock wave interaction, where two shock waves cross in spacetime. Moreover, no general construction has been given that proves that such points exist, and meet the regularity assumptions of our theorem. Nevertheless, all the evidence points to the fact that points of shock wave interaction exist, have the structure we assume in SSC, and in fact cannot be avoided in solutions consisting of, say, an outgoing spherical shock wave (the blast wave of an explosion) evolving inside an incoming spherical shock wave (the leading edge of an implosion). Indeed, the existence theory of Temple and Groah [3] lends strong support to this claim, establishing existence of weak solutions of the Einstein-Euler equations in spherically symmetric spacetimes. The theory applies to arbitrary numbers of initial shock waves of arbitrary strength, existence is established beyond the point of shock wave interaction, and the regularity assumptions of our theorem are within the regularity class to which the Groah-Temple theory applies. Moreover, the recent work of Vogler and Temple gives a numerical simulation of a class of solutions in which two shock waves emerge from a point of

interaction where two compression waves focus into a discontinuity in density and velocity, and the numerics demonstrate that the structure of the emerging shock waves meets all of the assumptions of our theorem at the point of interaction. Still, as far as we know, there is no rigorous mathematical construction of an exact solution of the Einstein equations consisting of two interacting shock waves. Taken on whole, we interpret the above considerations as a definitive physical proof that points of shock wave interaction exist in GR, and meet the regularity assumptions of our theorem. The conclusion of our theorem then, is that such points must exist where the gravitational metric tensor cannot be smoothed to $C^{1,1}$ by $C^{1,1}$ coordinate transformation.

In Section 2 we start with the preliminaries, followed by the set up of the basic framework in which we address shock waves in GR, in Section 3. In particular we define a point of regular shock wave interaction in SSC. Before we present our main method and results in Section 6, 7, 8, 9 and 10, we discuss in Section 4 the inconclusiveness in deriving a regularity from the procedure of guessing coordinates and using the Einstein equations to read off the regularity. In Section 5 we introduce the weak formulation of the Einstein equations in spherical symmetry and prove that the metric coefficient to the spheres of symmetry is always $C^{1,1}$, showing that the lack of metric smoothness is confined to the (t, r) -plane.

We begin the presentation of our main method by introducing the precise sense in which functions and metrics are said to be only $C^{0,1}$ across a hypersurface in Section 6. We end Section 6 by introducing a canonical form for functions Lipschitz continuous across a hypersurface. The canonical form isolates the Lipschitz regularity from the C^1 regularity of functions in a neighborhood of the hypersurface.

In Section 7 we show that the property of the metric tensor being $C^{0,1}$ across a shock surface is covariant under C^2 coordinate transformations, but not under $C^{1,1}$

transformations and we use this lack of covariance to derive conditions on the Jacobians of general $C^{1,1}$ coordinate transformations necessary and sufficient to lift the regularity of a metric tensor from $C^{0,1}$ to C^1 at points on a shock surface. This condition enables us to represent all such Jacobians in terms of the canonical form introduced in Section 6, unique up to addition of an arbitrary C^1 function. Specifically, we isolate the properties of the Jacobian which enable it to cancel out all of the discontinuities present in the derivatives of the metric in the original coordinates, under transformation to the new coordinates. The result is a canonical form for the Jacobians of all coordinate transformations that can possibly lift the regularity of the gravitational metric tensor to C^1 .

In Section 8, we give a new constructive proof of Israel's theorem for spherically symmetric spacetimes, by showing directly that the Jacobians expressed in our canonical form do indeed smooth the gravitational metric to $C^{1,1}$ at points on a single shock surface, thereby reproving Israel's Theorem within the framework of our newly developed method. The essential difficulty is to prove that the freedom to add an arbitrary C^1 -function to our canonical form, is sufficient to guarantee that we can meet the integrability condition on the Jacobian required to integrate it up to an actual coordinate system. The main point is that this is achievable within the required C^1 gauge freedom if and only if the RH jump (c.f. [10]) conditions and the Einstein equations hold at the shock interface.

The main step towards Theorem 1.1 is then achieved in Section 9 where we prove that at a point of regular shock wave interaction in SSC there exists no coordinate transformations in the (t, r) -plane, (that is, transformations that keep the angular part fixed), that lift the metric regularity to C^1 . The central method herein is that the C^1 gauge freedom in our canonical forms is insufficient to satisfy the integrability condition on the Jacobians, without forcing the determinant of the Jacobian to vanish at the point of interaction. In Section 10 we extend this result to the full atlas

of spacetime, thereby proving Theorem 1.1. Since we do not know how to make mathematical and physical sense of coordinate transformations less regular than $C^{1,1}$ in general relativity, we conclude that points of shock wave interaction represent a new kind of singularity in spacetime, which we call regularity singularity. We derive the loss of locally inertial frames in Section 11.

2. PRELIMINARIES

In this preliminary section we discuss the basic framework of General Relativity, in particular we introduce the coupled Einstein-Euler equations and the phenomenon of shock waves as solutions of the latter. Furthermore, we give a short review of Israel's result [6, 11], introduce spherically symmetric spacetimes and discuss the Einstein equations in Standard Schwarzschild Coordinates, together with the metric regularity.

The framework of General Relativity is a four dimensional manifold M , together with a Lorentz metric g (i.e. a metric tensor with signature $(-1, 1, 1, 1)$). Requiring spacetime to be a manifold reflects Einstein's original insight of general covariance [1], that is, all physical equations must be formulated as tensor equations, while the signature of the metric induces a notion of time and causality. Around each point $p \in M$ there exists an open neighborhood \mathcal{N}_p (called a coordinate patch) and a homeomorphism

$$x = (x^0, \dots, x^3) : \mathcal{N}_p \rightarrow \mathbb{R}^4$$

that defines coordinates on its image in \mathbb{R}^4 . The collection of all such neighborhoods (that cover the whole spacetime) and corresponding homeomorphisms is called an *atlas*. If the intersection of two coordinate patches is nonempty, $x \circ y^{-1}$ defines a mapping from an open region in \mathbb{R}^4 to an open region in \mathbb{R}^4 (often referred to as a coordinate transformation) and one obtains a notion of the differentiability of the mapping $x \circ y^{-1}$ in terms of partial derivatives in \mathbb{R}^4 . Given that all coordinate transformations in the atlas are C^k -diffeomorphisms, then one calls the manifold M a C^k -manifold and its atlas a C^k -atlas. In General Relativity and Riemannian geometry the manifold is usually assumed to be C^2 , however, in this paper we consider $C^{1,1}$ manifolds, since this low level of regularity is crucial in order to address shock wave solutions of the Einstein Euler equations in an appropriate way. We show in section 7 that lowering the regularity to $C^{1,1}$ is the crucial step that allows for a smoothing

of the metric in the presence of a single shock wave, (c.f. Theorem 8).

In this thesis we use the Einstein summation convention, that is, we sum over repeated upper and lower indices, and we use the type of index to indicate in what coordinates we consider a tensor to be expressed in. For instance T_ν^μ denotes a $(1, 1)$ -tensor in coordinates x^μ and T_j^i denotes the same tensor in coordinates x^j , under a change of coordinates both are related by the covariant transformation rule

$$(2.1) \quad T_\nu^\mu = J_i^\mu J_\nu^j T_j^i,$$

where we define the *Jacobian* J as follows

$$(2.2) \quad J_j^\mu := \frac{\partial x^\mu}{\partial x^j}.$$

The transformation law of the form (2.1) defines a tensor, expressing its covariance. A particular tensorial transformation is the one of the metric tensor

$$(2.3) \quad g_{\mu\nu} = J_\mu^i J_\nu^j g_{ij},$$

which is crucial for the problem we address in this article, namely the question whether there exists a coordinate transformation (or equivalently a Jacobian) that lifts the metric regularity from $C^{0,1}$ to $C^{1,1}$ or not. We use the convention to raise and lower tensor indices with the metric, for example

$$T_j^i = g_{jl} T^{li},$$

and denote with g^{ij} the inverse of the metric, defined via the equation

$$(2.4) \quad g_{il} g^{lj} = \delta_i^j.$$

For a set of functions J_β^μ to be a Jacobian it is necessary and sufficient to satisfy (see appendix A for details)

$$(2.5) \quad J_{i,j}^\mu = J_{j,i}^\mu$$

$$(2.6) \quad \det(J_j^\mu) \neq 0,$$

where $f_{,j} := \frac{\partial f}{\partial x^j}$ denotes partial differentiation with respect to the coordinate x^j . Condition (2.5) ensures that J_j^μ is integrable to coordinates x^μ and condition (2.6) ensures that tensors do not become singular (in the sense of a vanishing determinant) under a change of coordinates, as well as the invertibility of the coordinate functions. We refer to the PDE (2.5) as the *integrability condition*.

We now discuss the *Einstein Field equations* that govern the gravitational field through coupling the spacetime curvature to the energy (and matter) contained in it. The Einstein Field equations read [1]

$$(2.7) \quad G^{ij} = \kappa T^{ij}$$

where $\kappa := 8\pi\mathcal{G}$ incorporates Newton's gravitational constant \mathcal{G} into the equation and

$$(2.8) \quad G^{ij} = R^{ij} - \frac{1}{2}Rg^{ij} + \Lambda g^{ij}$$

is the Einstein tensor, Λ denotes the cosmological constant. Our method applies regardless the choice of $\Lambda \in \mathbb{R}$, since the term Λg^{ij} in the Einstein equations (2.7) is continuous. The Ricci tensor R_{ij} is defined to be the trace of the Riemann tensor $R_{ij} = R_{ilj}^l$ and the trace of the Ricci tensor $R = R^l_l$ is called the scalar curvature. The metric tensor g_{ij} enters the Einstein equations through the Christoffel symbols

$$\Gamma_{jk}^i = \frac{1}{2}g^{li}(g_{jl,k} + g_{kl,j} - g_{jk,l}),$$

since the Riemann curvature tensor can be expressed as

$$R_{jkl}^i = \Gamma_{jk,l}^i - \Gamma_{jl,k}^i + \Gamma_{ml}^i \Gamma_{jk}^m - \Gamma_{mk}^i \Gamma_{jl}^m.$$

Taking into account the symmetry of the metric tensor $g_{ij} = g_{ji}$, (and the Ricci tensor $R_{ij} = R_{ji}$), the Einstein equations are a set of 10 second order differential equations on g_{ij} . For the Einstein tensor to be defined in a strong sense a C^2 metric regularity is necessary, however, at the level of shock waves only Lipschitz continuity is given in general and one must introduce the Einstein tensor in a weak (distributional) sense.

We describe the weak form of the Einstein equations in Section 5 in the case of a spherically symmetric spacetime in SSC, see [11] for a treatment of the weak formalism in a general manifold.

By construction, the Einstein tensor is divergence free

$$(2.9) \quad G^{ij}{}_{;j} = 0,$$

where the semicolon in (2.9) denotes covariant differentiation, that is

$$(2.10) \quad v^i{}_{;j} := v^i{}_{,j} + \Gamma^i_{lj} v^l$$

for a vector field v^i . Through (2.9) the Einstein equations ensure conservation of energy in the matter source

$$(2.11) \quad T^{ij}{}_{;j} = 0,$$

which was one of the guiding principles Einstein followed in the construction of the Einstein tensor [1]. In the case of a perfect fluid

$$(2.12) \quad T^{ij} = (p + \rho)u_i u_j + p g_{ij},$$

(in units where $c = 1$), (2.11) are the general relativistic *Euler* equations, with ρ being the density, p the pressure and u^i the tangent vector of the fluid flow (see for example [13]). After imposing an equation of state $p = p(\rho)$ and fixing the parametrization in the tangent vectors of the fluid flow u^i , for example such that $u^j u_j = -1$, the system closes in the sense that the number of unknowns (ρ and u^i) matches the number of equations. If in some coordinates the metric at a point p equals the Minkowski metric η_{ij} to second order, that is

$$(2.13) \quad \begin{aligned} g_{ij}(p) &= \eta_{ij} \\ \text{and } g_{ij,l}(p) &= 0, \end{aligned}$$

then (2.11) reduce to the special relativistic Euler equations

$$T^{ij}{}_{,j} = 0$$

at the point p . Considering g_{ij} , ρ and u^j as unknowns, the Einstein equations (2.7) together with the general relativistic Euler equation (2.11), form the coupled *Einstein-Euler* equations. Imposing an equation of state $p = p(\rho)$ the system closes and the coupled Einstein-Euler equations form a set of fourteen partial differential equations in fourteen unknowns.

The Euler equations (2.11) are a system of conservation laws, which do not only allow for discontinuous (shock wave) initial data, but moreover, as Riemann himself first showed, shock waves form from smooth solutions that are sufficiently compressive, [10, 8]. This makes the study of shock waves unavoidable for perfect fluids, arguing that the result of this paper is fundamental to General Relativity. Being discontinuous, the solutions can satisfy the conservation law only in the weak sense. To obtain the weak formulation of the equations, multiply the equation with a smooth test function of compact support and integrate the equation afterwards in order to shift the derivatives to the test function using the divergence theorem (c.f. [11]). In the case of the Euler equations the weak form reads

$$(2.14) \quad \int_M T^{ij} \varphi_{;j} d\mu_M,$$

where $\varphi \in C_0^\infty(M)$ is a test function and $d\mu_M$ is the volume element. Once the discontinuity forms across a hypersurface Σ (the so-called shock surface) the solution satisfies the Rankine Hugoniot jump conditions (or RH condition),

$$(2.15) \quad [T^{\mu\nu}]N_\nu = 0,$$

where N^ν is normal to the hypersurface Σ and $[u] := u_L - u_R$ denotes the difference of the left and right limit (to Σ) and is referred to as the jump in u across Σ . In fact, if T^{ij} is a strong solution everywhere away from the hypersurface and satisfies the jump conditions (2.15), then this is equivalent to T^{ij} being a weak solution in the whole region. Thus, one can avoid using the weak formalism and instead use the jump conditions and the strong solution everywhere away from the shock surface. In this

article we work on the shock solutions only in this way, not using the weak formalism. We finally want to point out that we do not specifically use a perfect fluid source in our methods, but only assume that T^{ij} is bounded and continuous away from some hypersurfaces, across which it satisfies the Rankine Hugoniot jump condition (2.15), (see Definition 3.1 for details).

For the Einstein tensor to be defined in a strong almost everywhere sense one needs a $C^{1,1}$ regularity, however, at the level of shock waves in SSC the gravitational metric tensor is Lipschitz continuous and can only satisfy the Einstein equations in a weak (distributional) sense, (we derive this below, see also [3]). Israel proved in his 1966 paper [6] that in the presence of a *single* smooth shock surface one recovers the $C^{1,1}$ metric regularity by a coordinate transformation if and only if the energy momentum tensor is bounded almost everywhere. If so, the Rankine Hugoniot jump conditions hold everywhere on the shock surface, moreover the RH conditions are even equivalent to the existence of coordinates x^i where the metric is $C^{1,1}$ if spacetime is spherically symmetric, as shown by Smoller and Temple [11]. The theorem is stated for a n -dimensional Riemannian manifold but applies to Lorentz manifold analogously as long that the shock surface is not null. The essence of the proof is to construct Gaussian Normal Coordinates with respect to the shock surface Σ , that is, we first arrange by a smooth coordinate transformation that locally $\Sigma = \{p \in M : x^n(p) = 0\}$, then the coordinate vector $\frac{\partial}{\partial x^n}$ of the coordinates x^i is normal to the surface. We now define coordinates by mapping a point $p \in M$ (sufficiently close to Σ) to \mathbb{R}^n as follows:

$$(2.16) \quad x^\alpha(p) = (s, x^{n-1}(q), \dots, x^1(q)),$$

where s is the arc-length parameter of a geodesic curve γ starting at the point $q \in \Sigma$ in the direction $\frac{\partial}{\partial x^n}$ normal to Σ , with $\gamma(s) = p$, and $x^\alpha(q) = x^i(q)$ for all $\alpha = i \in \{1, \dots, n-1\}$. Said differently, we define new coordinates x^α by exchanging the n -th coordinate x^n by the geodesic arc-length parameter. The coordinates (2.16)

are called Gaussian Normal Coordinates. Computing now the Einstein tensor in coordinates (2.16), it turns out that each component of the resulting Einstein tensor contains only a single “critical”⁸ second order derivative $g_{\alpha\beta,nn}$, while all other terms in the Einstein equations are in L^∞ and thus $g_{\alpha\beta,nn} \in L^\infty$. Since all other second order derivatives are bounded we conclude that $g_{\alpha\beta} \in C^{1,1}$. (Here we use that Lipschitz continuity of a function is equivalent for it to be in the Sobolev-space $W^{1,\infty}$, containing all functions with almost everywhere bounded first order weak derivatives. Either of these imply that the function is differentiable almost everywhere.) In Section 8 we give a new constructive proof of Israel’s result, based on the method we introduce in Section 7.

In the following we introduce the spherically symmetric spacetimes, and restrict our attention to these. Many exact solutions of fundamental interest to general relativity are spherically symmetric, including the Schwarzschild, Oppenheimer-Volkoff, Reissner-Nordstrom and Friedmann-Robertson-Walker spacetimes, [5]. Assuming spherical symmetry, one can describe many astrophysical objects (like stars or black holes) to a very high accuracy [18]. A spherically symmetric spacetime is a spacetime that allows for two spacelike Killing vector field X^i and Y^i , such that the subspaces parameterized by the flow of the Killing vectors have a positive constant curvature [18]. We refer to those subspaces as the spaces of symmetry, in fact, in suitable coordinates the spaces of symmetry are given by a family of two spheres of smoothly varying radii. A vector field X^j is called a Killing vector if it satisfies Killing’s equations

$$(2.17) \quad X_{i;j} + X_{j;i} = 0.$$

Killing’s equation ensures that the flow of a solution X^i is an isometry of spacetime, since a vector field X^i solves (2.17) if and only if the Lie derivative of the metric in

⁸By this we mean a second order derivative of the metric components that might be distributional. Since the vector $\frac{\partial}{\partial x^n}$ is normal to Σ , only $g_{ij,nn}$ might be distributional, while all other second order metric derivatives are at most discontinuous.

direction of X^i vanishes [5, 19, 18],

$$(2.18) \quad L_X g = 0,$$

which is exactly the condition that the flow of X^i is an isometry of the metric.

A very interesting and from an analytical perspective maybe the most important feature of spherically symmetric spacetimes is that one can always define coordinates ϑ and φ such that the metric takes on the form [18]

$$(2.19) \quad ds^2 = -Adt^2 + Bdr^2 + 2Edtdr + Cd\Omega^2,$$

where

$$d\Omega^2 = d\varphi^2 + \sin^2(\vartheta) d\vartheta^2$$

is the line element on the two-sphere and the metric coefficients A , B , C and E only depend on t and r . This simplifies the metric a lot, because its original ten free components reduce to four and the ten Einstein equations reduce to four independent ones accordingly (c.f. Section 5). One can simplify the metric further by introducing a new “radial” variable $r' := \sqrt{C}$ and removing the off-diagonal element by an appropriate coordinate transformation in the (t, r') -plane [18], denoting the resulting coordinates again by t and r the new metric reads

$$(2.20) \quad ds^2 = -Adt^2 + Bdr^2 + r^2d\Omega^2.$$

The coordinates in which the metric is given by (2.20) are called Standard Schwarzschild Coordinates (SSC), in these coordinates the metric has only two free components and the Einstein equations simplify significantly:

$$(2.21) \quad B_r + B \frac{B-1}{r} = \kappa AB^2 r T^{00}$$

$$(2.22) \quad B_t = -\kappa AB^2 r T^{01}$$

$$(2.23) \quad A_r - A \frac{1+B}{r} = \kappa AB^2 r T^{11}$$

$$(2.24) \quad B_{tt} - A_{rr} + \Phi = -2\kappa AB r^2 T^{22},$$

with

$$\Phi := -\frac{BA_t B_t}{2AB} - \frac{B_t^2}{2B} - \frac{A_r}{r} + \frac{AB_r}{rB} + \frac{A_r^2}{2A} + \frac{A_r B_r}{2B}.$$

The first three Einstein equations in SSC are central in the method we develop and in the proof of Theorem 9.1.

We now discuss the atlas of a spherically symmetric spacetime, that is the set of all local diffeomorphisms on a spacetime M . Since the Killing equation is covariant the requirement of spherical symmetry does not impose any restriction on the atlas. However, it is desirable to preserve the metric structure (2.19) under a change of coordinates in order to keep the analysis of the Einstein equations and the local structure of spacetime simple. A particularly natural class of coordinate transformations, that preserve the metric structure (2.19) are the transformations in the (t, r) -plane, (referring to the variables t and r in (2.19)), that keep the angular variables ϑ and φ fixed. In Theorem 9.1 we only consider the atlas of $C^{1,1}$ regular coordinate transformations in the (t, r) -plane and in Theorem 10.1 we extend the result to the full $C^{1,1}$ atlas of a spherically symmetric spacetime.

