Expanding wave solutions of the Einstein equations that induce an anomalous acceleration into the Standard Model of Cosmology

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We derive a system of three coupled equations that implicitly defines a continuous one-parameter family of expanding wave solutions of the Einstein equations, such that the Friedmann universe associated with the pure radiation phase of the Standard Model of Cosmology is embedded as a single point in this family. By approximating solutions near the center to leading order in the Hubble length, the family reduces to an explicit one-parameter family of expanding spacetimes, given in closed form, that represents a perturbation of the Standard Model. By introducing a comoving coordinate system, we calculate the correction to the Hubble constant as well as the exact leading order quadratic correction to the redshift vs. luminosity relation for an observer at the center. The correction to redshift vs. luminosity entails an adjustable free parameter that introduces an anomalous acceleration. We conclude (by continuity) that corrections to the redshift vs. luminosity relation observed after the radiation phase of the Big Bang can be accounted for, at the leading order quadratic level, by adjustment of this free parameter. The next order correction is then a prediction. Since nonlinearities alone could actuate dissipation and decay in the conservation laws associated with the highly nonlinear radiation phase and since noninteracting expanding waves represent possible time-asymptotic wave patterns that could result, we propose to further investigate the possibility that these corrections to the Standard Model might be the source of the anomalous acceleration of the galaxies, an explanation not requiring the cosmological constant or dark energy.

Expansion waves and shock waves are fundamental to conservation laws because, even when dissipative terms are neglected, nonlinearities alone can cause noninteracting wave patterns to emerge from interactive solutions via the mechanism of shockwave dissipation. In this article, we construct a one-parameter family of noninteracting expanding wave solutions of the Einstein equations in which the Standard Model of Cosmology (during the pure radiation epoch) is embedded as a single point.

Our initial insight was the discovery of a set of coordinates in which the critical ($k = 0$) Fiedmann–Robertson–Walker spacetime with pure radiation sources ($p = \rho c^2/3$), referred to here simply as FRW, goes over to a standard Schwarzchild metric form (barred coordinates) in such a way that the metric components depend only on the single self-similar variable $\bar{r}/\bar{t}$ (cf. ref. 1). From this we set out to find the general equations for such self-similar solutions. In this paper we show that the partial differential equations (PDEs) for a spherically symmetric spacetime in Standard Schwarzchild coordinates (SSC) reduce, under the assumption $p = \rho c^2/3$, to a new system of three ordinary differential equations* in the same self-similar variable $\bar{r}/\bar{t}$. After removing one scaling parameter and imposing regularity at the center, we prove that there exists implicitly within the three-parameter family, a continuous one parameter family of self-similar solutions of the Einstein equations that extends the FRW metric.

Because different solutions in the family expand at different rates, our expanding wave equations introduce a acceleration parameter a, and suitable adjustment of parameter a will speed up or slow down the expansion rate. By normalization, $a = 1$ corresponds to the neutral FRW spacetime, $a < 1$ slows it down, and $a > 1$ speeds it up. Using special properties of the spacetime metrics near $a = 1$, we find an exact expression for the leading order (quadratic) correction to the redshift vs. luminosity relation of the standard model that can occur during the radiation phase of the expansion. By the continuity of the subsequent evolution with respect to the acceleration parameter, it follows that the leading order correction implied by an arbitrary anomalous acceleration observed at any time after the radiation phase of the Big Bang can be accounted for by suitable adjustment of the acceleration parameter.

Our proposal for further investigation, then, is to obtain the correction to redshift vs. luminosity induced by the expanding waves at present time by evolving forward, up through the $p = 0$ stage of the Standard Model, the correction induced by the expanding wave perturbations at the end of the radiation phase. Matching the leading order correction to the data will fix the choice of acceleration parameter, and the higher order corrections at that choice of acceleration parameter are then a verifiable prediction of the theory. The point to be made here is that decay to a noninteracting expansion wave would most likely occur during the radiation phase of the expansion because this is when the sound speed and modulus of genuine nonlinearity (GN) [in the sense of Lax (3)] are maximal (4). That is, by standard theory of hyperbolic conservation laws, GN is a measure of the magnitude of nonlinear compression that drives decay via shockwave dissipation, even when dissipative terms are neglected in the equations (cf. refs. 3, 5, and 6). That is why we focus on expanding wave solutions during the radiation phase. After this phase, the pressure drops to $p \approx 0$, and the resulting equations (for dust) have a zero modulus of GN. Thus significant decay should not occur after the uncoupling of radiation from matter. However, even though a self-similar expanding wave created when $p = \rho c^2/3$ should evolve into a noninteracting expansion wave during the $p \approx 0$ phase, there is no reason to believe that the solution would remain self-similar after the radiation phase. Moreover, we see no reason at this stage to assume that these noninteracting expansion waves should describe all of spacetime. As a consequence, the global analysis of solutions, while interesting, is of secondary interest to the purpose of this article, which is to explore the possibility that we might lie near the center of such an expansion

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[1]We have departed from our usual convention of listing authors alphabetically in order to recognize B.T.'s extraordinary contribution to this particular article.