Let us address the issue of the metric regularity in the presence of shock waves in the coupled Einstein Euler equations. A shock wave is a weak solution T^{ij} of the Euler equations that is discontinuous across a timelike hypersurface Σ (the so-called shock surface), in particular it is a strong solution away from the surface and the discontinuity across Σ satisfies the jump conditions (2.15). However, turning towards the first three Einstein equations in SSC (2.21)-(2.23), it is straightforward to read off that the metric cannot be any smoother than Lipschitz continuous if the matter source $T^{ij} \in L^\infty$ is discontinuous. In this paper we henceforth assume that the gravitational metric in SSC is Lipschitz continuous, since this provides a consistent framework to address shock waves in General Relativity and this assumption agrees

with various examples of solutions to the coupled Einstein Euler equations, for instance see [11] or [3]⁹. Moreover, Lipschitz continuity arises naturally in the general problem of matching two spacetimes across a hypersurface, as first considered by Israel in [6]. As mentioned above, Israel proved the rather remarkable result that whenever a metric is Lipschitz continuous across a smooth single shock surface Σ and has a almost everywhere bounded Einstein tensor, then there always exists a coordinate transformation defined in a neighborhood of Σ , that smooths the components of the gravitational metric to $C^{1,1}$. Part of the precise result is that the gravitational metric is smoothed to $C^{1,1}$ in Gaussian Normal Coordinates if and only if the second fundamental form of the metric is continuous across the surface. The latter is an invariant condition meaningful for metrics Lipschitz continuous across a hypersurface, and is often referred to in the literature as the *junction condition*, c.f. [18]. In [11], Smoller and Temple showed that in spherically symmetric spacetimes, the junction conditions hold across radial surfaces if and only if the single $[T^{ij}]n_in_j = 0$, implied by (2.15), holds. Thus, for example, single radial shock surfaces can be no smoother than Lipschitz continuous in SSC, but can be smoothed to $C^{1,1}$ by coordinate transformation. However, it has remained an open problem whether or not such a theorem applies to the more complicated $C^{0,1}$ SSC solutions proven to exist by Groah and Temple [3]. Our purpose here is to show that such solutions cannot be smoothed to C^1 in a neighborhood of a point of regular shock wave *interaction*, a notion we make precise in Section 3.

⁹It is not clear to us if solutions with a lower regularity could exist. Even though B must be in $C^{0,1}$ by (2.21) one cannot rule out that A_t might be unbounded, but we expect this case to be rather pathological. Moreover, if A_t were unbounded, the metric would not be $C^{1,1}$ in any coordinate system, that could be reached within a $C^{1,1}$ atlas, since for Lipschitz continuous Jacobians J_μ^j and $g_{ij} \in C^{1,1}$ the metric in coordinates x^μ must be Lipschitz continuous, due to the covariant transformation rule (2.3).

3. A POINT OF REGULAR SHOCK WAVE INTERACTION IN SSC

In this section we first set up our basic framework of radial shock surfaces and then give the definition of a point of regular shock wave interaction in SSC, that is, a point p where two distinct shock waves enter or leave the point p with distinct speeds. Throughout this paper we restrict attention to radial shock waves,¹⁰ that is, shock surfaces Σ that can (locally) be parameterized by

$$(3.1) \quad \Sigma(t, \vartheta, \varphi) = (t, x(t), \vartheta, \varphi),$$

and across which the stress-energy-momentum tensor T is discontinuous. Note that if t is timelike, then all timelike shock surfaces in SSC can be so parameterized. Our subsequent methods apply to spacelike and timelike surfaces alike, (inside or outside a black hole, c.f. [12]), in the sense that t can be timelike or spacelike (depending on the signs of the metric coefficient), but without loss of generality and for ease of notation we restrict to timelike surfaces in the remainder of this paper, surfaces parameterized as in (3.1).

For radial hypersurfaces in SSC, the angular variables play a passive role, and the essential issue regarding smoothing the metric components by $C^{1,1}$ coordinate transformation lies in the atlas of coordinate transformations acting on the (t, r) -plane. (In fact, for the proof of Theorem 8.1 and 9.1 it suffices to consider the (t, r) -plane only and it is not that difficult to extend the result to the full atlas and thus obtain Theorem 1.1.) Therefore it is sufficient to work with the so-called shock curve γ , that is, the shock surface Σ restricted to the (t, r) -plane,

$$(3.2) \quad \gamma(t) = (t, x(t)),$$

with normal 1-form

$$(3.3) \quad n_\nu = (\dot{x}, -1).$$

¹⁰The results in Section 5 do not explicitly rely on the assumption of radial shock surfaces.

Then, for radial shock surfaces (3.1) in SSC, the RH jump conditions (2.15) take the simplified form

$$(3.4) \quad [T^{00}] \dot{x} = [T^{01}]$$

$$(3.5) \quad [T^{10}] \dot{x} = [T^{11}]$$

In Section 9 we prove the main result of this paper by establishing that SSC metrics that are only Lipschitz continuous across *two* intersecting shock curves, cannot be smoothed to $C^{1,1}$ by a $C^{1,1}$ coordinate transformation. To establish our basic framework, suppose two timelike shock surfaces Σ_i are parameterized in SSC by

$$(3.6) \quad \Sigma_i(t, \theta, \phi) = (t, x_i(t), \theta, \phi),$$

for $i = 1, 2, 3, 4$, where Σ_1 and Σ_2 are defined for $t \leq 0$ and Σ_3 and Σ_4 are defined for $t \geq 0$, described in the (t, r) -plane by,

$$(3.7) \quad \gamma_i(t) = (t, x_i(t)),$$

with normal 1-forms

$$(3.8) \quad (n_i)_\nu = (\dot{x}_i, -1).$$

For the proof of our main result (Theorem 9.1) it suffices to restrict attention to the lower ($t < 0$) or upper ($t > 0$) part of a shock wave interaction that occurs at $t = 0$. That is, it suffices to impose conditions on either either the lower or upper half plane

$$\mathbb{R}_-^2 = \{(t, r) : t < 0\},$$

or

$$\mathbb{R}_+^2 = \{(t, r) : t > 0\},$$

respectively, whichever half plane contains two shock waves that intersect at p with distinct speeds. (We denote with $\overline{\mathbb{R}_\pm^2}$ the closure of \mathbb{R}_\pm^2 .) Thus, without loss of

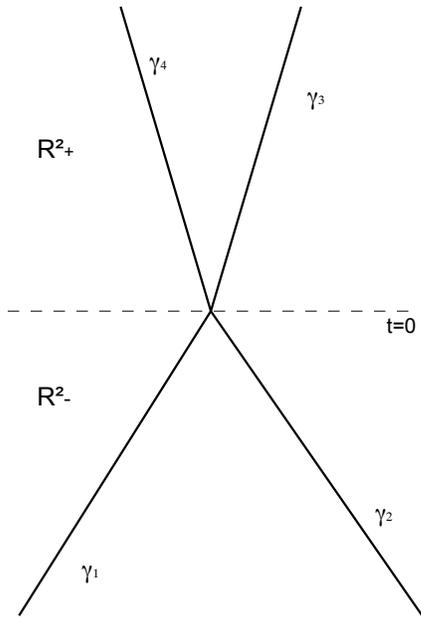


FIGURE 1. Example of intersecting shock curves.

generality, let $\gamma_i(t) = (t, x_i(t))$, ($i = 1, 2$), be two shock curves in the lower (t, r) -plane that intersect at a point $(0, r_0)$, $r_0 > 0$, i.e.

$$(3.9) \quad x_1(0) = r_0 = x_2(0).$$

With this notation, we can now give a precise definition of what we call a point of regular shock wave interaction in SSC. By this we mean a point p where two distinct shock waves enter or leave the point p with distinct speeds. The structure makes precise what one would generically expect, namely, that the metric is a smooth solution of the Einstein equations away from the interacting shock curves, the metric is Lipschitz continuous and the RH jump conditions hold across each shock curve, and derivatives are continuous up to the boundary on either side. In particular, the definition reflects the regularity of shock wave solutions of the coupled Einstein-Euler equations consistent with the theory in [3] and confirmed by the numerical simulation in [17]. Without loss of generality we assume a lower shock wave interaction in \mathbb{R}_-^2 .

Definition 3.1. Let $r_0 > 0$, and let $g_{\mu\nu}$ be an SSC metric in $C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$, where $\mathcal{N} \subset \mathbb{R}^2$ is a neighborhood of a point $p = (0, r_0)$ of intersection of two timelike shock curves $\gamma_i(t) = (t, x_i(t)) \in \mathbb{R}^2$, $t \in (-\epsilon, 0)$. Assume the shock speeds $\dot{x}_i(0) = \lim_{t \rightarrow 0} \dot{x}_i(t)$ exist and are distinct, $\dot{x}_1(0) \neq \dot{x}_2(0)$, and let $\hat{\mathcal{N}}$ denote the neighborhood consisting of all points in $\mathcal{N} \cap \mathbb{R}^2_-$ not in the closure of the two intersecting curves $\gamma_i(t)$. Then we say that p is a point of regular shock wave interaction in SSC if:

- (i) The pair (g, T) is a strong solution of the SSC Einstein equations (2.21)-(2.24) in $\hat{\mathcal{N}}$, with $T^{\mu\nu} \in C^0(\hat{\mathcal{N}})$ and $g_{\mu\nu} \in C^2(\hat{\mathcal{N}})$.
- (ii) The limits of T and of metric derivatives $g_{\mu\nu, \sigma}$ exist on both sides of each shock curve $\gamma_i(t)$ for all $-\epsilon < t < 0$.
- (iii) The jumps in the metric derivatives $[g_{\mu\nu, \sigma}]_i(t)$ are C^1 function with respect to t for $i = 1, 2$ and for $t \in (-\epsilon, 0)$.
- (iv) The limits

$$\lim_{t \rightarrow 0} [g_{\mu\nu, \sigma}]_i(t) = [g_{\mu\nu, \sigma}]_i(0)$$

exist for $i = 1, 2$.

- (v) The metric g is continuous across each shock curve $\gamma_i(t)$ separately, but no better than Lipschitz continuous in the sense that, for each i there exists μ, ν such that

$$[g_{\mu\nu, \sigma}]_i(n_i)^\sigma \neq 0$$

at each point on γ_i , $t \in (-\epsilon, 0)$ and

$$\lim_{t \rightarrow 0} [g_{\mu\nu, \sigma}]_i(n_i)^\sigma \neq 0.$$

- (vi) The stress tensor T is bounded on $\mathcal{N} \cap \overline{\mathbb{R}^2_-}$ and satisfies the RH jump conditions

$$[T^{\nu\sigma}]_i(n_i)_\sigma = 0$$

at each point on $\gamma_i(t)$, $i = 1, 2$, $t \in (-\epsilon, 0)$, and the limits of these jumps exist up to p as $t \rightarrow 0$.

The structure pinned down in Definition 3.1 reflects the regularity of shock wave solutions of the coupled Einstein Euler equations, in fact, this structure is even forced

upon us if we require it to include generic shock wave solutions, (e.g. the solutions in [11, 3, 14, 15, 16, 17]). It is consistent with the theory in [3] and consistent with the numerical simulation in [17]. Furthermore, the structure we defined is general enough to include the case of two interacting shock waves of opposite families that cross at the point p , (two such shock waves typically change their speeds discontinuously at the point of interaction), but also general enough to include the cases of two outgoing shock waves created by the focusing of compressive rarefaction waves (like those simulated in [17]), or two incoming shock waves of the same family that interact at p to create an outgoing shock wave of the same family and an outgoing rarefaction wave of the opposite family, (c.f. [10] for a detailed discussion of shock wave interactions).

The structure we assume in Definition 3.1 reflects the structure of various shock wave solutions of the Einstein-Euler equations [11, 12, 13, 14, 15], and the theory in [3] and numerics in [17] confirm that points of shock wave interaction exhibit the structure identified in Definition 3.1. However, even though the Groah-Temple theory [3] establishes existence of $C^{0,1}$ shock waves before and after interaction, and the work of Vogler numerically simulates the detailed structure of the metric at a point of shock wave interaction, the mathematical theory still lacks a complete mathematical proof that establishes rigorously the detailed structure of shock wave interactions summarized in Definition 3.1. Such a proof would be very interesting, and remains to be done.

We end this section by discussing the hypotheses in Definition 3.1 in more detail. Note that we do not explicitly assume a perfect fluid source in Definition 3.1, only an energy momentum tensor $T^{\mu\nu}$ that satisfies the Rankine Hugoniot jump condition. We restricted to timelike shock curves to ease notation but, in fact, our main result and Definition 3.1 applies to timelike and spacelike shock wave interactions alike. (However, our method of proof breaks down for null hypersurfaces.) Our assumption in (i) reflects the standard hypothesis in the theory of shock waves, namely that

solutions are smooth away from the shock curves [10, 8, 11]. A loss of C^2 metric regularity away from the shock surfaces would in general give rise to discontinuities in the matter source, being a potential source of new shock surfaces. In order to isolate the problem in its simplest and cleanest setting we want to exclude these pathological cases. Furthermore, requiring the existence of the limit towards p in (vi) as well as imposing the RH jump conditions to hold across each shock curve is exactly what is assumed in the theory of shock waves in flat space. A loss of continuity in $T^{\mu\nu}$ along the shock surfaces would give rise to phenomena similar to shock wave interactions, which one could always avoid by restricting to a smaller neighborhood of p . (Note that we do not require $T^{\mu\nu}$ to match up continuously at p .) The structure assumed on the metric tensor in (ii) and (iv) is consistent with the assumption of a bounded energy momentum tensor in the Einstein equations, since a blow up in any of those limits violates in general the boundedness of T .

4. THE OBSTACLES TO COORDINATE SMOOTHING AT POINTS OF SHOCK WAVE INTERACTION

The central point in Israel's proof that metrics can be smoothed at points on single shock surfaces, is that Gaussian Normal Coordinates at the shock smooth the metric to $C^{1,1}$. This raises the question as to whether such coordinates exist at points of shock wave interaction. The results of this paper demonstrate that no such coordinates exist, but in this section we illustrate the difficulty and indicate why the direct approach of Israel is inconclusive. (This section does not contain any results important for the main part of this dissertation (Section 6-11) and can be skipped.)

Around a point p of shock wave interaction Gaussian Normal Coordinates (with respect to any of the two shock surfaces) are not a suitable choice, since across the intersection the metric tensor cannot be C^2 tangential to any of the shock surfaces, but this is crucial for Israel's method to work [11]. Moreover, at the point of interaction the unit normal vectors of the shock surfaces are (in general) discontinuous, but to construct Gaussian Normal Coordinates with respect to some hypersurface it is crucial for the unit vectors normal to this surface to be continuous, in order to reach every point in a sufficiently small neighborhood around p with a geodesic curve perpendicular to the surface (c.f. Section 2). Thus, in physically relevant cases (e.g. shocks governed by the Euler equation) Gaussian Normal Coordinates cannot be constructed around a point of shock wave interaction. We conclude that to smooth the metric around a point of shock wave interaction Gaussian Normal Coordinates are not suitable and one should choose different coordinates.

We now choose specific coordinates (different from Gaussian Normal Coordinates) and study the Einstein equations to illustrate that this procedure is inconclusive regarding the question of whether one can or cannot lift the metric regularity to $C^{1,1}$. It was pointed out in [4] that the change of coordinates to SSC causes a loss of regularity, since one defines the metric coefficient C to be the radial variable r , thus

derivatives of C enter the Jacobian and the metric in SSC. If the original metric is only $C^{1,1}$, then the metric in SSC (2.20) is only Lipschitz continuous. From this one might expect that the high C^∞ regularity of $C = r^2$ forces the coefficient A and B to assume a very low regularity and allowing C to be only $C^{1,1}$ regular could yield a coordinate system in which the Einstein Equation reduce to a form that yields a $C^{1,1}$ regularity for all metric coefficients. Furthermore, for the analysis of the Einstein equations to be feasible we want to choose coordinates in which the metric assumes a simple structure, namely

$$(4.1) \quad ds^2 = A(\tau, x) (dx^2 - d\tau^2) + C(\tau, x)d\Omega^2,$$

(see Appendix B for the existence of a coordinate transformation to this metric). The Einstein equations in those coordinates take on the form

$$(4.2) \quad \begin{aligned} C_{xx} &= (l.o.t.) + \kappa A^2 CT^{00} \\ C_{\tau x} &= (l.o.t.) - \kappa A^2 CT^{01} \\ C_{\tau\tau} &= (l.o.t.) + \kappa A^2 CT^{11} \\ A_{\tau\tau} - A_{xx} &= (l.o.t.) + 2\kappa A^2 CT^{22}, \end{aligned}$$

where *(l.o.t.)* denotes “lower order terms” in the sense that those terms are in L^∞ for a Lipschitz continuous metric, since *(l.o.t.)* contains only first order derivatives of the metric and the metric itself. Assuming now that the metric (4.1) is Lipschitz continuous and that $T^{\alpha\beta}$ is in L^∞ we conclude from the first three equations in (4.2) that $C \in C^{1,1}$. All second order derivatives on the metric coefficient A appear in the fourth equation and one would expect only this equation to yield a higher regularity for A . However, this is a wave equation and the regularity depends crucially on the initial data and on the right hand side. Treating the right hand side as a source not depending on A , the method of characteristics suggests that in order to obtain $A \in C^{1,1}$ one needs $C^{1,1}$ initial values, as well as a right hand side that is Lipschitz continuous. We do not see any reason why any of these requirements should hold, unless there

exist a coordinate system in which the metric is $C^{1,1}$. Moreover, it seems impossible to align the derivatives with the shock surfaces in some way such that the fourth equation yields a higher regularity in the above coordinates. This suggests that the analysis of the Einstein equations (4.2) is inconclusive regarding the metric regularity.

The above consideration illustrate a fundamental obstacle one faces (in most coordinate systems) analyzing the Einstein equations regarding the metric regularity around a point of shock wave interaction: Unlike in the case of a single shock wave it is not possible to align the coordinate to the shock surfaces in such a way that the critical (discontinuous) derivatives are isolated and it is not clear what the regularity of neither the initial values nor the source terms are. In summary, we conclude that the method of choosing coordinates and analyzing the Einstein equations is inconclusive. However, the above considerations suggest that the metric coefficient C is always in $C^{1,1}$, in fact, we prove this in Section 5.

5. SMOOTHNESS CLASS OF THE RADIAL COMPONENT $C(t, r)$

We now consider the spherically symmetric Einstein equations in coordinates where the metric is of the general box-diagonal form (2.19), in order to prove that the metric component C to the spheres of symmetry always assumes a $C^{1,1}$ regularity. The strategy is as follows: Computing the Einstein equations in those coordinates, we solve for the second order derivatives of C in the first three Einstein equations, which enables us to isolate each of those derivatives and concluding that $C \in C^{1,1}$, with respect to the coordinates at hand. In addition we define the weak formulation of the Einstein equations in those coordinates and discuss its equivalence to different weak formalisms. Even though this section contains some results and a short analysis of the Einstein equations, it is of no relevance for the main part of this dissertation (Section 6-11) and can be skipped.

Suppose for the moment that the metric is C^2 in given coordinates $(t, r, \vartheta, \varphi)$, defined on some open region \mathcal{N} . For a metric of the form (2.19) we compute the first three Einstein Field equations to be given by

$$\begin{aligned}
 E^2 C_{tt} + 2AEC_{tr} + A^2 C_{rr} &= l.o.t. + |g|C\kappa T_{00} \\
 BEC_{tt} + (AB - E^2) C_{tr} - AEC_{rr} &= l.o.t. + |g|C\kappa T_{01} \\
 (5.1) \quad B^2 C_{tt} - 2BEC_{tr} + E^2 C_{rr} &= l.o.t. + |g|C\kappa T_{11}
 \end{aligned}$$

where the indices t and r denote respective partial derivation, $|g| := -AB - E^2$ is the determinant of the upper metric entries and *l.o.t.* denotes “lower order terms” in the derivatives on the metric, that is, first order derivatives of the metric and the metric components themselves. (We used Maple to compute the Einstein tensor.) Considering this as a linear system in C_{ij} and using Gaussian elimination we write (5.1) equivalently as

$$\begin{aligned}
 C_{rr} &= l.o.t. + \kappa C |g| T^{00} \\
 C_{tr} &= l.o.t. - \kappa C |g| T^{01} \\
 (5.2) \quad C_{tt} &= l.o.t. + \kappa C |g| T^{11}.
 \end{aligned}$$

It is quite remarkable that the operation performed on the energy momentum tensor T_{ij} within the Gaussian elimination turn out to be exactly the raising of the indices via the metric $T^{ij} = g^{ik}g^{jl}T_{kl}$. The advantage of (5.2) above (5.1) is that it is straightforward to read off the regularity of C . For an almost everywhere bounded energy momentum tensor $T^{\mu\nu} \in L^\infty(\mathcal{N})$ and $g_{\mu\nu} \in C^{0,1}(\mathcal{N})$ the right hand side in (5.2) is in L^∞ and we conclude that C is in the Sobolev space $W^{2,\infty}$, which is equivalent to $C \in C^{1,1}(\mathcal{N})$ (see [2]). We will prove this result from the weak formulation below.