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*As far as we are aware the only other known way the PDEs for metrics in SSC with perfect fluid sources reduce to ordinary differential equations (ODEs), is the time independent case when they reduce to the Oppenheimer–Volkoff equations (2).

wave and to derive the resulting consequence to the redshift vs. luminosity.

Based on this we propose to investigate whether the observed anomalous acceleration of the galaxies might be due to the fact that we are looking outward into an expansion wave.[†] This would provide an explanation for the anomalous acceleration within classical general relativity without recourse to the ad hoc assumption of dark energy with its unphysical antigravitational properties.[‡] Because these expanding waves have a center of expansion when $a \neq 1$, this would violate the so-called Copernican Principle, a simplifying assumption generally taken in cosmology (compare with ref. 8 and our discussion in Concluding Remarks; see also ref. 9).[§] But most importantly, we emphasize that our anomalous acceleration parameter is not put in ad hoc, but rather is derived from first principles starting from a theory of noninteracting expansion waves (cf. ref. 10). The purpose of this note is to summarize our results and describe the physical interpretations.

An Expanding Wave Coordinate System for the FRW Spacetime

We consider the Standard Model of Cosmology during the pure radiation phase, after inflation, modeled by an FRW spacetime. In comoving coordinates this metric takes the form (2),

$$ds^2 = -dt^2 + R(t)^2 dr^2 + \bar{r}^2 d\Omega^2, \qquad [2.1]$$

where $\bar{r} = Rr$ measures arclength distance at fixed time t and $R \equiv R(t)$ is the cosmological scale factor. Assuming a comoving perfect fluid with equation of state $p = \rho c^2/3$, the Einstein equations give (cf. ref. 1)

$$H(t) = \frac{\dot{R}}{R} = \frac{1}{2t}, \qquad [2.2]$$

where H is the Hubble constant. Note that H and $\bar{r}$ are scale-independent relative to the scaling law $r \to \alpha r$, $R \to \frac{1}{\alpha}R$ of the FRW metric Eq. **2.1**; cf. ref. 1.

The next theorem gives a coordinate transformation that takes Eq. **2.1** to the SSC form,

$$ds^2 = -B(\bar{t},\bar{r})d\bar{t}^2 + \frac{1}{A(\bar{t},\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega^2, \qquad [2.3]$$

such that A and B depend only on $\bar{r}/\bar{t}$.

Theorem 1. *Assume* $p = \frac{\rho c^2}{3}$ and $k = 0$. *Then the FRW metric*

$$ds^2 = -dt^2 + R(t)^2 dr^2 + \bar{r}^2 d\Omega^2,$$

under the change of coordinates

$$\bar{t} = \psi_0 \left\{ 1 + \left[\frac{R(t)r}{2t} \right]^2 \right\} t, \qquad [2.4]$$

$$\bar{r} = R(t)r, \qquad [2.5]$$

transforms to the SSC-metric

$$ds^2 = -\frac{d\bar{t}^2}{\psi_0^2(1 - v(\xi)^2)} + \frac{d\bar{r}^2}{1 - v(\xi)^2} + \bar{r}^2 d\Omega^2, \qquad [2.6]$$

where

$$\xi \equiv \frac{\bar{r}}{\bar{t}} = \frac{2v}{1 + v^2}, \qquad [2.7]$$

and v is the SSC velocity given by

$$v = \frac{1}{\sqrt{AB}} \frac{\bar{u}^1}{\bar{u}^0}. \qquad [2.8]$$

Here $\bar{u} = (\bar{u}^0, \bar{u}^1)$ gives the $(\bar{t},\bar{r})$ components of the 4-velocity of the sources in SSC coordinates. [We include the constant ψ_0 to later account for the time rescaling freedom in Eq. **2.3** (compare with equation 2.18 on page 85 of ref. 1).]