As a set of second order differential equations the Einstein equations can be solved by a Lipschitz continuous metric only in the weak sense. We now set up and discuss the weak formulation of the Einstein Field Equations in spherical symmetry for a metric of the form (2.19). To obtain a weak form of (5.2) we multiply the equations with a smooth test function ϕ of compact support, integrate the resulting expressions and use the divergence theorem to shift all second order derivatives to the test-function, such that only first order metric derivatives are left [11, 2]. One can perform the integration with respect to a flat or “curved” volume form, that is, $dx^0\dots dx^3$ or $\sqrt{|\det(g)|}dx^0\dots dx^3$, since both are related through a multiplication of the equations in (5.2) by a factor of $\sqrt{|\det(g)|}$. (Moreover, it is shown in [11] that a weak solution $g_{\alpha\beta}$ of the Einstein equations in a given coordinate system is also a weak solution in any other coordinate system that can be reached within a $C^{1,1}$ atlas.) However, in general we prefer using the curved volume form $\sqrt{|\det(g)|}dx^0\dots dx^3$, since then the integral transforms as a scalar and only then the weak form of the relativistic Euler equation is given by (2.14). From Proposition 5.2 we conclude that the first three Einstein equations (5.2) actually hold in a strong sense in any coordinate system and we only need a weak formulation of the fourth equation. The fourth Einstein equations reads

$$(5.3) \quad -A_{rr} + 2E_{tr} + B_{tt} = l.o.t.,$$

where *l.o.t.* includes all terms that are in L^∞ , thus by the above consideration also containing second order derivatives of C . The *quasi-linear* structure of the Einstein equations ensures that one can always define a weak formulation for Lipschitz continuous metrics, since one can shift all second order derivatives via the divergence theorem, obtaining only first order derivatives on the metric and the test function. We obtain the weak form of the fourth Einstein equation (5.3) by the above mentioned procedure, it is given by

$$(5.4) \quad \int_{\mathcal{N}} (A_r \phi_r - 2E_t \phi_r - B_t \phi_t - (l.o.t)\phi) d\mu = 0,$$

where $\phi \in C_0^\infty(\mathcal{N})$ is a smooth test-function with compact support defined on some open set \mathcal{N} and $d\mu$ is either the volume form or some flat measure.

In the remainder of this section we prove that C is always $C^{1,1}$ regular in the presence of an almost everywhere bounded source and that (5.2) always hold in the strong sense. For completeness we prove the following Lemma, that shows that a function with almost everywhere bounded second order derivatives is in $C^{1,1}$.

Lemma 5.1. *Let Ω be an open set in \mathbb{R}^n with a C^1 boundary, and let $f^{\alpha\beta} \in L^\infty(\Omega)$ for $\alpha, \beta \in \{1, \dots, n\}$. Suppose $C \in C^{0,1}(\Omega)$ with $\frac{\partial^2 C}{\partial x^\alpha \partial x^\beta} = f^{\alpha\beta} \quad \forall \alpha, \beta = 1, \dots, n$ in the weak sense, then $C \in C^{1,1}(\Omega)$ and the above second order derivatives hold in the strong almost everywhere sense.*

Proof. The regularity assumption on the boundary of Ω is a sufficient condition to prove the equivalence of the space $C^{0,1}(\Omega)$ of Lipschitz continuous functions and the Sobolev space $W^{1,\infty}(\Omega)$, containing all functions with weak derivative in L^∞ , where we take the continuous representative of each equivalence class [2]. Both imply that C is differentiable point-wise almost everywhere. It remains to show that C is in $W^{2,\infty}$, the Sobolev space of functions with almost everywhere bounded second order weak

derivatives. For a function $C \in W^{1,\infty}(\Omega)$ the weak formulation of $\frac{\partial^2 C}{\partial x^\alpha \partial x^\beta} = f$ reads

$$\int_{\Omega} C \varphi_{\alpha\beta} d\mu = \int_{\Omega} f^{\alpha\beta} \varphi d\mu \quad \forall \varphi \in C_0^\infty(\Omega),$$

where $d\mu$ denotes some Borel measure and $C_0^\infty(\Omega)$ the space of smooth functions with compact support in Ω . This shows that $f^{\alpha\beta} \in L^\infty$ is the weak derivative of C with respect to $\frac{\partial^2}{\partial x^\alpha \partial x^\beta}$ and in summary all second order weak derivatives of C are in $L^\infty(\Omega)$, yielding that C is an element in $W^{2,\infty}(\Omega)$. \square

We now conclude, from Einstein's equation (5.2), that the metric coefficient C is indeed always $C^{1,1}$ regular in all coordinates where the metric assumes the general spherically symmetric form (2.19).

Proposition 5.2. *Let x^α be coordinates such that the spherically symmetric metric $g_{\alpha\beta}$ is of the box-diagonal form (2.19). Assume $g_{\alpha\beta}$ is Lipschitz continuous in coordinates x^α and solves the Einstein equations (5.2) weakly and assume that the energy-momentum tensor is bounded almost everywhere, (that is, $T^{\alpha\beta} \in L^\infty$). Then $C \in C^{1,1}(\Omega)$ with respect to partial differentiation in coordinates x^α and the first three equations in 2.7 hold point-wise almost everywhere.*

Proof. Einstein's equations in the form (5.2) have a right hand side that is bounded almost everywhere. Applying Lemma 5.1 regarding the metric coefficient C on some coordinate neighborhood proves the Theorem. \square

The above proposition shows that the metric coefficient C is always sufficiently smooth for the first three Einstein equations to hold strongly, that is, point-wise almost everywhere. This shows that the lack of $C^{1,1}$ regularity is confined to the metric coefficients A , B and E , being entirely isolated to the (t, r) -plane. This suggests that in order to lift the metric regularity to $C^{1,1}$ it suffices to consider coordinate transformations in the (t, r) -plane. We close this section with the following Corollary, showing that C must be in $C^{1,1} \setminus C^2$ in order to allow for all metric-components to

be in $C^{1,1}$.

Corollary 5.3. *Assume the hypotheses of Proposition 5.2 hold and that the energy momentum tensor $T^{\alpha\beta}$ is discontinuous, then the metric can only be $C^{1,1}$ in a given coordinate system, if the metric coefficient C in (2.19) is not in C^2 .*

Proof. Suppose $g_{\alpha\beta} \in C^{1,1}$ and assume that $C \in C^2$. Then (5.2) yields that for $T^{\alpha\beta}$ to be discontinuous, some of the *lower order terms* must be discontinuous. This can only be the case if some metric coefficients have discontinuous first derivatives, contradicting the ingoing assumptions that $g_{\alpha\beta} \in C^{1,1}$. \square

6. FUNCTIONS $C^{0,1}$ ACROSS A HYPERSURFACE

In order to prove that the gravitational metric cannot be smoothed from $C^{0,1}$ to $C^{1,1}$ by a $C^{1,1}$ coordinate transformation at points of shock wave interaction in general relativity, we first define what it means for a function and a metric to be $C^{0,1}$ across a hypersurface. We then study the relation of a metric being $C^{0,1}$ across a hypersurface to the Rankine Hugoniot jump condition through the Einstein equations and derive a set of three equations that is central in our methods in Sections 7, 8 and 9. We end this section deriving a canonical form functions $C^{0,1}$ across a shock surface can be represented in, which is fundamental in the proof of our main result.

The following definition makes precise what it means for a function and a metric to be $C^{0,1}$ across a (non-null) hypersurface. This helps us to prove in Section 7 that the central expression of this definition is covariant under C^2 but, in general, not under $C^{1,1}$ coordinate transformations. It is exactly this loss of covariance that opens up the possibility to smooth out the metric, in fact, in Section 7 we use this lack of covariance to set up a necessary and sufficient condition on the Jacobian for lifting the metric regularity to $C^{1,1}$, which fails for C^2 coordinate transformations.

Definition 6.1. *Let Σ be a smooth (timelike) hypersurface in some open set $\mathcal{N} \subset \mathbb{R}^d$. We call a function f “Lipschitz continuous across Σ ”, (or $C^{0,1}$ across Σ), if $f \in C^{0,1}(\mathcal{N})$, f is smooth¹¹ in $\mathcal{N} \setminus \Sigma$, and limits of derivatives of f exist and are smooth functions on each side of Σ separately. We call a metric $g_{\mu\nu}$ Lipschitz continuous across Σ in coordinates x^μ if all metric components are $C^{0,1}$ across Σ .*

The main point of the above definition is that we assume smoothness of f , (or $g_{\mu\nu}$), away and tangential to the hypersurface Σ . Note that the continuity of f across Σ

¹¹For us, “smooth” means enough continuous derivatives so that smoothness is not an issue. Thus here, $f \in C^2(\mathcal{N} \setminus \Sigma)$ suffices. In our proof of Theorem 8.1 a C^3 metric regularity away from the shock curve is convenient.

implies the continuity of all derivatives of f tangent to Σ , i.e.,

$$(6.1) \quad [f,_{\sigma}]v^{\sigma} = 0,$$

for all v^{σ} tangent to Σ . Moreover, Definition 6.1 allows for the normal derivative of f to be discontinuous, that is,

$$(6.2) \quad [f,_{\sigma}]n^{\sigma} \neq 0,$$

where n^{σ} is normal to Σ with respect to some (Lorentz-) metric $g_{\mu\nu}$ defined on \mathcal{N} .

In the following we clarify the implications of Definition 6.1, particularly (6.1), together with the Einstein equations on the RH jump conditions (3.4), (3.5). Consider a spherically symmetric spacetime metric (1.1) given in SSC, assume that the first three Einstein equations (2.21)-(2.23) hold, and assume that the stress tensor T is discontinuous across a smooth radial shock surface described in the (t, r) -plane by $\gamma(t)$ as in (3.1)-(3.3). To this end, condition (6.1) across γ applied to each metric component $g_{\mu\nu}$ in SSC (2.20) reads

$$(6.3) \quad [B_t] = -\dot{x}[B_r],$$

$$(6.4) \quad [A_t] = -\dot{x}[A_r].$$

On the other hand, the first three Einstein equations in SSC (2.21)-(2.23) imply

$$(6.5) \quad [B_r] = \kappa AB^2 r [T^{00}],$$

$$(6.6) \quad [B_t] = -\kappa AB^2 r [T^{01}],$$

$$(6.7) \quad [A_r] = \kappa AB^2 r [T^{11}].$$

Now, using the jumps in Einstein equations (6.5)-(6.7), we find that (6.3) is equivalent to the first RH jump condition (3.4),¹² while the second condition (6.4) is independent of equations (6.5)-(6.7), because A_t does not appear in the first order SSC equations (2.21)-(2.23). The result, then, is that in addition to the assumption that the metric

¹²This observation is consistent with Lemma 9, page 286, of [11], where only one jump condition need be imposed to meet the full RH relations.

be $C^{0,1}$ across the shock surface in SSC, the RH conditions (3.4) and (3.5) together with the Einstein equations (6.5)-(6.7), yield only one additional condition over and above (6.3) and (6.4), namely,

$$(6.8) \quad [A_r] = -\dot{x}[B_t].$$

The RH jump conditions together with the Einstein equations will enter our method in Section 7 only through the three equations (6.8), (6.3) and (6.4). In particular, our proofs of Theorems 8.1 and 9.1 below will only rely on (6.8), (6.3) and (6.4).

The following lemma provides a canonical form for any function f that is Lipschitz continuous across a *single* shock curve γ in the (t, r) -plane, under the assumption that the vector n^μ , normal to γ , is obtained by raising the index in (3.3) with respect to a Lorentzian metric g that is $C^{0,1}$ across γ . A direct consequence of Definition 6.1 is then that n^μ varies C^1 in directions tangent to γ . (Again, since we consider spherically symmetric spacetimes, we suppress the angular coordinates).

Lemma 6.2. *Suppose f is $C^{0,1}$ across a smooth curve $\gamma(t) = (t, x(t))$ in the sense of Definition 6.1, $t \in (-\epsilon, \epsilon)$, in an open subset \mathcal{N} of \mathbb{R}^2 . Then there exists a function $\Phi \in C^1(\mathcal{N})$ such that*

$$(6.9) \quad f(t, r) = \frac{1}{2}\varphi(t) |x(t) - r| + \Phi(t, r),$$

if and only if

$$(6.10) \quad \varphi(t) = \frac{[f, \mu]n^\mu}{n^\sigma n_\sigma} \in C^1(-\epsilon, \epsilon),$$

where

$$(6.11) \quad n_\mu(t) = (\dot{x}(t), -1)$$

is a 1-form normal to the tangent vector $v^\mu(t) = \dot{\gamma}^\mu(t)$. In particular, it suffices that indices are raised and lowered by a Lorentzian metric $g_{\mu\nu}$ which is $C^{0,1}$ across γ .

Proof. We first prove the explicit expression for φ in (6.21). Suppose there exists a $\Phi \in C^1(\mathcal{N})$ satisfying (6.16), defining $X(t, r) := x(t) - r$ this implies that

$$(6.12) \quad [f, \mu]n^\mu = \frac{1}{2}\varphi[H(X)]X_{,\mu}n^\mu \quad \text{with} \quad H(X) := \begin{cases} -1 & \text{if } X < 0 \\ +1 & \text{if } X > 0 \end{cases},$$

where we use the fact that $\frac{d}{dX}|X| = H(X)$ and that the jumps of the continuous functions Φ_μ vanish, that is, $[\Phi, \mu] = 0$. Observing that $[H(X)] = 2$ and that $X_{,\mu}n^\mu = n_\mu n^\mu$ by (6.21), we conclude that

$$(6.13) \quad [f, \mu]n^\mu = \varphi n_\mu n^\mu.$$

Solving this equation for φ we obtain the expression claimed in (6.21).

We now prove the reverse direction. Suppose φ , defined in (6.21), is given and is in C^1 . To show the existence of $\Phi \in C^1(\mathcal{N})$ define

$$(6.14) \quad \Phi = f - \frac{1}{2}\varphi|X|,$$

then (6.16) holds and it remains to prove the C^1 regularity of Φ . It suffices to prove

$$[\Phi, \mu]n^\mu = 0 = [\Phi, \mu]v^\mu,$$

since $\Phi \in C^1(\mathcal{N} \setminus \gamma)$ follows immediately from its above definition and the C^1 regularity of f and φ away from γ . By assumption f satisfies (6.1) and thus

$$[\Phi_\mu]v^\mu = -\varphi X_{,\mu}v^\mu,$$

which vanishes since $v^\mu(t) = (1, \dot{x}(t))^T$. The expression for φ , defined in (6.21), together with

$$X_{,\mu}n^\mu = n_\mu n^\mu$$

show that

$$[\Phi_\mu]n^\mu = \varphi n_\mu n^\mu - \varphi X_{,\mu}n^\mu = 0.$$

This completes the proof. \square

In words, the canonical form (6.16) separates off the $C^{0,1}$ kink of f across γ into the function $|x(t) - r|$, from its more regular C^1 behavior incorporated into the functions φ , which gives the strength of the jump, and Φ , which encodes the remaining C^1 behavior of f . Taking the jump in the normal derivative of f across γ shows that φ gives the strength of the jump because the dependence on Φ cancels out. Note finally that the regularity assumption on the metric across γ is required for $\varphi(t)$ to be well defined in (6.10), and also to get the C^1 regularity of $n^\sigma n_\sigma$ tangent to γ . In Section 8 below we prove Israel's Theorem for a single shock surface by constructing a $C^{1,1}$ coordinate transformation using 6.16 of Lemma 6.2 as a canonical form for the Jacobian derivatives of the transformation. However, to get Israel's theorem in both directions, we need the following refinement of Lemma 6.2 to a lower regularity away from the shock curve γ :

Corollary 6.3. *Let \mathcal{N} be an open neighborhood of a smooth curve $\gamma(t) = (t, x(t))$ in \mathbb{R}^2 , $t \in (-\epsilon, \epsilon)$, and suppose f is in $C^{0,1}(\mathcal{N})$.¹³ Then there exists a function $\Phi \in C^{0,1}(\mathcal{N})$ with*

$$(6.15) \quad [\Phi_t] \equiv 0 \equiv [\Phi_r],$$

such that

$$(6.16) \quad f(t, r) = \frac{1}{2}\varphi(t)|x(t) - r| + \Phi(t, r),$$

if and only if

$$(6.17) \quad \varphi(t) = \frac{[f, \mu]n^\mu}{n^\sigma n_\sigma} \in C^{0,1}(-\epsilon, \epsilon),$$

where n_μ is defined by (6.21) and indices are raised and lowered by a Lorentzian metric $g_{\mu\nu}$ which is $C^{0,1}$ across γ . In particular, φ has discontinuous derivatives wherever $f \circ \gamma$ does.

¹³This is a weaker condition on f than in Definition 6.1 and Lemma 6.2 because we do not assume C^1 regularity away from the curve γ .

Proof. The Corollary follows by the same arguments as in the proof of Lemma 6.2, since in that proof the C^1 regularity of Φ only enters in the sense of (6.15) and (6.1) also holds for a Lipschitz continuous function f , where the derivatives $f_{,\mu}$ exist almost everywhere. Moreover, φ has discontinuous derivatives wherever $f \circ \gamma$ does, due to (6.17) and the C^1 metric regularity tangential to γ . \square

In Section 9 we need a canonical form analogous to (6.16) for two shock curves, but such that it allows for the Jacobian to be in the weaker regularity class $C^{0,1}$ away from the shock curves. To this end, suppose timelike shock surfaces described in the (t, r) -plane by, $\gamma_i(t) = (t, x_i(t))$, such that (3.6) - (3.8) applies. To cover the generic case of shock wave interaction, we assume each $\gamma_i(t)$ is C^2 away from $t = 0$ with the lower/upper-limit of the tangent vectors existing up to $t = 0$. For our main results (Theorem 9.1 and 1.1) it suffices to consider the upper ($t > 0$) or lower part ($t < 0$) of a shock wave interaction (at $t = 0$) separately, whichever part contains two shock waves that interact with distinct speeds. In the following we restrict without loss of generality to the lower part of a shock wave interactions, that is, to $\mathbb{R}_-^2 = \{(t, r) : t < 0\}$.

Corollary 6.4. *Let $\gamma_i(t) = (t, x_i(t))$ be two smooth curves defined on $I = (-\epsilon, 0)$, for some $\epsilon > 0$, such that the limits $\lim_{t \rightarrow 0^-} \gamma_i(t) = (0, r_0)$ and $\dot{x}_i(0) = \lim_{t \rightarrow 0^-} \dot{x}_i(t)$ both exist for $i = 1, 2$. Let \mathcal{N} be an open neighborhood of $p = (0, r_0)$ in \mathbb{R}^2 and suppose f is in $C^{0,1}(\mathcal{N})$. Then there exists a $C^{0,1}$ function Φ defined on $\mathcal{N} \cap \mathbb{R}_-^2$, with*

$$(6.18) \quad [\Phi_t]_i \equiv 0 \equiv [\Phi_r]_i, \quad i = 1, 2,$$

such that

$$(6.19) \quad f(t, r) = \frac{1}{2} \sum_{i=1,2} \varphi_i(t) |x_i(t) - r| + \Phi(t, r),$$

for all (t, r) in $\mathcal{N} \cap \mathbb{R}_-^2$, if and only if

$$(6.20) \quad \varphi_i(t) = \frac{[f_{,\mu}]_i(n_i)^\mu}{(n_i)^\mu(n_i)_\mu} \in C^{0,1}(I),$$

where

$$(6.21) \quad (n_i)_\mu(t) = (\dot{x}_i(t), -1)$$

is a 1-form normal to the tangent vector $v_i^\mu(t) = \dot{\gamma}_i^\mu(t)$, (for $i = 1, 2$), and indices are again raised and lowered by a Lorentzian metric $g_{\mu\nu}$ which is $C^{0,1}$ across γ . In particular, φ_i has discontinuous derivatives wherever $f \circ \gamma_i$ does.

Proof. The Corollary follows by the same arguments as in the proof of Corollary 6.3 on each of the curves γ_i , ($i = 1, 2$), defining

$$(6.22) \quad \Phi := f - \frac{1}{2} \sum_{i=1,2} \varphi_i |X_i|,$$

for $X_i(t, r) := x_i(t) - r$, instead of (6.14) and using that $H(X_i)$ is discontinuous across γ_i , but $[H(X_i)]_l = 0$ for $l \neq i$. (In particular, f meets condition (6.1) across each of the curves γ_i , with derivatives $f_{,\mu}$ defined almost everywhere.) \square

In Section 9, we use the canonical form (6.19) to characterize all Jacobians in the (t, r) -plane, that could possibly lift the metric regularity from $C^{0,1}$ to $C^{1,1}$ on a deleted neighborhood of the point p of shock wave interaction, unique up to addition of a function $\Phi \in C^{0,1}$ satisfying (6.18). It is precisely the regularity condition (6.18) that forces the determinant of those Jacobians to vanish at p if one takes the limit of the Jacobians towards p , hence resulting in a singular metric at p in the new coordinates.

7. A NECESSARY AND SUFFICIENT CONDITION FOR SMOOTHING METRICS

In this section we derive a necessary and sufficient pointwise condition on the Jacobians of a coordinate transformation that it lift the regularity of a $C^{0,1}$ metric tensor to $C^{1,1}$ in a neighborhood of a point on a single shock surface Σ . In Section 8 we use this condition to prove that such transformations exist in a neighborhood of a point on a single shock surface, and in Section 9 we use this pointwise condition on each of two intersecting shock surfaces to prove that no such coordinate transformation exists in a neighborhood of a point of shock wave interaction.

We begin with the covariant transformation law

$$(7.1) \quad g_{\alpha\beta} = J_{\alpha}^{\mu} J_{\beta}^{\nu} g_{\mu\nu},$$

for the metric components at a point on a hypersurface Σ for a general $C^{1,1}$ coordinate transformation $x^{\mu} \rightarrow x^{\alpha}$, where, as customary, the indices indicate the coordinate system. If not otherwise stated indices μ, ν and σ always refer to SSC, (c.f. Lemma 7.1). Let J_{α}^{μ} denote the Jacobian of the transformation

$$J_{\alpha}^{\mu} = \frac{\partial x^{\mu}}{\partial x^{\alpha}}.$$

Assume now, that the metric components $g_{\mu\nu}$ are only Lipschitz continuous with respect to x^{μ} across Σ . Then differentiating (7.1) with respect to $\frac{\partial}{\partial x^{\gamma}}$ and subtracting the left-limits towards Σ from the right-limit we obtain

$$(7.2) \quad [g_{\alpha\beta,\gamma}] = J_{\alpha}^{\mu} J_{\beta}^{\nu} [g_{\mu\nu,\gamma}] + g_{\mu\nu} J_{\alpha}^{\mu} [J_{\beta,\gamma}^{\nu}] + g_{\mu\nu} J_{\beta}^{\nu} [J_{\alpha,\gamma}^{\mu}],$$

where $[f] = f_L - f_R$ denotes the jump in the quantity f across the shock surface Σ . Thus, since both g and J_{α}^{μ} are in general Lipschitz continuous across Σ , the jumps appear only on the derivatives. Equation (7.2) gives a necessary and sufficient condition for the metric g to be $C^{1,1}$ in x^{α} coordinates. Namely, $g_{\alpha\beta}$ is in $C^{1,1}$ if and only if

$$(7.3) \quad [g_{\alpha\beta,\gamma}] = 0$$

for every $\alpha, \beta, \gamma \in \{0, \dots, 3\}$, while (7.2) implies that (7.3) holds if and only if

$$(7.4) \quad [J_{\alpha,\gamma}^\mu] J_{\beta}^\nu g_{\mu\nu} + [J_{\beta,\gamma}^\nu] J_{\alpha}^\mu g_{\mu\nu} + J_{\alpha}^\mu J_{\beta}^\nu [g_{\mu\nu,\gamma}] = 0.$$

Equation (7.4) is a necessary and sufficient condition for smoothing the metric by the means of a coordinate transformation, (c.f. Corollary 7.2).

If the coordinate transformation is C^2 , so that J_{α}^μ is C^1 , then the jumps in $J_{\alpha,\beta}^\mu$ vanish, and (7.2) reduces to

$$[g_{\alpha\beta,\gamma}] = J_{\alpha}^\mu J_{\beta}^\nu J_{\gamma}^\sigma [g_{\mu\nu,\sigma}],$$

which is tensorial because the non-tensorial terms cancel out in the jump $[g_{\alpha\beta,\gamma}]$. Since tensor transformations preserve the zero tensor, it is precisely the lack of covariance in (7.2) for $C^{1,1}$ transformations that provides the necessary degrees of freedom, (the jumps $[J_{\alpha,\gamma}^\mu]$ in the first derivatives of the Jacobian), that make it possible for a Lipschitz metric to be smoothed by coordinate transformation at points on a single shock surface, proving that there is no hope of lifting the metric regularity by coordinate transformations that are C^2 .

Equation (7.4) is an inhomogeneous linear system in the jumps in the derivatives of the Jacobians $[J_{\alpha,\gamma}^\mu]$. Solving this system provides a necessary condition on the jumps in the derivatives of the Jacobian for lifting the metric regularity from $C^{0,1}$ to $C^{1,1}$, however, a solution of (7.4) alone does not provide in general a Jacobian, since it does not yet ensure that there actually exist $C^{0,1}$ functions J_{α}^μ that take on the values $[J_{\alpha,\gamma}^\mu]$ and that these functions J_{α}^μ satisfy the integrability condition (2.5), which is necessary and sufficient for integrating J_{α}^μ up to coordinates. It is therefore crucial to impose in addition to (7.4) that the $[J_{\alpha,\gamma}^\mu]$ satisfy an appropriate integrability condition, namely

$$(7.5) \quad [J_{\alpha,\beta}^\mu] = [J_{\beta,\alpha}^\mu].$$

In the following we solve the linear system obtained from (7.4) and (7.5) for $[J_{\alpha,\gamma}^\mu]$ using Gaussian elimination in Lemma 7.1. This system is nonsingular and thus yields a necessary and sufficient condition on the jumps in the Jacobian derivatives for lifting the regularity of a metric in SSC from $C^{0,1}$ to $C^{1,1}$ in a neighborhood of a point on a single shock surface, provided J_α^μ is the actual Jacobian of a coordinate transformation, as we prove in Corollary 7.2. In Section 8 we prove that the Einstein equations together with the RH jump conditions are necessary and sufficient for such a Jacobian to exist in a neighborhood of a point on a single radial shock surface, thereby giving an alternative constructive proof of Israel's Theorem in spherically symmetric spacetimes. Such a Jacobian fails to exist if two radial shock surfaces intersect, as we prove in Section 9.

We now restrict to spherically symmetric spacetimes in SSC coordinates, and reduce (7.4) to six equations in six unknowns among $[J_{\alpha,\gamma}^\mu]$. To this end, suppose we are given a single radial shock surface Σ in SSC locally parameterized by

$$(7.6) \quad \Sigma(t, \theta, \phi) = (t, x(t), \theta, \phi),$$

described in the (t, r) -plane by the corresponding shock curve

$$(7.7) \quad \gamma(t) = (t, x(t)).$$

For such a hypersurface in Standard Schwarzschild Coordinates (SSC), the angular variables play a passive role, and the essential issue regarding smoothing the metric components by $C^{1,1}$ coordinate transformation, lies in the atlas of (t, r) -coordinate transformations. Thus we restrict to the atlas of (t, r) -coordinate transformations, (that keep the angular coordinates fixed), for a general $C^{0,1}$ metric in SSC, c.f. (2.20). Now, (7.4) and (7.5) form a linear inhomogeneous 8×8 system in eight unknowns $[J_{\alpha,\gamma}^\mu]$. The following lemma shows that this system is uniquely solvable, states the solution $[J_{\alpha,\gamma}^\mu]$ explicitly in terms of a SSC metric $g_{\mu\nu}$ only Lipschitz continuous across Σ , and expresses the $[J_{\alpha,\gamma}^\mu]$'s as functions in SSC restricted to the shock

curves $\gamma(t) = (t, x(t))$.

Lemma 7.1. *Let*

$$(7.8) \quad g_{\mu\nu} dx^\mu dx^\nu = -A(t, r) dt^2 + B(t, r) dr^2 + r^2 d\Omega^2,$$

be a given metric expressed in SSC, let Σ denote a single radial shock surface (7.6) across which g is only Lipschitz continuous. Then the unique solution $[J_{\alpha,\gamma}^\mu]$ of (7.4) which satisfies the integrability condition (7.5), is given by:

$$(7.9) \quad \begin{aligned} [J_{0,t}^t] &= -\frac{1}{2} \left(\frac{[A_t]}{A} J_0^t + \frac{[A_r]}{A} J_0^r \right); & [J_{0,r}^t] &= -\frac{1}{2} \left(\frac{[A_r]}{A} J_0^t + \frac{[B_t]}{A} J_0^r \right); \\ [J_{1,t}^t] &= -\frac{1}{2} \left(\frac{[A_t]}{A} J_1^t + \frac{[A_r]}{A} J_1^r \right); & [J_{1,r}^t] &= -\frac{1}{2} \left(\frac{[A_r]}{A} J_1^t + \frac{[B_t]}{A} J_1^r \right); \\ [J_{0,t}^r] &= -\frac{1}{2} \left(\frac{[A_r]}{B} J_0^t + \frac{[B_t]}{B} J_0^r \right); & [J_{0,r}^r] &= -\frac{1}{2} \left(\frac{[B_t]}{B} J_0^t + \frac{[B_r]}{B} J_0^r \right); \\ [J_{1,t}^r] &= -\frac{1}{2} \left(\frac{[A_r]}{B} J_1^t + \frac{[B_t]}{B} J_1^r \right); & [J_{1,r}^r] &= -\frac{1}{2} \left(\frac{[B_t]}{B} J_1^t + \frac{[B_r]}{B} J_1^r \right). \end{aligned}$$

(We use the notation $\mu, \nu \in \{t, r\}$ and $\alpha, \beta \in \{0, 1\}$, so that t, r are used to denote indices whenever they appear on the Jacobian J_α^μ .)

Proof. We first setup the 8×8 linear system formed from (7.4) and (7.5) for the case of Jacobians in the (t, r) -plane and the metric tensor in (7.8). For Jacobians in the (t, r) -plane the only nonzero terms $[J_{\alpha,\beta}^\mu]$ are those on the left hand side in (7.9), that is, for $\mu \in \{t, r\}$ and $\alpha, \beta \in \{0, 1\}$. Then the integrability condition (7.5) reduces to

$$(7.10) \quad \begin{aligned} [J_{0,1}^t] &= [J_{1,0}^t] \\ [J_{0,1}^r] &= [J_{1,0}^r]. \end{aligned}$$

Employing (7.10) the original eight unknowns $[J_{\alpha,\beta}^\mu]$ reduce to six and it remains to setup and solve (7.4) for the metric in SSC (7.8) and Jacobians in the (t, r) -plane. A major simplification of considering Jacobians acting on the (t, r) -plane is that the angular coefficients of the metric in SSC (7.8) drop out and (7.4) reduces to the following 6×6 linear system:

$$(7.11) \quad A\vec{v} = \vec{w}$$

with unknowns

$$(7.12) \quad \vec{v} = {}^T([J_{0,0}^t, [J_{0,1}^t, [J_{1,1}^t, [J_{0,0}^r, [J_{0,1}^r, [J_{1,1}^r]),$$

where T denotes the transpose. The right hand side in (7.11) is given by

$$(7.13) \quad \begin{aligned} \vec{w} &= {}^T(w_1, w_2, w_3, w_4, w_5, w_6) \\ &= {}^T(w_{000}, w_{010}, w_{110}, w_{001}, w_{011}, w_{111}), \end{aligned}$$

for

$$(7.14) \quad w_{\alpha\beta\gamma} = J_\alpha^0 J_\beta^0 [A, \gamma] - J_\alpha^1 J_\beta^1 [B, \gamma].$$

The matrix A reads

$$(7.15) \quad A = \begin{pmatrix} 2a & 0 & 0 & 2d & 0 & 0 \\ c & a & 0 & b & d & 0 \\ 0 & 2c & 0 & 0 & 2b & 0 \\ 0 & 2a & 0 & 0 & 2d & 0 \\ 0 & c & a & 0 & b & d \\ 0 & 0 & 2c & 0 & 0 & 2b \end{pmatrix},$$

with $a := -J_0^0 A$, $b := J_1^1 B$, $c := -J_1^0 A$ and $d := J_0^1 B$. Observe that we already incorporated the integrability condition (7.10) into the linear system (7.11) by considering only 6 unknowns in (7.12).

(7.11) has a unique solution given by, (we used MAPLE to compute this),

$$(7.16) \quad \begin{aligned} [J_{0,0}^t] &= \frac{2w_2d - bw_1 - dw_4}{2AB|J|}; \\ [J_{0,1}^t] &= \frac{dw_3 - bw_4}{2AB|J|}; \\ [J_{1,1}^t] &= \frac{dw_6 + bw_3 - 2bw_5}{2AB|J|}; \\ [J_{0,0}^r] &= \frac{aw_4 - w_2a + cw_1}{2AB|J|}; \\ [J_{0,1}^r] &= \frac{cw_4 - aw_3}{2AB|J|}; \\ [J_{1,1}^r] &= \frac{2cw_5 - aw_6 - cw_3}{2AB|J|}, \end{aligned}$$

where $|J| = \det(J_\alpha^\mu) = J_0^t J_1^r - J_1^t J_0^r$ denotes the determinant of the Jacobian.

It remains to perform a change of coordinates in the expression in (7.16) to SSC, in order to recover the expressions in (7.9).¹⁴ From the chain rule and the continuity of the inverse Jacobian J_γ^σ we get

$$[J_{\alpha,\nu}^\mu] = [J_{\alpha,\gamma}^\mu] J_\nu^\gamma$$

and (for $\sigma = t, r$)

$$[g_{\mu\nu,\gamma}] = [g_{\mu\nu,\sigma}] J_\gamma^\sigma.$$

Employing furthermore the definition of w_i in (7.14) together with (7.16) and the jumps in the integrability condition (7.5) it is a straightforward (but lengthy) calculation that leads to (7.9).

In order to exemplify the procedure we outline the calculation leading to the expression for $[J_{0,t}^t]$ in (7.9):

$$\begin{aligned} [J_{0,t}^t] &= [J_{0,0}^t] J_t^0 + [J_{0,1}^t] J_t^1 \\ (7.17) \quad &= [J_{0,0}^t] \frac{J_1^r}{|J|} - [J_{0,1}^t] \frac{J_1^t}{|J|}, \end{aligned}$$

where we used Cramer's rule in the last equality. Substituting now (7.16) together with the definition of w_i in (7.14) and the integrability condition (7.5) a straightforward calculation leads to

$$[J_{0,t}^t] = -\frac{1}{2} \left(\frac{[A_t]}{A} J_0^t + \frac{[A_r]}{A} J_0^r \right),$$

as claimed in (7.9). Performing similar computations for the remaining expression finishes the proof. \square

¹⁴Equation (7.16) together with (7.5) yields a necessary and sufficient condition on the first derivatives of the Jacobian for smoothing the metric to $C^{1,1}$, however, since we need to consider the shock curve γ in the coordinates we start in, (that is, in SSC), we must express the above condition in terms of the given coordinates x^μ .

Condition (7.4) is a necessary and sufficient condition for $[g_{\alpha\beta,\gamma}] = 0$ at a point on a smooth single shock surface. Because Lemma 7.1 tells us that we can uniquely solve (7.4) for the jumps in the Jacobian derivatives, it follows that a necessary and sufficient condition for $[g_{\alpha\beta,\gamma}] = 0$ is that the jumps in the Jacobian derivatives be exactly the functions of the jumps in the original SSC metric components recorded in (7.9). It is remarkable that, in order to lift the metric regularity, the Jacobian must mirror the regularity of the metric in order to compensate for all discontinuous first order derivatives of the metric by its own discontinuous first order derivatives. (This is why we expect that only $C^{1,1}$ transformations can possibly lift the metric regularity from $C^{0,1}$ to $C^{1,1}$, and that $C^{1,\alpha}$ is not an appropriate choice for the atlas regularity if $\alpha \neq 1$.) In light of this, Lemma 7.1 immediately implies the following corollary:

Corollary 7.2. *Let p be a point on a single smooth shock curve γ , and let $g_{\mu\nu}$ be a metric tensor in SSC, which is $C^{0,1}$ across γ (in the sense of Definition 6.1). Suppose J_α^μ is the Jacobian of an actual coordinate transformation defined on a neighborhood \mathcal{N} of p and suppose J_α^μ is $C^{0,1}$ across γ . Then the metric in the new coordinates $g_{\alpha\beta}$ is in $C^{1,1}(\mathcal{N})$ if and only if J_α^μ satisfies (7.9).*

Proof. We first prove that $g_{\alpha\beta} \in C^{1,1}(\mathcal{N})$ implies (7.4). Suppose there exist coordinates x^α such that the metric in the new coordinates $g_{\alpha\beta}$ is in $C^{1,1}$, then

$$[g_{\alpha\beta,\gamma}] = 0 \quad \forall \alpha, \beta, \gamma \in \{0, \dots, 3\}.$$

This directly implies (7.4) and since J_α^μ are the Jacobians of an actual coordinate transformation they satisfy the integrability condition (7.5) as well. By Lemma 7.1 the jumps in the derivatives of the Jacobian $[J_{\alpha,\gamma}^\mu]$ then satisfy (7.9).

We now prove the opposite direction. Suppose that the Jacobians J_α^μ satisfy (7.9), then by Lemma 7.1 they satisfy the smoothing condition (7.4), which implies that all metric derivatives $g_{\alpha\beta,\gamma}$ match continuously across the shock curve γ , that is, $[g_{\alpha\beta,\gamma}] = 0$ for all $\alpha, \beta, \gamma \in \{0, \dots, 3\}$. Since $g_{\mu\nu}$ and J_α^μ are assumed to be C^2

away from γ it follows that $g_{\alpha\beta} \in C^1(\mathcal{N})$. In fact, the C^2 regularity of $g_{\mu\nu}$ and J_α^μ away from γ , together with the existence of the limit towards γ of all second order derivatives¹⁵ already implies $g_{\alpha\beta} \in C^{1,1}(\mathcal{N})$, since for any $q \in \Sigma$ one can bound

$$(7.18) \quad |g_{\alpha\beta,\gamma}(t, r) - g_{\alpha\beta,\gamma}(q)| < c |(t, r)|,$$

by a finite constant¹⁶ c , for instance set

$$c = \sum_{\substack{\alpha,\beta,\gamma,\delta \\ =0,1,2,3}} \left(\sup_{q' \in \Sigma'} |(g_{\alpha\beta,\gamma\sigma})_L(q')| + \sup_{q' \in \Sigma'} |(g_{\alpha\beta,\gamma\sigma})_R(q')| \right) + \sum_{\substack{\alpha,\beta,\gamma \\ =0,1,2,3}} \sup_{(t,r) \in \mathcal{N}'} |g_{\alpha\beta,\gamma}(t, r)|,$$

for $\mathcal{N}' \subset \mathcal{N}$ a compact neighborhood of q and $\Sigma' = \Sigma \cap \mathcal{N}'$. \square

We conclude that (7.9) is a necessary and sufficient condition for a coordinate transformation to lift the regularity of an SSC metric from $C^{0,1}$ to $C^{1,1}$ around a point on a single smooth shock surface. The condition relates the jumps in the derivatives of the Jacobian to the jumps in the metric derivatives across the shock. This establishes the rather remarkable result that there is no algebraic obstruction to lifting the regularity in the sense that the jumps in the Jacobian derivatives can be *uniquely* solved for in terms of the jumps in the metric derivatives, precisely when the integrability condition (7.5) is imposed. The condition is a statement purely about spherically symmetric spacetime metrics in SSC because neither the RH conditions nor the Einstein equations have yet been imposed. But we know by Israel's theorem that the RH conditions must be imposed to conclude that smoothing transformations exist. The point then, is that to prove the existence of coordinate transformations that lift the regularity of SSC metrics to $C^{1,1}$ at $p \in \Sigma$, we must prove that there exists a set of functions J_α^μ defined in a neighborhood of p , such that (7.9) holds at p , and such that the integrability condition (2.5), (required for J_α^μ to be the Jacobian of a coordinate transformation), holds in a whole neighborhood containing p . In Section 8 we give an alternative proof of Israel's Theorem by showing that such J_α^μ exist in a

¹⁵The existence of limits of all second order derivatives (in SSC) of $g_{\mu\nu}$ and J_α^μ to Σ , is implicitly assumed by requiring J_α^μ to be $C^{0,1}$ across Σ .

¹⁶The continuity of $g_{\alpha\beta,\gamma}$ at q is crucial in (7.18), since otherwise the left hand side in (7.18) does not tend to zero for $x \rightarrow 0$, (since $g_{\alpha\beta,\gamma}$ is not well-defined at q).

neighborhood of a point p on a smooth single shock surface if and only if the Einstein equations and the RH jump condition hold, and in Section 9 we prove that no such functions can exist in a neighborhood of a point p of shock wave interaction, unless $\det(J_\alpha^\mu) = 0$ at p .

8. METRIC SMOOTHING ON SINGLE SHOCK SURFACES AND A CONSTRUCTIVE
PROOF OF ISRAEL'S THEOREM

We have shown in Corollary 7.2 that (7.9) is a necessary and sufficient condition on a Jacobian derivative J_α^μ for lifting the SSC metric regularity to $C^{1,1}$ in a neighborhood of a shock curve. We now address the issue of how to obtain such Jacobians of actual coordinate transformations defined in a whole neighborhood of a point on a single shock surface. For this we need to find a set of functions J_α^μ that satisfies (7.9), as well as the integrability condition (2.5) in a whole neighborhood. In this section we show that this can be accomplished in the case of *single* shock surfaces, thereby giving an alternative constructive proof of Israel's Theorem for spherically symmetric spacetimes:

Theorem 8.1. (Israel's Theorem) *Suppose $g_{\mu\nu}$ is an SSC metric that is $C^{0,1}$ across a radial shock surface γ in the sense of Definition 6.1, such that it solves the Einstein equations (2.21) - (2.24) strongly away from γ , and assume $T^{\mu\nu}$ is everywhere bounded and in C^0 away from γ . Then around each point p on γ there exists a $C^{1,1}$ coordinate transformation of the (t, r) -plane, defined in a neighborhood \mathcal{N} of p , such that the transformed metric components $g_{\alpha\beta}$ are $C^{1,1}$ functions of the new coordinates, if and only if the RH jump conditions (3.4) - (3.5) hold on γ in a neighborhood of p .*

The main step is to construct Jacobians acting on the (t, r) -plane that satisfy the smoothing condition (7.9) on the shock curve, the condition that guarantees $[g_{\alpha\beta, \gamma}] = 0$. The following lemma gives an explicit formula for functions J_α^μ satisfying (7.9). The point then is that, in the case of single shock curves, both the RH jump conditions and the Einstein equations are necessary and sufficient for such functions J_α^μ to exist.