We now assume $p = \rho c^2/3$ and that solutions depend only on ξ. In the next section we show how the Einstein equations for metrics taking the SSC form Eq. **2.3** reduce to a system of three ODEs. A subsequent lengthy calculation then shows that FRW is a special solution of these equations.

The Expanding Wave Equations

Putting the SSC metric ansatz into MAPLE the Einstein equations $G = \kappa T$ reduce to the four partial differential equations:

$$\left\{ -r\frac{A_r}{A} + \frac{1 - A}{A} \right\} = \frac{\kappa B}{A} r^2 T^{00} \qquad [3.1]$$

$$\frac{A_t}{A} = \frac{\kappa B}{A} r T^{01} \qquad [3.2]$$

$$\left\{ r\frac{B_r}{B} - \frac{1 - A}{A} \right\} = \frac{\kappa}{A^2} r^2 T^{11} \qquad [3.3]$$

$$-\left\{ \left(\frac{1}{A} \right)_{tt} - B_{rr} + \Phi \right\} = 2\frac{\kappa B}{A} r^2 T^{22}, \qquad [3.4]$$

where

$$\Phi = \frac{B_t A_t}{2A^2 B} - \frac{1}{2A}\left(\frac{A_t}{A} \right)^2 - \frac{B_r}{r} - \frac{BA_r}{rA} + \frac{B}{2}\left(\frac{B_r}{B} \right)^2 - \frac{B}{2}\frac{B_r}{B}\frac{A_r}{A}. \qquad [3.5]$$

On smooth solutions, Eqs. **3.1–3.4** are equivalent to Eqs. **3.1–3.3** together with $Div_j T^{j1} = 0$, where $Div_j T^{j1} = 0$ can be written in the locally inertial form,

$$\left\{ T_M^{01} \right\}_{,t} + \left\{ \sqrt{AB} T_M^{11} \right\}_{,r} = -\frac{1}{2}\sqrt{AB} \left\{ \frac{4}{r} T_M^{11} + \frac{(1 - A)}{Ar}(T_M^{00} - T_M^{11}) + \frac{2\kappa r}{A}(T_M^{00} T_M^{11} - (T_M^{01})^2) - 4rT^{22} \right\}, \qquad [3.6]$$

where T_M denotes the Minkowski stress tensor (11). Assuming that A and B depend only on $\xi = \bar{r}/\bar{t}$, Eqs. **3.1–3.3** and **3.6** are equivalent to the three ODEs, Eqs. **3.1**, **3.3**, and **3.6**, together with the one constraint, which represents the consistency condition obtained by equating A_ξ from Eqs. **3.1** and **3.2**. Setting

$$\kappa w \equiv \frac{\kappa}{3} \rho \bar{r}^2 (1 - v^2)^{-1},$$

and

$$G = \frac{\xi}{\sqrt{AB}},$$

[†] B.T. originally proposed the idea that a secondary expansion wave reflected backwards from the cosmic shock wave constructed in ref. 7 might account for the anomalous acceleration of the galaxies discussed in footnote ‡ and in National Science Foundation proposal DMS-060-3754.

[‡] Temple B, Numerical Refinement of a Finite Mass Shock-Wave Cosmology. Numerical Relativity: American Mathematical Society National Meeting, January 5–7, 2007, New Orleans. Notes available at: www.math.ucdavis.edu/~temple/talks/NumericalShockWaveCosTalk.pdf.

[§] See also ref. 10, which appeared while this work was under review.

a (long) calculation shows that the three ODEs can be written in the form

$$\xi A_\xi = -\left[\frac{4(1-A)v}{(3+v^2)G-4v}\right] \quad \textbf{[3.7]}$$

$$\xi G_\xi = -G\left\{\left(\frac{1-A}{A}\right)\frac{2(1+v^2)G-4v}{(3+v^2)G-4v}-1\right\} \quad \textbf{[3.8]}$$

$$\xi v_\xi = -\left(\frac{1-v^2}{2\{\cdot\}_D}\right)\left\{(3+v^2)G-4v+\frac{4\left(\frac{1-A}{A}\right)\{\cdot\}_N}{(3+v^2)G-4v}\right\}, \quad \textbf{[3.9]}$$

where[¶]

$$\{\cdot\}_N = \left\{-2v^2+2(3-v^2)vG-(3-v^4)G^2\right\} \quad \textbf{[3.10]}$$

$$\{\cdot\}_D = \left\{(3v^2-1)-4vG+(3-v^2)G^2\right\}, \quad \textbf{[3.11]}$$

and the constraint becomes

$$\kappa w = \frac{(1-A)G}{(3+v^2)G-4v}. \quad \textbf{[3.12]}$$

In summary, we have the following theorem

Theorem 2. *Assume that $A(\xi)$, $G(\xi)$ and $v(\xi)$ solve the ODEs (Eqs.* **3.7–3.9***), and use the constraint (Eq.* **3.12***) to define the density by*

$$\kappa\rho = \frac{3(1-v^2)(1-A)G}{(3+v^2)G-4v}\frac{1}{\bar{r}^2}. \quad \textbf{[3.13]}$$