Lemma 8.2. *Let p be a point on a single shock curve γ across which the SSC metric $g_{\mu\nu}$ is Lipschitz continuous in the sense of Definition 6.1. Then there exists a set of*

functions $J_\alpha^\mu \in C^{0,1}(\mathcal{N})$ satisfying the smoothing condition (7.9) on $\gamma \cap \mathcal{N}$ if and only if (6.8) holds on $\gamma \cap \mathcal{N}$. Furthermore, all J_α^μ that are in $C^{0,1}(\mathcal{N})$ and satisfy (7.9) on $\gamma \cap \mathcal{N}$ are given by

$$(8.1) \quad \begin{aligned} J_0^t(t, r) &= \frac{[A_r]\phi(t) + [B_t]\omega(t)}{4A \circ \gamma(t)} |x(t) - r| + \Phi(t, r) \\ J_1^t(t, r) &= \frac{[A_r]\nu(t) + [B_t]\zeta(t)}{4A \circ \gamma(t)} |x(t) - r| + N(t, r) \\ J_0^r(t, r) &= \frac{[B_t]\phi(t) + [B_r]\omega(t)}{4B \circ \gamma(t)} |x(t) - r| + \Omega(t, r) \\ J_1^r(t, r) &= \frac{[B_t]\nu(t) + [B_r]\zeta(t)}{4B \circ \gamma(t)} |x(t) - r| + Z(t, r), \end{aligned}$$

for arbitrary functions $\Phi, \Omega, Z, N \in C^{0,1}(\mathcal{N})$, where

$$(8.2) \quad \phi = \Phi \circ \gamma, \quad \omega = \Omega \circ \gamma, \quad \nu = N \circ \gamma, \quad \zeta = Z \circ \gamma.$$

Moreover, each arbitrary function $U = \Phi, \Omega, Z$ or N satisfies

$$(8.3) \quad [U_r] = 0 = [U_t].$$

Proof. Suppose there exists a set of functions $J_\alpha^\mu \in C^{0,1}(\mathcal{N})$ satisfying (7.9), then their continuity implies that tangential derivatives along γ match across γ , that is

$$(8.4) \quad [J_{\alpha,t}^\mu] = -\dot{x}[J_{\alpha,r}^\mu]$$

for all $\mu \in \{t, r\}$ and $\alpha \in \{0, 1\}$. Imposing (8.4) in (7.9) and using (6.3) - (6.4) yields (6.8). (For instance imposing $[J_{0,t}^t] = -\dot{x}[J_{0,r}^t]$ in (7.9) yields

$$-\frac{1}{2} \left(\frac{[A_t]}{A} J_0^t + \frac{[A_r]}{A} J_0^r \right) = \frac{\dot{x}}{2} \left(\frac{[A_r]}{A} J_0^t + \frac{[B_t]}{A} J_0^r \right)$$

and using (6.3) - (6.4) most terms cancel out, leaving behind $[A_r] = -\dot{x}[B_t]$ as claimed.)

To prove the opposite direction it suffices to show that all t and r derivatives of J_α^μ , defined in the above ansatz (8.1), satisfy (7.9) for all $\mu \in \{t, r\}$ and $\alpha \in \{0, 1\}$.

This follows directly from (6.3), (6.4) and (6.8), upon noting that (8.2) implies the identities

$$(8.5) \quad \phi = J_0^t \circ \gamma, \quad \nu = J_1^t \circ \gamma, \quad \omega = J_0^r \circ \gamma, \quad \zeta = J_1^r \circ \gamma .$$

This proves the existence of functions J_α^μ satisfying (7.9).

Furthermore, applying Corollary 6.3 (which allows Φ to have the lower regularity $\Phi \in C^{1,1}$ but imposes the jumps (9.4) along γ), confirms that all such functions can be written in the canonical form (8.1), which proves the supplement. \square

To complete the proof of Israel's Theorem, we must prove the existence of coordinate transformations $x^\mu \rightarrow x^\alpha$ that lift the $C^{0,1}$ regularity of $g_{\mu\nu}$ to $C^{1,1}$. It remains, now, to show that the functions J_α^μ defined above in ansatz (8.1) can be integrated to coordinate functions, i.e., that they satisfy the integrability condition (2.5) in a whole neighborhood. This is accomplished in the following two lemmas. The first lemma gives an equivalent form of the integrability condition (2.5) that is suitable for the Jacobian ansatz (8.1), as it gives a PDE on the free functions Φ , Ω , Z and N in SSC. The main step in the proof of the following lemma is to express (2.5) in SSC.

Lemma 8.3. *The functions J_α^μ defined in (8.1) satisfy the integrability condition (2.5) if and only if the free functions Φ, Ω, N and Z satisfy the following system of two PDE's:*

$$(8.6) \quad (\dot{\alpha}|X| + \Phi_t) (\beta|X| + N) + \Phi_r (\epsilon|X| + Z) - (\alpha|X| + \Phi) (\dot{\beta}|X| + N_t) \\ - N_r (\delta|X| + \Omega) + fH(X) = 0$$

$$(8.7) \quad (\dot{\delta}|X| + \Omega_t) (\beta|X| + N) + \Omega_r (\epsilon|X| + Z) - (\dot{\epsilon}|X| + Z_t) (\alpha|X| + \Phi) \\ - Z_r (\delta|X| + \Omega) + hH(X) = 0 ,$$

where $X(t, r) = x(t) - r$, $H(X)$ denotes the Heaviside step function, and

$$(8.8) \quad \begin{aligned} \alpha &= \frac{[A_r]\phi(t) + [B_t]\omega(t)}{4A \circ \gamma(t)}; \\ \beta &= \frac{[A_r]\nu(t) + [B_t]\zeta(t)}{4A \circ \gamma(t)}; \\ \delta &= \frac{[B_t]\phi(t) + [B_r]\omega(t)}{4B \circ \gamma(t)}; \\ \epsilon &= \frac{[B_t]\nu(t) + [B_r]\zeta(t)}{4B \circ \gamma(t)}; \end{aligned}$$

and

$$(8.9) \quad \begin{aligned} f &= (\beta\delta - \alpha\epsilon) |X| + \alpha\dot{x}N - \beta\dot{x}\Phi + \beta\Omega - \alpha Z \\ h &= (\beta\delta - \alpha\epsilon) \dot{x}|X| + \delta\dot{x}N - \epsilon\dot{x}\Phi + \epsilon\Omega - \delta Z, \end{aligned}$$

where, $\alpha, \beta, \delta, \epsilon, f$ and h are all C^1 functions of t .

Proof. Recall the integrability condition (2.5),

$$(8.10) \quad J_{\alpha,\beta}^\mu = J_{\beta,\alpha}^\mu.$$

A set of functions J_α^μ is integrable to coordinate functions if and only if they satisfy the above PDE, (c.f. Appendix A). The partial differentiation in (8.10) is expressed in coordinates x^α , the coordinate system we are trying to construct, but our Jacobian ansatz (8.1) is given in SSC, therefore we need to perform a change of coordinates in the above PDE (8.10). From the chain rule we find that (8.10) implies

$$(8.11) \quad J_{\alpha,\nu}^\mu J_\beta^\nu = J_{\beta,\nu}^\mu J_\alpha^\nu,$$

(where x^ν denote SSC), in fact, (8.11) is equivalent to (8.10), as we prove in the following. Once the equivalence of (8.11) and (8.10) is proven the equivalence of (8.10) and (8.6) follows immediately by substituting the Jacobians (8.1) into the above integrability condition (8.11) and separating all discontinuous terms into the functions f and h defined in (8.9).

We now prove that the integrability condition in SSC (8.11) implies (8.10), (thus both equations are equivalent). Our strategy is to show that (8.11) implies the integrability condition

$$(8.12) \quad J_{\mu,\nu}^\alpha = J_{\nu,\mu}^\alpha,$$

on the inverse Jacobian J_μ^α . If (8.12) hold then we can integrate the function J_μ^α up to coordinates x^α , (c.f. Appendix, Lemma A.1), and since x^α is a bijective function in SSC x^μ , we are able to introduce the functions

$$(8.13) \quad J_\alpha^\mu = \frac{\partial x^\mu}{\partial x^\alpha}.$$

Moreover, the functions defined in (8.13) do coincide with the pointwise linear algebraic inverse of J_μ^α , (by the chain rule). The functions introduced in (8.13) satisfy the original integrability condition 8.10 by the commutativity of partial derivatives, thereby proving Lemma 8.3, (once we proved that (8.11) imply (8.12)).

We now show that (8.11) implies (8.12). Suppose that 8.11 holds, which is equivalent to

$$(8.14) \quad J_1^r J_{0,r}^t - J_0^r J_{1,r}^t = J_0^t J_{1,t}^r - J_1^t J_{0,t}^r$$

$$(8.15) \quad J_1^r J_{0,r}^r - J_0^r J_{1,r}^r = J_0^t J_{1,t}^r - J_1^t J_{0,t}^r.$$

It remains to show that the expression on the left hand side of 8.12 equal the ones on the right hand side. We are going to first use Cramer's rule to get the algebraic inverse J_μ^α of J_α^μ , substitute its components into (8.12) and then apply (8.14) - (8.15) to the resulting expressions. In more detail, the linear algebraic inverse J_μ^α of J_α^μ is given by

$$(8.16) \quad \begin{pmatrix} J_t^0 & J_r^0 \\ J_t^1 & J_r^1 \end{pmatrix} = \frac{1}{|J|} \begin{pmatrix} J_1^r & -J_1^t \\ -J_0^r & J_0^t \end{pmatrix},$$

where $|J|$ denotes the determinant of J_α^μ . A straightforward computation yields

$$J_{t,r}^0 = \left(\frac{J_1^r}{|J|} \right)_r$$

$$= \frac{J_1^t}{|J|^2} (J_1^r J_{0,r}^r - J_0^r J_{1,r}^r) - \frac{J_1^r}{|J|^2} (J_1^t J_{0,r}^t - J_0^t J_{1,r}^t),$$

and exchanging the first term using the second integrability condition (8.15) and the second term using (8.14) we get

$$\begin{aligned} J_{t,r}^0 &= \frac{J_1^t}{|J|^2} (J_0^t J_{1,t}^r - J_1^t J_{0,t}^r) - \frac{J_1^r}{|J|^2} (J_0^t J_{1,t}^t - J_1^t J_{0,t}^t) \\ &= - \left(\frac{J_1^t}{|J|} \right)_t \\ &= J_{r,t}^0. \end{aligned}$$

Similarly we compute

$$\begin{aligned} J_{t,r}^1 &= - \left(\frac{J_0^r}{|J|} \right)_r \\ &= \frac{J_0^r}{|J|^2} (J_1^r J_{0,r}^t - J_0^r J_{1,r}^t) - \frac{J_0^t}{|J|^2} (J_1^r J_{0,r}^r - J_0^r J_{1,r}^r) \end{aligned}$$

and substitute (8.14) for the first and (8.15) for the second term leads to

$$\begin{aligned} J_{t,r}^1 &= \frac{J_0^r}{|J|^2} (J_0^t J_{1,t}^t - J_1^t J_{0,t}^t) - \frac{J_0^t}{|J|^2} (J_0^t J_{1,t}^r - J_1^t J_{0,t}^r) \\ &= \left(\frac{J_0^t}{|J|} \right)_t \\ &= J_{r,t}^1. \end{aligned}$$

This proves that (8.11) implies (8.12) and therefore completes the proof. \square

The proof of Israel's Theorem is complete once we prove the existence of solutions Φ, Ω, N and Z of (8.6), (8.7) that are $C^{0,1}$, such that they satisfy (8.3). For this it suffices to choose N and Z arbitrarily, so that (8.6) - (8.7) reduce to a system of 2 linear first order PDE's for the unknown functions Φ and Ω . The condition (8.3) essentially imposes that Φ, Ω, N and Z be C^1 across the shock γ . Since (8.6), (8.7) are linear equations for Φ and Ω , they can be solved along characteristics, and so the only obstacle to solutions Φ and Ω with the requisite smoothness to satisfy condition (8.3), is the presence of the Heaviside function $H(X)$ on the right hand side of (8.6), (8.7). Lemma 8.3 thus isolates the discontinuous behavior of equations (8.6), (8.7) in the functions f and h , the coefficients of H . Israel's theorem is now a consequence of the following lemma which states that these coefficients of $H(X)$ vanish precisely

when the RH jump conditions hold on γ .

Lemma 8.4. *Assume the SSC metric $g_{\mu\nu}$ is $C^{0,1}$ across γ , (in the sense of Definition 6.1), and solves the first three Einstein equations strongly away from γ . Then the coefficients f and g of $H(X)$ in (8.6), (8.7) vanish on γ if and only if the RH jump conditions (2.15) hold on γ , (in the sense of (6.3)-(6.8)).*

Proof. The first terms of $f \circ \gamma$ and $g \circ \gamma$ in (8.9) drop out, since $X \circ \gamma = 0$. (Alternatively, (6.5)-(6.7) yield

$$16AB(\beta\delta - \alpha\epsilon) = (\kappa AB^2 r)^2 ([T^{00}][T^{11}] - [T^{10}]^2) (\nu\omega - \phi\zeta),$$

which vanishes by the jump conditions (3.4)-(3.5).)

It follows that only the second terms in $f \circ \gamma$ and $g \circ \gamma$ remain, that is,

$$(8.17) \quad \begin{aligned} f \circ \gamma &= \alpha \dot{x}\nu - \beta \dot{x}\phi + \beta\omega - \alpha\zeta \\ h \circ \gamma &= \delta \dot{x}\nu - \epsilon \dot{x}\phi + \epsilon\omega - \delta\zeta, \end{aligned}$$

where we used (8.2). Employing the definition of α and β (8.8) in (8.17) a straightforward computation shows that

$$f \circ \gamma = 0$$

is equivalent to

$$(8.18) \quad ([A_r] + \dot{x}[B_t]) (\phi\zeta - \nu\omega) = 0.$$

Using now that

$$(8.19) \quad (\phi\zeta - \nu\omega) = \det(J_\alpha^\mu \circ \gamma) \neq 0,$$

we conclude that $f \circ \gamma = 0$ if and only if (6.8), which is equivalent to the second Rankine Hugoniot jump condition (3.5) via the Einstein equations (6.6)-(6.7).

Furthermore, substituting the definition of δ and ϵ (in (8.8)) into (8.17) gives that

$$h \circ \gamma = 0$$

if and only if

$$(8.20) \quad ([B_t] + \dot{x}[B_r]) (\phi\zeta - \nu\omega) = 0.$$

Using again (8.19) yields the equivalence of (8.20) to (6.4), (that is, $[B_t] + \dot{x}[B_r] = 0$), and we conclude that $h \circ \gamma = 0$ is equivalent to the first RH jump condition through the Einstein equations (6.5)-(6.6). This completes the proof. \square

In summary, Lemma 8.4 shows that $\Phi, \Omega \in C^1(\mathcal{N})$ (or that (8.3) holds) if and only if the Rankine Hugoniot jump conditions (3.4)-(3.5) hold. We can now complete the proof of Israel's Theorem.

Proof. (Theorem 8.1)

The metric coefficient C is $C^{1,1}$ regular in every coordinate system that can be reached from SSC by a $C^{1,1}$ coordinate transformation in the (t, r) -plane, since then, C transforms as a scalar and the smoothness of $C(t, r) = r^2$ in SSC is at worst reduced to $C^{1,1}$. It remains to construct Jacobians acting on the (t, r) -plane that smooth the remaining metric coefficients A and B .

We first prove that if there exist coordinates x^α such that $g_{\alpha\beta}$ is in $C^{1,1}$ then the RH jump condition (3.4)-(3.5) hold. According to the considerations in section 6 the first RH jump condition is already implied by the continuity of the metric $g_{\mu\nu}$, (c.f. (6.3)), and it remains to prove that (6.8) holds on γ . Now $g_{\alpha\beta} \in C^{1,1}$ implies, (by Lemma 8.2), that the Jacobians of the coordinate transformation from SSC to x^α are of the canonical form (8.1). Moreover, the functions Φ, Ω, N and Z satisfy the integrability condition (8.6) as well as the regularity condition (8.3), which immediately implies that $f \circ \gamma = 0$ and $h \circ \gamma = 0$ by taking the jumps of the integrability condition (8.6) and using (8.3). By Lemma 8.4 we conclude that (6.8) and therefore the second RH

jump condition (3.5) holds.

We now prove the reverse implication. Suppose the RH jump conditions (3.4)-(3.5) hold on $\gamma \cap \mathcal{N}$, then by Lemma 8.2 there exist a set of functions J_α^μ defined in (8.1) on \mathcal{N} that satisfy the smoothing condition (7.9) and thus lift the metric regularity from $C^{0,1}$ to $C^{1,1}$. It remains to prove that there exist $C^{0,1}$ functions Φ , Ω , N and Z , that satisfy the integrability condition (8.6) and the regularity condition (8.3), to obtain a metric C^1 in the new coordinates x^α . In order to apply Corollary 7.2 and obtain the higher $C^{1,1}$ metric regularity (not just C^1), we need to show that one can construct Jacobians with a C^2 regularity away from γ . In the following we first show that a given solution of (8.6) is C^1 across γ and C^2 away from γ , and then we prove the existence of functions Φ , Ω , N and Z that satisfy (8.6). Without loss of generality we solve (8.6) for Φ and Ω , which allows us to choose N and Z arbitrarily as long that N and Z are sufficiently smooth and satisfy (8.3) together with

$$(8.21) \quad \zeta \neq \dot{x}\nu$$

on the shock curve, where $\nu = N \circ \gamma$ and $\zeta = Z \circ \gamma$. Condition (8.21) implies that the characteristic curves close to γ are not parallel to the shock curve, since the vector (ν, ζ) is tangent to the characteristic curve on γ , (c.f. [7] for a definition of characteristic curves)¹⁷.

We now prove the C^1 regularity of the solution (Φ, Ω) . Away from the shock curve γ and tangential to γ all coefficients in (8.6) are smooth, thus its solution (Φ, Ω) is smooth away from and tangential to γ . (Note that we assume N , Z and the initial data prescribed to be smooth.) It remains to prove a C^1 regularity across the shock curve, i.e. that (8.3) holds. A given (weak) solution (Φ, Ω) of (8.6) is Lipschitz continuous, since the weak form of (8.6) yields that weak derivatives in direction of

¹⁷Roughly speaking, for a scalar linear first order equation the characteristic curves are defined to be the flow of the vectorfield given by the coefficients of the first order derivatives, for example, the coefficients of Φ_t and Φ_r in the first equation in (8.6).

the characteristic curves across γ are bounded. Furthermore, by assumption the RH jump condition hold on γ and thus Lemma 8.4 yields that

$$(8.22) \quad f \circ \gamma = 0 = h \circ \gamma.$$

Now, using (8.22), the continuity of the coefficients in (8.6) and the continuity of (Φ, Ω) , the integrability condition (8.6) implies that there are no jumps in the derivatives in direction of the characteristics, that is,

$$(8.23) \quad \begin{aligned} [\Phi_t]\nu + [\Phi_r]\zeta &= 0 \\ [\Omega_t]\nu + [\Omega_r]\zeta &= 0, \end{aligned}$$

where $(\nu(t), \zeta(t))$ is a tangent vector of the characteristic curve of (8.6) at $\gamma(t)$. Since by (8.21) the characteristic curves are not parallel to the shock curve, the continuity of (Φ, Ω) across γ imposes another independent condition (c.f. (6.1)), namely, that derivatives tangent to the shock curve match up continuously:

$$(8.24) \quad \begin{aligned} [\Phi_t] + [\Phi_r]\dot{x} &= 0 \\ [\Omega_t] + [\Omega_r]\dot{x} &= 0. \end{aligned}$$

Thus, (8.23) together with (8.24) imply that (Φ, Ω) meets (8.3).

We now prove the existence of a $C^{0,1}$ solution of (8.6). Picking smooth initial values on the shock curve, i.e. $\Phi \circ \gamma = \phi$ and $\Omega \circ \gamma = \omega$, the derivatives $\dot{\phi}$ and $\dot{\omega}$ that appear in (8.6) (inside the terms $\dot{\alpha}$ and $\dot{\delta}$) are given functions and (8.6) is a linear coupled first order PDE in Φ and Ω . Now, with the shock curve γ being non-characteristic, we can apply the method described in [7], (chapter 2.5, pp. 46 - 48), except that we use a L^∞ -norm whenever a maximum norm is employed in [7], since the coefficients in (8.6) are in general only in $C^{0,1}$. (Note that all coefficients in (8.6) are in $C^{0,1}$, since (8.22) together with the Lipschitz continuity of (Φ, Ω) imply that $fH(X)$ and $hH(X)$ are continuous.) For smooth initial data assigned on the shock curve γ the method in [7] yields existence of a $C^{0,1}$ solution (Φ, Ω) of the integrability condition (8.6) on some open set containing γ . Moreover, we can pick the initial values and

the free functions, ϕ, \dots, ζ , such that $\det(J) \neq 0$ everywhere on the shock curve and once we proved continuity of the solution it is non-zero in a neighborhood \mathcal{N} of γ . In summary, under the assumption that the RH jump conditions hold on γ , we can integrate the Jacobians J_α^μ to coordinate functions that smooth the metric $g_{\mu\nu}$ to $C^{1,1}$. This completes the proof of Theorem 8. \square

9. SHOCK WAVE INTERACTIONS AS REGULARITY SINGULARITIES IN GR;
TRANSFORMATIONS IN THE (t, r) -PLANE

The main step in the proof of Theorem 1.1 is to prove that there do not exist $C^{1,1}$ coordinate transformations of the (t, r) -plane in a neighborhood of a point p of regular shock wave interaction in SSC that lifts the regularity of the metric g from $C^{0,1}$ to $C^{1,1}$ in a neighborhood of p . We then prove in Section 10 that no such transformation can exist within the full $C^{1,1}$ atlas that transforms all four variables of the spacetime, i.e., including the angular variables. We formulate the main step precisely for lower shock wave interactions in \mathbb{R}_-^2 in the following theorem, which is the topic of this section. A corresponding result applies to upper shock wave interactions in \mathbb{R}_+^2 , as well as two wave interactions in a whole neighborhood of p .

Theorem 9.1. *Suppose that p is a point of regular shock wave interaction in SSC, in the sense of Definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there does not exist a $C^{1,1}$ coordinate transformation $x^\alpha \circ (x^\mu)^{-1}$ of the (t, r) -plane, defined on $\mathcal{N} \cap \mathbb{R}_-^2$ for a neighborhood \mathcal{N} of p in \mathbb{R}^2 , such that the metric components $g_{\alpha\beta}$ are C^1 functions of the coordinates x^α in $\mathcal{N} \cap \mathbb{R}_-^2$ and such that the metric has a non-vanishing determinant at p , (that is, such that $\lim_{q \rightarrow p} \det(g_{\alpha\beta}(q)) \neq 0$).¹⁸*

In the remainder of this section we present the proof of Theorem 9.1, which mirrors the constructive proof of Israel's Theorem in Section 8 in that it uses the extension of our Jacobian ansatz (8.1) to the case of two interacting shock waves. But now, to prove non-existence, we must show the ansatz is general enough to include all $C^{0,1}$ Jacobians that could possibly lift the regularity of the metric. Our strategy is to assume for contradiction that there does exist a Jacobian of a coordinate transformation that takes a $C^{0,1}$ SSC metric $g_{\mu\nu}$ to a metric $g_{\alpha\beta}$ which is $C^{1,1}$ in a neighborhood of the

¹⁸Note that Theorem 1.1 states the non-existence of coordinates on an entire neighborhood \mathcal{N} of p in \mathbb{R}^2 , but here we have prove the stronger result that such coordinates do not exist on the upper or lower half planes separately.

point p , and then use that the necessary and sufficient condition (7.9) of Corollary 7.2 holds in a deleted neighborhood of the two shock curves that enter the interaction at p . We then restrict to the half plane where two shock waves enter p at distinct speeds, and use the smoothing condition (7.9) to construct a canonical form for the Jacobians in a neighborhood of p , that generalizes (8.1) to the case of two shock curves, with the weaker assumption of $C^{0,1}$ regularity on the functions Φ , Ω , Z , N . We conclude the proof by showing that this canonical form is inconsistent with the assumption that $\det(g_{\alpha\beta}) \neq 0$ at p , by using the continuity of the Jacobians up to p .

To implement these ideas, the main step is to show that the canonical form (6.19) of Corollary 6.4 can be applied to the Jacobians J_α^μ in the presence of a shock wave interaction. The result is recorded in the following lemma:

Lemma 9.2. *Let p be a point of regular shock wave interaction in SSC in the sense of Definition 3.1, corresponding to the SSC metric $g_{\mu\nu}$ defined on $\mathcal{N} \cap \overline{\mathbb{R}^2_-}$. Then there exists a set of functions $J_\alpha^\mu \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ satisfying the smoothing condition (7.9) on $\gamma_i \cap \mathcal{N}$, $i = 1, 2$, if and only if (6.8) holds on each shock curve $\gamma_i \cap \mathcal{N}$. In this case, all J_α^μ in $C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ assume the canonical form*

$$\begin{aligned}
 J_0^t(t, r) &= \sum_i \alpha_i(t) |x_i(t) - r| + \Phi(t, r), \\
 J_1^t(t, r) &= \sum_i \beta_i(t) |x_i(t) - r| + N(t, r), \\
 J_0^r(t, r) &= \sum_i \delta_i(t) |x_i(t) - r| + \Omega(t, r), \\
 J_1^r(t, r) &= \sum_i \epsilon_i(t) |x_i(t) - r| + Z(t, r),
 \end{aligned}
 \tag{9.1}$$

where

$$\begin{aligned}
 \alpha_i(t) &= \frac{[A_r]_i \phi_i(t) + [B_t]_i \omega_i(t)}{4A \circ \gamma_i(t)}, \\
 \beta_i(t) &= \frac{[A_r]_i \nu_i(t) + [B_t]_i \zeta_i(t)}{4A \circ \gamma_i(t)}, \\
 \delta_i(t) &= \frac{[B_t]_i \phi_i(t) + [B_r]_i \omega_i(t)}{4B \circ \gamma_i(t)},
 \end{aligned}$$

$$(9.2) \quad \epsilon_i(t) = \frac{[B_t]_i \nu_i(t) + [B_r]_i \zeta_i(t)}{4B \circ \gamma_i(t)},$$

with

$$(9.3) \quad \phi_i = \Phi \circ \gamma_i, \quad \omega_i = \Omega \circ \gamma_i, \quad \zeta_i = Z \circ \gamma_i, \quad \nu_i = N \circ \gamma_i,$$

and where $\Phi, \Omega, Z, N \in C^{0,1}(\mathcal{N} \cap \overline{\mathbb{R}^2_-})$ have matching derivatives on each shock curve $\gamma_i(t)$,

$$(9.4) \quad [U_r]_i = 0 = [U_t]_i,$$

for $U = \Phi, \Omega, Z, N, t \in (-\epsilon, 0)$.

Proof. The proof is analogous to the proof of the single shock version in Lemma 8.2. In more detail, suppose that there exist $C^{0,1}$ functions J_α^μ that meet the smoothing condition (7.9) on each γ_i , ($i = 1, 2$), then the continuity implies that (c.f. (6.1) and (8.4))

$$(9.5) \quad [J_{\alpha,t}^\mu]_i = -\dot{x}_i [J_{\alpha,r}^\mu]_i.$$

Substituting the expressions for $[J_{\alpha,\nu}^\mu]_i$ from the smoothing condition (7.9), (w.r.t. γ_i), into the above equation (9.5) and using that the metric is Lipschitz continuous across γ_i , (that is, (6.3)-(6.4) hold w.r.t. γ_i), we finally get that (6.8) holds on γ_i . Now, (6.3) together with (6.8) imply the RH jump condition on γ_i .

We now prove the opposite direction. Suppose that the RH jump condition holds on each γ_i , $i = 1, 2$, then (6.3), (6.4) and (6.8) hold on each shock curve. Now, a straightforward computation using (6.3), (6.4) and (6.8) shows that the Jacobian ansatz defined in (9.1) meets the smoothing condition (7.9) on each shock curve γ_i , therefore proving the existence of functions satisfying (7.9).

From Corollary 6.4 we conclude that if $C^{0,1}$ functions J_α^μ satisfying the smoothing condition (7.9) on each shock curve exist, then there exist functions $\Phi, \Omega, Z, N \in$

$C^{0,1}(\mathcal{N})$ satisfying (9.4), such that the J_α^μ can be written as in (8.2). This completes the proof. \square

The essence of the canonical form (9.1) is that the jumps in derivatives across the shock waves have been taken out of the functions Φ, Ω, Z, N in (9.4). We now have a canonical form for all functions J_α^μ that meet the necessary and sufficient condition (7.9) for $[g_{\alpha\beta,\gamma}] = 0$. However, for J_α^μ to be proper Jacobians that can be integrated to a coordinate system, we must use the free functions Φ, Ω, Z and N to meet the integrability condition (2.5). The following lemma gives an equivalent integrability condition in SSC to (2.5) in the spirit of Lemma 8.3, tailored to our canonical ansatz of the Jacobian (9.2).

Lemma 9.3. *For the ansatz (9.1) the integrability condition (2.5) is equivalent to*

$$(9.6) \quad A\varphi_t + B\varphi_r + C\varphi + \sum_{i=1,2} D_i \frac{d}{dt} (\varphi \circ \gamma_i) + \sum_{i=1,2} F_i H(X_i) = 0,$$

where $X_i(t, r) = x_i(t) - r$ and $H(\cdot)$ denotes the Heaviside step function. A, B, C and D_i are (2×2) -matrix valued continuous functions, (in fact, A and B are both diagonal), and φ denotes any of the pairs ${}^T(\Phi, \Omega)$, ${}^T(\Phi, Z)$, ${}^T(N, Z)$ or ${}^T(N, \Omega)$. The vector valued function $F_i = {}^T(f_i, h_i)$ is Lipschitz continuous and continuous on the shocks up to p , where on the shocks the f_i and h_i are given by

$$(9.7) \quad \begin{aligned} f_i \circ \gamma_i &= (\alpha_i \dot{x}_i \nu_i - \beta_i \dot{x}_i \phi_i + \beta_i \omega_i - \alpha_i \zeta_i) \\ &\quad + (\alpha_i \dot{x}_i \beta_l - \beta_i \dot{x}_i \alpha_l + \beta_i \delta_l - \alpha_i \epsilon_l) |x_i(\cdot) - x_l(\cdot)| \\ h_i \circ \gamma_i &= (\delta_i \dot{x}_i \nu_i - \epsilon_i \dot{x}_i \phi_i + \epsilon_i \omega_i - \delta_i \zeta_i) \\ &\quad + (\delta_i \dot{x}_i \beta_l - \epsilon_i \dot{x}_i \alpha_l + \epsilon_i \delta_l - \delta_i \epsilon_l) |x_i(\cdot) - x_l(\cdot)|, \end{aligned}$$

for $i = 1, 2$ and $l \neq i$.

Proof. The proof resembles the proof of Lemma 8.3 in that the equivalence of the integrability condition (2.5) and the integrability condition in SSC (8.11)

$$J_{\alpha,\nu}^\mu J_\beta^\nu = J_{\beta,\nu}^\mu J_\alpha^\nu$$

follows by the same arguments, (in fact, this result holds independently of the Jacobian ansatz (8.1)). Furthermore, substituting the Jacobian ansatz for the intersecting shock curves (9.1) into the integrability condition in SSC (8.11) leads to a system of two first order PDE's similar to (8.6) except that we now sum over two shock curves instead of one. In particular, the resulting system of PDE's is of the form (9.6). The main difference to (8.6) is the appearance of additional mixed terms in the coefficients f and h of the discontinuous terms multiplying the Heaviside function $H(X)$, which if restricted to either of the shock curves γ_i are given by (9.7). \square

Note that depending on the choice of φ the matrix valued coefficient functions A and B in (9.6) might depend on φ itself, such that (9.6) is nonlinear. Nevertheless, there are choices of φ such that (9.6) is linear, for instance choosing $\varphi = {}^T(\Phi, \Omega)$ and picking $N, Z \in C^1(\mathcal{N})$ arbitrarily (not subject to the PDE's in (9.6)) leads to linear PDE's.

For our purposes here, the point of interest in the integrability condition (9.6) is that the coefficients f and h of the discontinuous Heaviside step functions $H(X)$, in contrast to the single shock case (8.6), now contain some additional mixed terms and those mixed terms do not vanish on the shock curves γ_i by the RH jump condition alone, but require a further constraint. However, taking the limit of this constraint to the point p of shock wave interaction yields exactly the condition that the Jacobian determinant $\det(J_\alpha^\mu)$ must vanish at p . Now, to finish the proof of Theorem 9.1 we show that as a consequence of (9.4), (that is, the free functions Φ, Ω, Z, N are C^1 regular at the shocks), the coefficient functions f_i and h_i must vanish on the shock curves. In summary, using the continuity of f_i and h_i the Jacobian determinant $\det(J_\alpha^\mu)$ must thus vanish at the point of shock interaction, which then implies $\det(g_{\alpha\beta}) = 0$.

The main step in the proof of Theorem 9.1 is recorded in Lemma 9.4. Before stating the lemma we compute the determinant of the Jacobian restricted to the shock curve

γ_i , using (9.3) together with the canonical form (9.1) we compute that it is given by

$$(9.8) \quad \det (J_\alpha^\mu \circ \gamma_i(t)) = (J_0^t J_1^r - J_1^t J_0^r) |_{\gamma_i(t)} = \phi_i(t)\zeta_i(t) - \nu_i(t)\omega_i(t).$$

Now, since the J_α^μ are continuous and since Definition 3.1 requires the limits

$$\lim_{t \rightarrow 0} [g_{\mu\nu,\sigma}]_i(t) = [g_{\mu\nu,\sigma}]_i(0)$$

to exist, we obtain the same limit along each shock curve γ_i , (for $i = 1, 2$), that is,

$$(9.9) \quad \begin{aligned} \phi_0 &= \lim_{t \rightarrow 0^-} \phi_i(t), \\ \omega_0 &= \lim_{t \rightarrow 0^-} \omega_i(t), \\ \zeta_0 &= \lim_{t \rightarrow 0^-} \zeta_i(t), \\ \nu_0 &= \lim_{t \rightarrow 0^-} \nu_i(t), \end{aligned}$$

and we finally get

$$(9.10) \quad \lim_{t \rightarrow 0^+} \det (J_\alpha^\mu \circ \gamma_i(t)) = \phi_0 \zeta_0 - \nu_0 \omega_0.$$

We are now ready to state the lemma and to realize its significance for the determinant of the Jacobian at p , (through (9.10)):

Lemma 9.4. *Let $p \in \mathcal{N}$ be a point of regular shock wave interaction in SSC in the sense of Definition 3.1. Then if the integrability condition*

$$(9.11) \quad J_{\alpha,\beta}^\mu = J_{\beta,\alpha}^\mu$$

holds in $\mathcal{N} \cap \mathbb{R}_-^2$ for the functions J_α^μ defined in (9.1), (so that Φ , Ω , N and Z satisfy (9.4)), then

$$(9.12) \quad \frac{1}{4B} \left(\frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_1 [B_r]_2 (\dot{x}_1 - \dot{x}_2) (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0.$$

holds at $t = 0$. (The coefficients A and B in (9.12) are evaluated at $p = (0, r_0)$ and ϕ_0, \dots, ω_0 defined according to (9.9).)

Proof. Suppose the functions J_α^μ defined in our Jacobian ansatz (9.1) solve the integrability condition (9.11). By Lemma 9.3 the functions $\Phi, \Omega, Z, N \in C^{0,1}(\mathcal{N})$ then satisfy the regularity constraint (9.4) and to solve the integrability condition in SSC (9.6). Furthermore, the coefficients in (9.6) are continuous in $\mathcal{N} \cap \mathbb{R}_-^2$ and thus taking the jump of the integrability condition (9.6) across each γ_i , $i = 1, 2$, gives

$$A[\varphi_t]_i + B[\varphi_r]_i + \sum_j D_j \left[\frac{d}{dt} \varphi \circ \gamma_j \right]_i = F_i[H(X_i)]_i,$$

and $[H(X_i)]_i = 2$ together with (9.4), (that is, $[\varphi_t]_i = 0 = [\varphi_r]_i$), yields

$$f \circ \gamma_i(t) = 0 = h \circ \gamma_i(t),$$

for all $t \in (-\epsilon, 0)$ and for $i = 1, 2$.

In the following we compute the limit of the equation $h \circ \gamma_i(t) = 0$ as t approaches 0 explicitly,¹⁹ which directly leads to (9.12) and finishes the proof. Without loss of generality we choose $i = 1$ (and thus $l = 2$ in (9.7)). Using (6.3) with respect to γ_1 , which holds since the metric is $C^{0,1}$ across γ_1 , the first term in (9.7) vanishes on γ_1 by the same arguments than in the proof of Lemma 8.4, explicitly we obtain

$$(9.13) \quad h_1 \circ \gamma_1 = (\delta_1 \dot{x}_1 \beta_2 - \epsilon_1 \dot{x}_1 \alpha_2 + \epsilon_1 \delta_2 - \delta_1 \epsilon_2) |x_1(\cdot) - x_2(\cdot)|.$$

Furthermore, for $t \neq 0$ our initial assumption that $\dot{x}_1(0) \neq \dot{x}_2(0)$ implies $|x_1(t) - x_2(t)| \neq 0$ for all $t \neq 0$ sufficiently close to 0 and thus $h \circ \gamma_1(t) = 0$ is equivalent to

$$(9.14) \quad \delta_1 \dot{x}_1 \beta_2 - \epsilon_1 \dot{x}_1 \alpha_2 + \epsilon_1 \delta_2 - \delta_1 \epsilon_2 = 0,$$

sufficiently close to 0 and by continuity (9.14) holds at $t = 0$ as well.

¹⁹Note, one can prove that $h \circ \gamma_i(t) = 0$ implies $f \circ \gamma_i(t) = 0$ and if the shock speed $\dot{x}_i(t) \neq 0$ does not vanish, then the inverse implication holds as well. However, computing the limit of $f \circ \gamma_i(t) = 0$ as t approaches 0 leads to (9.12) multiplied with the shock speeds at $t = 0$, which is not enough to prove Theorem 9.1, unless we assume the shock speeds are all nonzero.

We next compute the limit of (9.14) as t approaches 0 from below. From the definition of α_i and δ_i , (9.2), together with the RH jump condition in the form (6.3) and (6.8) we get the identity

$$(9.15) \quad \alpha_i = -\dot{x}_i \frac{B}{A} \delta_i,$$

for $i = 1, 2$, where A and B are evaluated at $p = (0, r_0)$. Now, from the definition of ϵ_i and β_i , (9.2), as well as (9.15) and the continuity of the Jacobian at p (9.9) we get that (9.14) at $t = 0$ is equivalent to

$$(9.16) \quad \begin{aligned} & \left(\left\{ \dot{x}_1 \frac{[A_r]_2}{A} - \frac{[B_t]_2}{B} \right\} \delta_1 - \left\{ \dot{x}_2 \frac{[A_r]_1}{A} - \frac{[B_t]_1}{B} \right\} \delta_2 \right) \nu_0 \\ & + \left(\left\{ \dot{x}_1 \frac{[B_t]_2}{A} - \frac{[B_r]_2}{B} \right\} \delta_1 - \left\{ \dot{x}_2 \frac{[B_t]_1}{A} - \frac{[B_r]_1}{B} \right\} \delta_2 \right) \zeta_0 = 0. \end{aligned}$$

Using (6.3) and (6.8) to eliminate $[B_t]_i$ and $[A_r]_i$ we find that the coefficients to δ_1 and δ_2 in (9.16) are related as follows:

$$(9.17) \quad \begin{aligned} \dot{x}_l \frac{[B_t]_i}{A} - \frac{[B_r]_i}{B} &= - \left(\frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_i, \\ \dot{x}_l \frac{[A_r]_i}{A} - \frac{[B_t]_i}{B} &= \dot{x}_i \left(\frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_i, \end{aligned}$$

for $i = 1, 2$ and $l \neq i$. Substituting (9.17) back into (9.16) yields

$$(9.18) \quad \left(\frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) ((\dot{x}_2 [B_r]_2 \delta_1 - \dot{x}_1 [B_r]_1 \delta_2) \nu_0 + (-[B_r]_2 \delta_1 + [B_r]_1 \delta_2) \zeta_0) = 0,$$

and using the definition of δ_i (9.2) at $t = 0$, that is,

$$\delta_i(t) = \frac{[B_t]_i \phi_0 + [B_r]_i \omega_0}{4B},$$

for $i = 1, 2$, in (9.18) we obtain that (9.14) is indeed equivalent to (9.12). This completes the proof. \square

Proof. (Theorem 9.1)

Assume there exists a $C^{1,1}$ -coordinate transformation in the (t, r) -plane mapping SSC to coordinates x^α such that $g_{\alpha\beta}$ is in $C^1(\mathcal{N} \cap \mathbb{R}_-^2)$ for some neighborhood \mathcal{N} of p . By Lemma 7.2, the Jacobian J_α^μ of the coordinate transformation satisfies the necessary and sufficient smoothing condition (7.9) on each of the shock curves

γ_i , ($i = 1, 2$), since the metric $g_{\mu\nu}$ satisfies the RH jump conditions (6.3) and (6.4) across each of the shock curves. Consequently, the Jacobians J_α^μ can be written in terms of the canonical form (9.1) with $C^{0,1}$ functions Φ, Ω, N and Z , which meet the regularity condition (9.4) that their derivatives are continuous across the shock curves. Furthermore, by assumption the J_α^μ are integrable to coordinates and thus they satisfy the integrability condition (2.5), (c.f. Appendix A). Applying Lemma 9.4 yields (9.12) at the point p of shock wave interaction, that is,

$$(9.19) \quad \frac{1}{4B} \left(\frac{\dot{x}_1 \dot{x}_2}{A} + \frac{1}{B} \right) [B_r]_1 [B_r]_2 (\dot{x}_1 - \dot{x}_2) (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0.$$

To finish the proof of Theorem 9.1, observe that the first three terms in (9.19) are nonzero by our assumption that shock curves are non-null, and have distinct speeds at $t = 0$. In more detail, by our assumptions in definition 3.1

$$(\dot{x}_1 - \dot{x}_2) \neq 0,$$

while

$$\frac{\dot{x}_i \dot{x}_l}{A} + \frac{1}{B} \neq 0$$

for time-like shock curves, which satisfy

$$\dot{x}_j^2 < \frac{A}{B}.$$

Finally,

$$[B_r]_i \neq 0,$$

since otherwise $[T^{\mu\nu}]_i = 0$ for all $\mu, \nu = 0, 1$ due to the Einstein equations (6.5)-(6.7), in contradiction to our assumptions about points of regular shock wave interaction in SSC, (c.f. Definition 3.1). We conclude that indeed

$$(9.20) \quad \phi_0 \zeta_0 - \nu_0 \omega_0 = 0$$

and using the explicit expression for the Jacobian at the point p of interaction (9.8)-(9.10) we finally conclude that

$$(9.21) \quad \det J_\alpha^\mu(p) = (\phi_0 \zeta_0 - \nu_0 \omega_0) = 0,$$

as claimed. This completes the proof of Theorem 9.1. \square

Theorem 9.1 is stated for the upper (\mathbb{R}_+^2) or the lower half plane (\mathbb{R}_-^2) separately. (In this way, one can apply the theorem to initial value problems at the $t = 0$ axis.) A straightforward consequence of Theorem 9.1 is the following corollary:

Corollary 9.5. *Suppose that p is a point of regular shock wave interaction in SSC, in the sense of Definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there does not exist a $C^{1,1}$ coordinate transformation $x^\alpha \circ (x^\mu)^{-1}$ of the (t, r) -plane, defined on a neighborhood \mathcal{N} of p in \mathbb{R}^2 , such that the metric components $g_{\alpha\beta}$ are C^1 functions of the coordinates x^α in \mathcal{N} and such that the metric has a non-vanishing determinant at p , (that is, $\det(g_{\alpha\beta}(p)) \neq 0$).*

Proof. Assume there exist a coordinate transformation of the (t, r) -plane defined on a neighborhood \mathcal{N} of p . Then, this coordinate transformation is also defined on $\mathcal{N} \cap \mathbb{R}_-^2$ and smooth the metric $g_{\alpha\beta}$ to C^1 . Applying Theorem 9.1 completes the proof. \square

We remark that at first there appears to be more than enough freedom to choose the free functions Φ, Ω, Z, N of the canonical form to arrange for the discontinuous term in the integrability condition to vanish. This together with the fact that the derivatives of J_α^μ are uniquely solvable in condition (7.9), lead us to believe until the very end that one could construct coordinates in which $g_{\alpha\beta}$ was $C^{1,1}$. But at the very last step, taking the limit of the integrability constraints to the limit of shock wave interaction p , we find that the condition (9.4), expressing that $[g_{\alpha\beta,\gamma}]$ vanishes at shocks, has the effect of freezing out all the freedom in Φ, Ω, Z, N , thereby forcing the determinant of the Jacobian to vanish at p . The answer was not apparent until the very last step, and thus we find the result remarkable, and most surprising!

10. SHOCK WAVE INTERACTIONS AS REGULARITY SINGULARITIES IN GR;
THE FULL ATLAS

For the proof of Theorem 1.1, we have established the nonexistence of $C^{1,1}$ coordinate transformations in the (t, r) -plane that can map a $C^{0,1}$ regular SSC metric $g_{\mu\nu}$ over to a $C^{1,1}$ metric $g_{\alpha\beta}$ in Theorem 9.1. It remains to extend this result to the full atlas of coordinate transformations that depend on all four coordinate variables, including the SSC angular variables. For completeness we state Theorem 1.1 again:

Theorem 10.1. *Suppose that p is a point of regular shock wave interaction in SSC, in the sense of Definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there does not exist a $C^{1,1}$ coordinate transformation $x^\alpha \circ (x^\mu)^{-1}$, defined on a neighborhood \mathcal{N} of p in \mathbb{R}^2 , such that the metric components $g_{\alpha\beta}$ are C^1 functions of the coordinates x^α in \mathcal{N} and such that the metric has a non-vanishing determinant at p , (that is, $\det(g_{\alpha\beta}(p)) \neq 0$).*

The proof method is as follows, we assume for contradiction that there are coordinates in which the metric is C^1 , in general the metric does have the full ten component and is not of the box diagonal form (2.19), that is, a metric of the form

$$(10.1) \quad ds^2 = -\bar{A}d\bar{t}^2 + \bar{B}d\bar{r}^2 + 2\bar{E}d\bar{t}d\bar{r} + \bar{C}d\bar{\Omega}^2,$$

where \bar{A} , \bar{B} , \bar{E} and \bar{C} depend on coordinates \bar{t} and \bar{r} only and

$$d\bar{\Omega} := d\bar{\vartheta}^2 + \sin^2(\bar{\vartheta})d\bar{\varphi}^2$$

is the induced metric on the unit sphere. However, (partially following the arguments in [18]), we can always transform a general spherically symmetric metric back to one of the box diagonal form (2.19) with the same induced metric on the spheres of symmetry than $g_{\mu\nu}$ and still preserve the C^1 metric regularity. The resulting metric is then related to $g_{\mu\nu}$ by a transformation in the (t, r) -plane, in contradiction to Theorem 9.1.

Before we begin with the proof of Theorem 10.1 we need to study the regularity of the Killing vector, which solve Killing's equations. In the following lemma we assume

that a $C^{0,1}$ solution of Killings equation is given, as it will be the case in the proof of Theorem 10.1 below. (The existence of a solution is a consequence of our incoming assumption of spherical symmetry and the $C^{0,1}$ regularity is a consequence of the fact that the isometries of the metric in SSC are smooth rotations acting on the spheres of symmetry, which then give rise to smooth Killing vectors, thus the Killing vectors are at least $C^{0,1}$ in any other coordinate system that can be reached within the atlas of $C^{1,1}$ coordinate transformations).

Lemma 10.2. *Let X_i be a $C^{0,1}$ solution of Killing's equation in a given coordinate system x^i , that is,*

$$(10.2) \quad X_{i,j} + X_{j,i} = 2\Gamma_{ij}^k X_k,$$

and suppose that the Γ_{ij}^k 's denote the Christoffel symbols of a C^1 metric g_{ij} , then X_i and the Killing vector $X^i = g^{ij} X_j$ are in C^1 in the coordinates x^i .

Proof. Suppose that with respect to partial differentiation in coordinates x^i the metric g_{ij} is C^1 and the Killing vector X^i is in $C^{0,1}$ and solves (10.2). The Lipschitz continuity of X_i implies that the derivatives $X_{i,j}$ exist almost everywhere and are at worst discontinuous, (c.f. [2]). Assume for contradiction that p is a point at which X_i is not in C^1 , then there exist a smooth hypersurface Σ containing p such that

$$[X_{i,j}] \neq 0,$$

for some $i, j \in \{0, \dots, 3\}$, where $[\cdot]$ denotes the jump at p across Σ . Since Σ is smooth all tangent vectors are smooth and tangential derivatives of X_i match up continuously across Σ , (c.f. (6.1)),

$$(10.3) \quad [X_{i,j}]v^j = 0$$

for any vector v^j tangent to Σ . (Note, in (10.3) the derivatives of X_i tangent to Σ could be discontinuous, however, the point is that their value agrees on both sides across Σ , since the difference quotient of X_i composed with some curve tangent to

Σ contains only expressions that match up continuously across Σ .) Pointwise, $[X_{i,j}]$ is a matrix and (10.3) implies that $[X_{i,j}]$ has a three dimensional nullspace at each point on Σ . Furthermore, taking the jump across Σ of Killing's equations (10.2) and using the continuity of the Christoffel symbols Γ_{ij}^k , (a consequence of our incoming assumption that g_{ij} is C^1), leads to

$$(10.4) \quad [X_{i,j}] + [X_{j,i}] = 0.$$

We conclude that $[X_{i,j}]$ is antisymmetric and thus has a vanishing trace,

$$\text{tr}([X_{i,j}]) = 0.$$

However, the trace of a matrix is the sum of its eigenvalues, but three of the eigenvalues of $[X_{i,j}]$ are zero, due to the three dimensional nullspace. Thus (10.4) implies that all eigenvalues vanish and therefore

$$(10.5) \quad [X_{i,j}] = 0 \quad \text{for all } i, j \in \{0, \dots, 3\}.$$

This contradicts the assumption that X_i is not in C^1 at p , and since p is an arbitrary point we conclude that X_i is in C^1 . Now, since by assumption g_{ij} is in C^1 we proved that $X^i = g^{ij} X_j$ is in C^1 . \square

Proof. (Theorem 10.1)

Assume for contradiction there exist coordinates x^j in which the metric g_{ij} is C^1 and has a non-vanishing determinant at p . In general g_{ij} is not of the box diagonal form (2.19). In the following we first prove that (a) one can always transform back to a metric $g_{\alpha\beta}$ in box diagonal form and preserve the metric regularity and (b) the resulting metric is then related to the SSC metric (in which p is a point of regular shock wave interaction) by a coordinate transformation in the (t, r) -plane. This then contradicts Theorem 9.1 and proves Theorem 10.1.

For the moment assume that (a) holds and that the metric g_{ij} is in the box diagonal form (10.1) with an induced metric $d\bar{\Omega} := d\bar{\vartheta}^2 + \sin^2(\bar{\vartheta})d\bar{\varphi}^2$, is in C^1 and has a non-vanishing determinant. This induced metric can be taken over to the line element $d\Omega := d\vartheta^2 + \sin^2(\vartheta)d\varphi^2$ of the SSC metric we started in by a coordinate transformation on the spheres of symmetry alone, that is, by a change of the angular variables only $(\bar{\vartheta}, \bar{\varphi}) \rightarrow (\vartheta, \varphi)$, (see [18], chapter 13.2, for a detailed proof). Since both induced metrics, $d\bar{\Omega}$ and $d\Omega$ are smooth, the coordinate transformation is smooth and preserves the C^1 regularity of the full metric. Now, the resulting metric,

$$(10.6) \quad ds^2 = -\bar{A}d\bar{t}^2 + \bar{B}d\bar{r}^2 + 2\bar{E}d\bar{t}d\bar{r} + \bar{C}d\bar{\Omega}^2,$$

can be taken over to our original SSC metric by a transformation in the (t, r) -plane, since both metrics, g_{ij} and this SSC metric, are assumed to be related by a coordinate transformation in the first place. (For instance, defining \bar{C} as the new radial coordinate and suitably rescaling the time coordinate to eliminate the resulting off-diagonal element we can map (10.6) to the SSC metric we started in.) However, this contradicts Theorem 9.1 since the metric (10.6) is in C^1 , has a non-vanishing determinant and can be reached from the SSC metric we started in by a coordinate transformation in the (t, r) -plane.

It remains to prove (a), that is, in a spherically symmetric spacetime there always exist a coordinate transformation that takes the C^1 metric g_{ij} over to a metric of box diagonal form (10.1) which is still in C^1 with respect to the new coordinates. We now construct this coordinate transformation, partially following the construction in [18], (chapter 13.5). (The result that this coordinate transformation preserve the C^1 metric regularity is our own contribution.)

We now prove that, if g_{ij} is in C^1 , then there exist a C^2 coordinate transformation such that all Killing vectors have two identically vanishing components, that is, in

the new coordinates all Killing vectors satisfy

$$(10.7) \quad X^i = 0 \quad \text{for } i = 0, 1.$$

To prove the existence of such a coordinate transformation it suffices to choose two linearly independent Killing vectorfields and two linearly independent vectorfields that are pointwise orthogonal to the two Killing vectorfields, such that the flow of these four vectorfields defines a coordinate system. (Such a set of vectorfields exists since, by assumption, the metric g_{ij} is related to a SSC metric through a coordinate transformation.) Integrating those vectorfields leads to a coordinate system in which (10.7) holds. The metric in the new coordinates has again a non-vanishing determinant, since the vectorfields are linearly independent in a neighborhood of p . Furthermore, if the Killing vectorfields are C^1 then the orthogonal vectorfields are C^1 , resulting in a C^2 coordinate transformation that preserves the C^1 metric regularity and maps to coordinates where (10.7) holds. Thus, it remains to prove that the Killing vectors in coordinates x^i are C^1 , which follows from Lemma 10.2, since g_{ij} is in C^1 and since the Killing vectors are $C^{0,1}$ regular. (In more detail, the Killing vectors are $C^{0,1}$ since the isometries of the metric in SSC are smooth rotations acting on the spheres of symmetry and, being the flux of Killing vectors, the Killing vectors in SSC are smooth as well and must be $C^{0,1}$ regular in any other coordinate system that can be reached within the atlas of $C^{1,1}$ coordinate transformations.) We denote the new coordinates again with x^i .

In the following we adapt the notation in [18], splitting up the coordinate indices as follows

$$(10.8) \quad \begin{aligned} u^i &= x^i & \text{for } i = 2, 3 \\ \text{and } v^a &= x^a & \text{for } a = 0, 1. \end{aligned}$$

We first prove that the metric takes on the form

$$(10.9) \quad ds^2 = g_{ab}(v) dv^a dv^b + f(v) \bar{g}_{ij}(u) du^i du^j,$$

where summation over a and b runs from 0 to 1, summation over i and j runs from 2 to 3, f is some C^1 function and $\bar{g}_{ij}(u)$ denotes the induced metric on the space of symmetry. A major advantage of the notation (10.8) is that in the coordinates we constructed above the metric satisfies

$$(10.10) \quad g_{ia} = 0,$$

for $a = 0, 1$ and $i = 2, 3$. From (10.10) and (10.7) we find that Killing's equation (10.2) (for indices $a, b \in \{0, 1\}$) implies

$$(10.11) \quad X^k \frac{\partial g_{ab}}{\partial u^k} = 0$$

and since this holds true for all Killing vectors we conclude that

$$(10.12) \quad \frac{\partial g_{ab}}{\partial u^k} = 0.$$

To verify (10.9) it remains to prove that $g_{ij}(v, u) = f(v)\bar{g}_{ij}(u)$ for some function f . This follows from Killing's equations, which imply that any symmetric 2-form $C_{ij}(u, v) du^i du^j$ on the spaces of symmetry agrees with the metric $\bar{g}_{ij}(u) du^i du^j$ up to a factor $f(v)$, (see [18], chapter 13.4, for a detailed derivation). This proves that in the coordinates x^i the metric is of the form (10.9).

For an induced metric on the spaces of symmetry with positive eigenvalues only and with a positive constant curvature K , the metric (10.9) can be written in the ("Euclidean") form

$$(10.13) \quad ds^2 = g_{ab}(v) dv^a dv^b + f(v) \left(d\vec{u}^2 + \frac{(\vec{u} \cdot d\vec{u})^2}{1 - \vec{u}^2} \right),$$

for $\vec{u} = (u^1, u^2)$. (By assumption a spherically symmetric spacetime has a positive constant curvature.) To finish the proof introduce angular variables via

$$(10.14) \quad \begin{aligned} u^1 &= \sin \vartheta \cos \varphi \\ u^2 &= \sin \vartheta \sin \varphi, \end{aligned}$$

in those coordinates the metric is of the box diagonal form (10.6) and (10.14) preserves the C^1 metric regularity. This proves (a) and completes the proof of Theorem 10.1. \square

A straightforward consequence of Theorem 10.1 is that at points of regular shock wave interaction the Einstein equations can only hold in the weak sense. From this we conclude that the weak formulation of the Einstein Field Equations is more fundamental and essential.

Corollary 10.3. *Suppose that p is a point of regular shock wave interaction in SSC, in the sense of Definition 3.1, for the SSC metric $g_{\mu\nu}$. Then there does not exist a coordinate system that can be reached from SSC in the class of $C^{1,1}$ coordinate transformations, such that the metric g is $C^{1,1}$ and solves the Einstein equations in the strong sense, that is, pointwise almost everywhere.*

Theorem 10.1 proves that the lack of C^1 regularity is invariant under all $C^{1,1}$ coordinate transformations and the atlas of $C^{1,1}$ coordinate transformations is generic to address shock waves in General Relativity for the following two reasons. Firstly, C^2 coordinate transformations cannot lift the metric regularity in the first place (c.f. Section 7), while a $C^{1,\alpha}$ atlas presumably fails to provide enough free parameters to the smoothing condition (7.4), for $\alpha \neq 1$, (it seems to be appropriate only for metric tensors in $C^{0,\alpha}$). Secondly, a $C^{1,1}$ atlas is natural as it preserves the weak formalism of the Einstein equations, while for any atlas regularity below C^1 , we expect that a weak formulation of the Einstein equations fails to exist (e.g., a $C^{0,1}$ atlas with resulting discontinuous metric components). Given this, points of regular shock wave interaction in SSC are singularities that cannot be removed by the means of $C^{1,1}$ coordinate transformations, that either have unbounded second order metric derivatives or a vanishing metric determinant at the point of shock interaction and which we therefore call *regularity singularities*.

11. THE LOSS OF LOCALLY INERTIAL FRAMES

Finally we shed light on the non-existence of locally inertial frames around a point of regular shock wave interaction, that we claimed in the Introduction. This is in vast contrast to the situation of only a single shock wave (c.f. [11]), where such coordinates always exist. We first clarify what we mean by a locally inertial frame:

Definition 11.1. *Let p be a point in a Lorentz manifold and let x^j be a coordinate system defined on a neighborhood of p . We call x^j a locally inertial frame around p if the metric g_{ij} in those coordinates satisfies:*

- (i) $g_{ij}(p) = \eta_{ij}$, where $\eta_{ij} = \text{diag}(-1, 1, 1, 1)$ denotes the Minkowski metric,
- (ii) $g_{ij,l}(p) = 0$ for all $i, j, l \in \{0, \dots, 3\}$,
- (iii) $g_{ij,kl}$ are bounded on every compact neighborhood of p .

We refer to a metric g_{ij} , that satisfies (1)-(3), as a locally Minkowskian (or locally flat or locally inertial) metric around p .

Condition (iii) of Definition 11.1 ensures that physical equations in GR (which are tensorial) differ from the corresponding equations in flat Minkowski space only by gravitational effects, since those are of second order in the metric derivatives. (In the standard literature the metric is assumed to be smooth, thus implying condition (iii) of Definition 11.1 right away.)

Now, by Theorem 1.1, there exist distributional and thus unbounded second order derivatives of the metric. Therefore, the following Corollary is a straightforward consequence of Theorem 1.1:

Corollary 11.2. *Let p be a point of regular shock wave interaction in SSC in the sense of Definition 3.1, then there does not exist a $C^{1,1}$ coordinate transformation such that the resulting metric g_{ij} is locally Minkowskian around p .*

Proof. Assume for contradiction that there exist a locally inertial frame x^i around the point p of regular shock wave interaction. Then the metric in coordinates x^i satisfies $g_{ij}(p) = \eta_{ij}$ and thus has a non-vanishing determinant, namely

$$\det g_{ij}(p) = 2 \neq 0.$$

By Theorem 10.1 the metric g_{ij} is not in C^1 , thus, there exist indices $i, j, l \in \{0, \dots, 3\}$ such that $g_{ij,l}$ is discontinuous and therefore there exist indices $i, j, l, k \in \{0, \dots, 3\}$ such that $g_{ij,lk}$ is distributional and not bounded. This completes the proof. \square

Corollary 11.2 proves that the gravitational metric cannot be locally Minkowskian at points of regular shock wave interaction in SSC since unbounded second order metric derivatives occur. However, at the present state it is not clear to the author if there exist coordinates in which the metric satisfies condition (i)-(ii) of Definition 11.1 at least and if those coordinates could play the role of locally inertial frames in some physically satisfying sense.

12. CONSTRUCTION OF A JACOBIAN ON A DELETED NEIGHBORHOOD AND THE GLUING CONDITIONS

The proof of Theorem 9.1 was constructive, providing a canonical form for Jacobians necessary and sufficient to smooth the metric to C^1 in a neighborhood of a point p of shock wave interaction, (c.f. (9.1)). In fact, for the proof of Theorem 9.1 it suffices to consider the upper or lower half plane only, $(\mathbb{R}_+^2$ or $\mathbb{R}_-^2)$, and the only obstruction to the existence of such Jacobians was a vanishing metric determinant at p . In this section we outline how to extend the Jacobian ansatz (9.1) to a deleted neighborhood²⁰ of p , ensuring its integrability and thereby smoothing the metric to C^1 in the new coordinates. The key point is that some “*gluing conditions*” need to be imposed at the surface $\{t = 0\}$ for a shock wave interaction at $t = 0$.²¹

As in the construction of the Jacobians in Theorem (8.1), requiring the Jacobians to be C^1 regular away from the shock surfaces is a generic condition for smoothing the metric, in order to avoid the appearance of additional points where the metric is not C^1 . On \mathbb{R}_\pm^2 , the regularity of the Jacobian J_α^μ away from the shocks is governed by the integrability condition (9.6) and the choice of the free functions in the Jacobian ansatz (9.1), (without loss of generality we take N and Z to be those free functions and solve (9.6) for Φ and Ω). However, at the surface $\{t = 0\}$ a C^1 regularity of the Jacobian imposes some additional conditions on Φ , Ω , N and Z . In more detail,

²⁰By a deleted neighborhood we mean a neighborhood of p from which some open set containing p is excluded. It is conceivable that one can construct an integrable Jacobian on a neighborhood of p from which only the point p is excluded. However, it is not clear if the integrability condition (9.6) is well-posed at the point p , if initial values are prescribed at the surface $t = 0$, due to the tangential derivatives $\frac{d\phi_i}{dt}$ in (9.6).

²¹ The purpose of this section is to introduce the idea of matching Jacobians constructed on \mathbb{R}_\pm^2 across $t = 0$ according to the gluing conditions, however, this section does not contain anything relevant for our main results Theorem 9.1 and 10.1.

denoting with $\{\cdot\}$ the jump across $t = 0$, that is,

$$\{u\}(r) = \lim_{t \searrow 0} u(t, r) - \lim_{t \nearrow 0} u(t, r)$$

for some function u , the conditions ensuring a C^1 regular Jacobian are

$$\begin{aligned} \{\Phi_t\} &= -(\dot{\alpha}_1 + \dot{\alpha}_2 - \dot{\alpha}_3 - \dot{\alpha}_4) |r_0 - r| \\ \{\Omega_t\} &= -(\dot{\delta}_1 + \dot{\delta}_2 - \dot{\delta}_3 - \dot{\delta}_4) |r_0 - r| \\ \{N_t\} &= -(\dot{\beta}_1 + \dot{\beta}_2 - \dot{\beta}_3 - \dot{\beta}_4) |r_0 - r| \\ (12.1) \quad \{Z_t\} &= -(\dot{\epsilon}_1 + \dot{\epsilon}_2 - \dot{\epsilon}_3 - \dot{\epsilon}_4) |r_0 - r| \end{aligned}$$

and

$$\begin{aligned} \{U\} &= 0 \\ (12.2) \quad \{U_r\} &= 0, \end{aligned}$$

where $U \in \{\Phi, \Omega, N, Z\}$ and $\dot{\alpha}_i, \dots, \dot{\epsilon}_i$ are the derivatives with respect to t of the Jacobian coefficients $\alpha_i, \dots, \epsilon_i$ in (9.2) evaluated at $t = 0$. We refer to the conditions (12.1) - (12.2) as “gluing conditions”. In the following lemma we prove that the Jacobian J_α^μ is C^1 across the surface $\{t = 0\}$ if and only if the gluing conditions (12.1) - (12.2) hold for all $r \neq r_0$.

Lemma 12.1. *Suppose the functions Φ, Ω, N and Z of the Jacobian ansatz 9.1 are in $C^1(\mathbb{R}_\pm^2)$, defined on a neighborhood of a point p of regular shock wave interaction in SSC, then J_α^μ is C^1 at the surface $\{(t, r) \in \mathbb{R}^2 : t = 0, r \neq r_0\}$ if and only if Φ, \dots, Z satisfy the gluing conditions (12.2)-(12.1).*

Proof. From the canonical form of the Jacobian 9.1, it is straightforward to compute that $\{J_0^t\} = 0$ if and only if

$$(12.3) \quad (\alpha_1(0) + \alpha_2(0) - \alpha_3(0) - \alpha_4(0)) |r_0 - r| + \{\Phi\}(r) = 0,$$

for all $r \neq r_0$. Now, at $t = 0$ the jumps of the metric derivatives across the shocks satisfy

$$(12.4) \quad [g_{\mu\nu, \sigma}]_1 + [g_{\mu\nu, \sigma}]_2 = [g_{\mu\nu, \sigma}]_3 + [g_{\mu\nu, \sigma}]_3,$$

for all $\mu, \nu, \sigma = 0, 1$, due to the C^1 regularity of the metric away from the shock curves, thus (12.3) is equivalent to

$$\{\Phi\}(r) = 0.$$

We conclude that $J_0^t \in C^0(\mathbb{R}^2)$ if and only if $\{\Phi\} = 0$. The proofs the remaining equivalences follow by similar arguments, using (12.4) whenever appropriate. This completes the proof. \square

The proof of the next lemma gives a construction of functions that are in $C^1(\mathbb{R}_\pm^2)$ and satisfy the gluing conditions (12.2)-(12.1). The key idea is to use a mollifier in time to approximate a function that is only Lipschitz continuous with respect to r , namely a function of the form $c|r - r_0|$ with c prescribed by the respective right hand side of equations (12.1).

Lemma 12.2. *Let $c \in \mathbb{R}$ be a constant, then there exist a function Φ in $C^1(\mathbb{R}_+^2 \cup \mathbb{R}_-^2) \cap C^{0,1}(\mathbb{R}^2)$ such that*

$$\begin{aligned} \{\Phi\} &= 0 \\ \{\Phi_r\} &= 0 \\ \{\Phi_t\} &= c|r - r_0|. \end{aligned} \tag{12.5}$$

Proof. Without loss of generality assume that $r_0 = 0$. Define

$$\phi^t(x) = \exp\left(\frac{t^2}{x^2 - t^2}\right),$$

then

$$(\phi^t \star |\cdot|)(r) = \int_{-\infty}^{\infty} \phi^t(r - x)|x|dx \tag{12.6}$$

is a smooth function that converges pointwise to the absolute value function $|\cdot|$ as t approaches 0, [2]. Now, define

$$\Phi(t, r) = \int_0^t \frac{c}{2} H(\tau) (\phi^\tau \star |\cdot|)(r) d\tau, \tag{12.7}$$

where H denotes the Heaviside step function, then $\Phi(0, r) = 0$ for all $r \in \mathbb{R}$ and thus $\{\Phi\} = 0 = \{\Phi_r\}$. Furthermore,

$$\Phi_t(t, r) = \frac{c}{2} H(t) (\phi^t \star |\cdot|) (r),$$

and it is straightforward to check that $\{\Phi_t\} = c|r|$, as claimed. \square

The gluing conditions (12.1) on Φ and Ω need to be consistent with the evolution of (Φ, Ω) imposed by the integrability condition (9.6). The next Lemma states the surprising result that the integrability conditions (9.6) always ensures that the gluing conditions (12.1) hold:

Lemma 12.3. *Let N and Z be in $C^1(\mathbb{R}_{\pm}^2)$ such that both functions satisfy the gluing conditions (12.2) across the surface $\{t = 0\}$ and such that*

$$(12.8) \quad (\beta_1 + \beta_2) |r - r_0| + \lim_{t \searrow 0} N(t, r) \neq 0.$$

Furthermore, suppose that Φ and Ω solve the integrability condition (9.6) on both \mathbb{R}_+^2 and \mathbb{R}_-^2 separately, and that $\{\Phi\} = 0 = \{\Omega\}$. Then Φ and Ω satisfy the gluing conditions (12.2).

Proof. First observe that the assumed continuity of Φ and Ω across the surface $\{t = 0\}$ implies that the tangential derivatives match up, that is, $\{\Phi_r\} = 0 = \{\Omega_r\}$, (by the same argument leading to (6.1)). It remains to prove that the derivatives of Φ and Ω with respect to time t match up continuously across the $t = 0$ surface. Subtracting the upper from the lower limit of the integrability condition (9.6), using that N and Z satisfy the gluing conditions (12.2) and using that $\{\Phi\} = 0 = \{\Omega\}$ and $\{\Phi_r\} = 0 = \{\Omega_r\}$, we get

$$(12.9) \quad \begin{aligned} \{\Phi_t\} a^<(r) + \left(\dot{\beta}_1 + \dot{\beta}_2 - \dot{\beta}_3 - \dot{\beta}_4 \right) a^<(r) |r_0 - r| &= 0 \\ \{\Omega_t\} a^<(r) + \left(\dot{\delta}_1 + \dot{\delta}_2 - \dot{\delta}_3 - \dot{\delta}_4 \right) a^<(r) |r_0 - r| &= 0, \end{aligned}$$

where

$$a^<(r) = (\beta_1 + \beta_2) |r - r_0| + \lim_{t \searrow 0} N(t, r).$$

Since $a^<$ is non-zero we conclude that $\{\Phi_t\} = 0 = \{\Omega_t\}$ and thus, the gluing condition (12.2) hold. This completes the proof. \square

For the construction of a Jacobian smoothing the metric to C^1 on a deleted neighborhood around p we now prescribe initial data on the surface $\{t = 0\}$ and solve the integrability condition (9.6) for (Φ, Ω) forward and backward in time. Then (Φ, Ω) match up continuously across $t = 0$ and choosing in addition N and Z such that the gluing conditions (12.2) and the technical assumption (12.8) hold, Lemma 12.3 yields that (Φ, Ω) satisfy the gluing conditions as well.²² Finally, we can choose $N \circ \gamma_i(t)$ and $Z \circ \gamma_i(t)$ such that the gluing conditions hold and $f_i(t) = 0 = h_i(t)$ for $i \in \{1, 2, 3, 4\}$ and thereby ensuring $(\Phi, \Omega) \in C^1(\mathbb{R}_\pm^2)$. This construction leads to a Jacobian that lifts the metric regularity to C^1 on a deleted neighborhood of p .

²²Initially we thought that arranging for the gluing condition would remove two free parameters, (e.g., ϕ_0 and ω_0), thus, we are very surprised that the integrability condition implies the gluing condition. This originally strengthened our believe that one can smooth the metric to C^1 and, in light of this, the discovery that the metric determinant must vanish was extremely unexpected and surprising.

13. DISCUSSION

Our results stated in Theorem 9.1 and Theorem 10.1 show that at points of regular shock wave interaction in SSC the lack of C^1 metric regularity cannot be removed by any $C^{1,1}$ coordinate transformation, in contrast to the situation of a single shock wave, (c.f. Theorem 8). We thus interpret points of regular shock wave interaction as spacetime singularities, (which we called “regularity singularities”), since we do not expect that one can extend the $C^{1,1}$ atlas to an even lower regularity. Namely, in view of the necessary (but not sufficient) smoothing condition 7.9, there is a unique solution $[J_{\alpha,\beta}^\mu]$ for (possibly) lifting the metric regularity to C^1 , however, a Jacobian which is Hölder but not Lipschitz continuous cannot meet this condition, since it does not “mirror” the $C^{0,1}$ metric regularity in an appropriate way. Furthermore, it is not obvious if the Einstein equations with a bounded source term are well-posed under general coordinate transformation with only Hölder continuous Jacobians. Moreover, lowering the atlas regularity to anything less smooth than C^1 would conflict with the weak formulation of the Einstein equations. For example, a $C^{0,1}$ atlas gives rise to coordinates in which the metric is discontinuous, now the nonlinear structure of the Einstein equations prohibits first order metric derivatives to be shifted to test functions, resulting in a breakdown of the weak formulation of the equations. We therefore conclude that points of regular shock wave interaction in SSC are spacetime singularities.

The scalar curvature is bounded at a regularity singularity, since the energy momentum tensor is bounded and the scalar curvature is given by the trace of the Einstein tensor, that is,

$$R = R^\mu_\mu = -G^\mu_\mu = -T^\mu_\mu.$$

Nevertheless, we believe that the unbounded second order metric derivatives could provide effects similar to an unbounded curvature. Moreover, points of regular shock wave interaction are not hidden from observation by an event horizon, which opens up the opportunity of direct measurement of effects that could resemble unbounded

scalar curvature. Maybe, such a measurable effect could be predicted by analyzing some differential equation (for example, the Maxwell or Dirac equation) defined on a background manifold containing a point of shock wave interaction in SSC. For instance, in a beam of electrons or photons emitted from the center of a supernovae, one might be able to measure some effect (e.g., a phase shift) when the outer shell of the star collapses onto the inner cooled off core. Regarding such a presumable effect, a regularity singularity provides an opportunity similar to a naked singularity, namely, a direct measurement of infinite second order metric derivatives.

At points of regular shock wave interaction in SSC the Einstein equation can only hold weakly and the metric has distributional second order derivatives. It is sometimes postulated to exclude points from spacetime at which the metric has a regularity lower than C^2 (c.f. [5]), however, we oppose to this requirement in cases where the Einstein equations can still be made sense of within the theory of distributions, since we expect weak solutions to have the same physical significance than strong ones. Thus, we consider points of shock wave interaction and regularity singularities to be included in spacetime. This point of view is further strengthened by Israel's Theorem, which shows that around each point q on any of the incoming or outgoing shock surfaces there exist a coordinate system such that the metric in the new coordinates is $C^{1,1}$ regular, except when q is the point p of shock wave interaction itself. Thus, there exists coordinates such that the Einstein equations hold strongly in a deleted neighborhood of the point p of shock wave interaction. In other words, for an appropriate choice of coordinates a strong solution of the Einstein equations evolves into a shock wave interaction, there it solves the equations only weakly, but then instantly continues evolution as a strong solution again. We take this as a definite physical proof that weak solutions have the same significance than strong ones.

When the first singularities in solutions of the Einstein equations were found, (notably the Schwarzschild, Reisner-Nordström and Kerr metrics), the idea was proposed that the appearance of those singularities are due to the assumption of spherical symmetry, but that they do not reflect real physical effects. It was not until the late sixties when Penrose and Hawking clarified that singularities do also arise in spacetimes without any symmetries, (c.f. [5]). An analogous question arises for the (regularity) singularity we discovered as well, since the assumption of a spherically symmetric metric and of radial shock surfaces enter our methods in Sections 7 - 9 heavily. However, we expect that removing our symmetry assumptions does not alter our results, since for a metric without any symmetries the smoothing condition (7.4) gives 64 equations in 64 unknowns $[J_{\alpha,\beta}^\mu]$ after imposing the integrability condition

$$[J_{\alpha,\beta}^\mu] = [J_{\beta,\alpha}^\mu].$$

We expect that (7.4) still has a unique solution, that one can construct an extended ansatz for the Jacobians similar to the one in spherical symmetry (9.1) and that a procedure analogous to the one in Section 9.1 proves again the existence of a regularity singularity.

To end this section we discuss the loss of locally inertial frames stated in Corollary 11.2 and the effects of incorporating viscosity terms. Conceptually, if one includes points of shock wave interaction into spacetime, this conflicts with Einstein's original postulate that gravity shall enter the equations of physics as a (bounded) second order correction only. At a regularity singularity this correction is unbounded, seemingly contradicting the basic framework of General Relativity. However, we expect this contradiction to be resolved in a more physical model, that is, after introducing viscosity into the equations. (Note that the issue of how to incorporate a relativistic viscosity that meets the speed of light bound is problematic, [18], but we believe viscosity in the framework of General Relativity to exist.) Namely, introducing (Navier Stokes type) viscosity terms into the perfect fluid source of the Einstein equation causes a

small scale smoothing of the discontinuities in the fluid variables, thus smearing out the shock profiles, such that the metric tensor is smoothed out correspondingly. Then locally inertial frames exist and gravity enters the equation of physics with second order. (In the limit to zero viscosity the discontinuities in the fluid source is recovered and a (regularity) singularity appears again in the metric.) However, even if viscosity is introduced, steep gradients in the fluid sources persist and we still expect to recover large second order metric derivatives that blow up as the viscosity tends to zero. From this we conclude that there should be some physical effect on the metric, even if viscosity terms in the fluid source are present.

14. CONCLUSION

Our results show that points of shock wave interaction give rise to a new kind of singularity which is different from the well known singularities of GR. The famous examples of singularities in GR are either non-removable singularities beyond physical spacetime, (for example the center of the Schwarzschild and Kerr metrics, and the Big Bang singularity in cosmology where the curvature cannot be bounded), or else they are removable in the sense that they can be transformed to locally inertial points of a regular spacetime under coordinate transformation, (for example, the apparent singularity at the Schwarzschild radius, the interface at vacuum in the interior Schwarzschild, Oppenheimer-Snyder, Smoller-Temple shock wave solutions, and any apparent singularity at smooth shock surfaces that become regularized by Israel's Theorem, [6, 11]). In contrast, points of shock wave interaction are non-removable singularities that propagate in physically meaningful spacetimes in GR, such that the curvature is uniformly bounded, but the spacetime is essentially *not* locally inertial at the singularity. For this reason we call these *regularity singularities*.

Since the gravitation metric tensor is not locally inertial at points of shock wave interaction, it begs the question as to whether there are general relativistic gravitational effects at points of shock wave interaction that cannot be predicted from the compressible Euler equations in special relativity alone. Indeed, even if there are dissipativity terms, like those of the Navier Stokes Equations,²³ which regularize the gravitational metric at points of shock wave interaction, our results assert that the steep gradients in the derivative of the metric tensor at small viscosity cannot be removed uniformly while keeping the metric determinant uniformly bounded away from zero, so one would expect the general relativistic effects at points of shock wave interaction to persist. Moreover, points of regular shock wave interaction are not hidden from observation by an event horizon, which opens up the opportunity of

²³The issue of how to incorporate a relativistic viscosity that meets the speed of light bound is problematic, [18].

direct measurement of effects that could resemble unbounded scalar curvature. We thus wonder whether shock wave interactions might provide a physical regime where new general relativistic effects might be observed, (see Section 13 for a more detailed discussion).

APPENDIX A. THE INTEGRABILITY CONDITION

In this section we review the (well known) equivalence of the existence of an integration factor and the integrability condition (2.5). Suppose we are given a coordinate transformation from coordinates x^μ to x^α , then the Jacobian is defined as

$$(A.1) \quad J_\mu^\alpha = \frac{\partial x^\alpha}{\partial x^\mu},$$

where the indices $\alpha = 0, 1$ and $\mu = t, r$ label a set of four functions. For a Jacobian in the (t, r) -plane all other components are either 0 or 1, thus, we restrict without loss of generality to the two dimensional case. In this paper we always use this notation, moreover, an index ν, μ, σ or ρ always refers to SSC $x^\mu = (t, r)$, while α, β or γ refer to coordinates $x^\alpha = (x^0, x^1)$, provided they exist.

Lemma A.1. *Let Ω be the square region $(a, b)^2 \subset \mathbb{R}^2$ with coordinates $x^\nu = (t, r)$. Suppose we are given a set of functions $J_\mu^\alpha(x^\nu)$ in $C^{0,1}(\Omega)$, C^1 away from some curve $\gamma(t) = (t, x(t))$, satisfying $\det(J_\mu^\alpha) \neq 0$. Then (i) and (ii) are equivalent:*

(i) *there exist locally invertible functions $x^\alpha(t, r) \in C^{1,1}(\Omega)$, for $\alpha = 0, 1$, such that*

$$\frac{\partial x^\alpha}{\partial x^\mu} = J_\mu^\alpha \text{ and}$$

(ii) *the set of functions $J_\mu^\alpha \in C^{0,1}(\Omega)$ satisfy the integrability condition*

$$(A.2) \quad J_{\mu,\nu}^\alpha = J_{\nu,\mu}^\alpha.$$

Proof. The implication from (i) to (ii) is trivial, since (weak) partial derivatives commute. We now prove that (ii) implies (i). Set for $(t, r) \in \Omega$

$$(A.3) \quad x^\alpha(t, r) = \int_a^r J_r^\alpha(t, x) dx + \int_a^t J_t^\alpha(\tau, a) d\tau,$$

then

$$\frac{\partial x^\alpha}{\partial r} = J_r^\alpha$$

follows by the definition of x^α and using the integrability condition A.2 we get

$$\frac{\partial x^\alpha}{\partial t}(t, r) = \int_a^r J_{r,t}^\alpha(t, x) dx + J_t^\alpha(t, a)$$

$$= \int_a^{x(t)} J_{t,r}^\alpha dx + \int_{x(t)}^r J_{t,r}^\alpha dx + J_t^\alpha(t, a).$$

Now that the integral is split up around the point of discontinuity $x(t) \in (a, r)$ we apply the fundamental theorem of calculus and obtain

$$\frac{\partial x^\alpha}{\partial t}(t, r) = J_t^\alpha(t, r).$$

The bijectivity of the function x^α on some open set follows from the Inverse Function Theorem, since

$$\det \left(\frac{\partial x^\mu}{\partial x^\alpha} \right) = \det (J_\alpha^\mu) \neq 0,$$

in Ω . □

APPENDIX B. THE EXISTENCE OF COORDINATES FROM SECTION 4

We now construct a coordinate transformation that transforms a general spherically symmetric metric (2.19) to a metric of the form

$$(B.1) \quad ds^2 = a(\tau, x) (dx^2 - d\tau^2) + C(\tau, x)d\Omega^2.$$

Following the procedure in [18] (to obtain Birkhoff's Theorem) we remove the off-diagonal component E of the general spherically symmetric metric (2.19) by defining new variables (t', r') , subject to a first order PDE (the integrability condition (2.5)) containing combinations of metric components as coefficients. We denote the new variables again by (t, r) and the resulting metric by

$$(B.2) \quad ds^2 = -A(t, r)dt^2 + B(t, r)dr^2 + C(t, r)d\Omega^2,$$

where the coefficient C transforms like a scalar. Analogously to the procedure of removing the off-diagonal metric component we define new coordinates, subject to the integrability condition (2.5), together with some algebraic conditions that ensure the resulting metric to assume the form (B.1). The method of characteristics then yield the existence of a coordinate transformation between a general spherically symmetric metric (2.19) and a metric of the form (B.1).

Suppose the metric (B.2) is given. Define new coordinates

$$(B.3) \quad \begin{aligned} d\tau &= \eta_1 dt + \eta_2 dr \\ dx &= \rho_1 dt + \rho_2 dr, \end{aligned}$$

subject to solve the integrability condition (2.5), in order to ensure that these definitions are integrable. Explicitly the system of PDEs read

$$(B.4) \quad \begin{aligned} \frac{\partial \eta_1}{\partial r} &= \frac{\partial \eta_2}{\partial t} \\ \frac{\partial \rho_1}{\partial r} &= \frac{\partial \rho_2}{\partial t}. \end{aligned}$$

In addition we impose the algebraic condition

$$a(-dt'^2 + dr'^2) = -Adt^2 + Bdr^2,$$

arranging that the transformed metric indeed assumes the form (B.1). Substituting the above definition (B.3) of the new coordinates yields

$$(B.5) \quad \begin{aligned} \rho_1 \rho_2 &= \eta_1 \eta_2 \\ \rho_1^2 - \eta_1^2 &= -\frac{A}{a} \\ \rho_2^2 - \eta_2^2 &= \frac{B}{a}. \end{aligned}$$

We solve this system for a , ρ_1 and η_1 , which yields

$$(B.6) \quad \begin{aligned} a &= \frac{A}{\eta_1^2 - \rho_1^2} \\ \eta_1 &= \pm \sqrt{\frac{A}{B}} \rho_2 \\ \rho_1 &= \pm \sqrt{\frac{A}{B}} \eta_2. \end{aligned}$$

Restricting to positive square roots and substituting (B.6) into the integrability condition (B.4) results in

$$(B.7) \quad \begin{aligned} \left(\sqrt{\frac{A}{B}} \rho \right)_r &= \eta_t \\ \left(\sqrt{\frac{A}{B}} \eta \right)_r &= \rho_t, \end{aligned}$$

where the indices r and t denote the respective partial differentiation, $\eta := \eta_2$ and $\rho := \rho_2$. Introducing new functions $\mu := \eta + \rho$ and $\nu := \eta - \rho$ the above system decouples

$$(B.8) \quad \begin{aligned} \mu_t &= \left(\sqrt{\frac{A}{B}} \mu \right)_r \\ \nu_t &= - \left(\sqrt{\frac{A}{B}} \nu \right)_r . \end{aligned}$$

Given non-characteristic initial values, that is all characteristic curves intersect the Cauchy surface once, this system of PDEs has a unique solution [7]. This proves the existence of the coordinate transformation from (2.19) to (B.1). If the general spherically symmetric metric (2.19) is in $C^{1,1}$, then the solution (μ, ν) and hence $a = \frac{B}{\rho^2 - \eta^2}$ is in $C^{1,1}$ as well.

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