Then the metric

$$ds^2 = -B(\xi)d\bar{t}^2 + \frac{1}{A(\xi)}d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

solves the Einstein equations with equation of state $p = \rho c^2/3$. Moreover, the transformation (Eq. **2.4***; Eq.* **2.5** *of the FRW metric) leads to the following special SSC relations that hold on the Standard Model during the radiation phase:*

$$\xi = \frac{2v}{\psi_0(1+v^2)}, \quad A = 1-v^2, \quad G = \psi_0\xi, \quad \textbf{[3.14]}$$

where ψ_0 is an arbitrary constant. Another (long) calculation verifies directly that Eq. **3.14** *indeed solves Eqs.* **3.7–3.9** *with the constraint Eq.* **3.13**.

We conclude that the Standard Model of cosmology during the radiation phase corresponds to a solution of the expanding wave equations Eqs. **3.7–3.9** and **3.13** with parameter ψ_0 accounting for the time-scaling freedom of the SSC metric (Eq. **2.3**). That is, the time-scaling $\bar{t} \to \psi_0\bar{t}$ preserves solutions of Eqs. **3.7–3.9** and the constraint (Eq. **3.12**). The next theorem states that modulo this scaling, distinct solutions of Eqs. **3.7–3.9** describe a two-parameter family of distinct spacetimes.

Theorem 3. *The replacement $\bar{t} \to \psi_0\bar{t}$ takes $A(\xi)$, $G(\xi)$, and $v(\xi)$ to $A(\xi/\psi_0)$, $G(\xi/\psi_0)$, and $v(\xi/\psi_0)$, and this scaling preserves solutions of Eqs.* **3.7–3.9** *and* **3.12**. *Moreover, this is the only scaling law in the sense that any two solutions of Eqs.* **3.7–3.9** *and* **3.12** *not related by the scaling $\bar{t} \to \psi_0\bar{t}$ describe distinct spacetimes.*

Since Eqs. **3.7–3.9** admit three (initial value) parameters and one scaling law, it follows that Eqs. **3.7–3.9** and **3.12** describes a two parameter family of distinct spacetimes. In the next section we show that by imposing regularity at the center there results a further reduction to a continuous one-parameter family of expanding wave solutions, such that one value of the parameter corresponds to the FRW metric with pure radiation sources.

¶We thank the referee for pointing out that the equations become autonomous under the change of variables $\xi = e^s$.

Leading Order Corrections to the Standard Model

To obtain the leading order corrections to the FRW metric implied by Eqs. **3.7–3.9** and **3.12**, we linearize the equations at $\xi = 0$ and then expand these equations to leading order in ξ. Modulo the scaling law, the resulting linearized equations admit one eigensolution that tends to infinity as $\xi \to 0$, and the other one satisfies $A(\xi) \to 1$ and $B(\xi) \to 1$ as $\xi \to 0$. Removing the singular solution and including the scaling law leaves a two-parameter family that includes the FRW Standard Model. The analysis leads to the following theorem.

Theorem 4. *The two-parameter family of bounded solutions of Eqs.* **3.7–3.9** *and* **3.12** *that extends FRW of the Standard Model is given in SSC in terms of the two parameters ψ_0 and a, to leading order in ξ, by*

$$ds^2 = -\frac{d\bar{t}^2}{\psi_0^2\left(1-\frac{a^2v_1^2(\xi)}{4}\right)} + \frac{d\bar{r}^2}{\left(1-\frac{a^2v_1^2(\xi)}{4}\right)} + \bar{r}^2 d\Omega^2, \quad \textbf{[4.1]}$$

where

$$A(\xi) = \left(1-\frac{a^2v_1^2(\xi)}{4}\right) + O(|a-1|\xi^4), \quad \textbf{[4.2]}$$

$$B(\xi) = \frac{1}{\psi_0^2\left(1-\frac{a^2v_1^2(\xi)}{4}\right)} + O(|a-1|\xi^4), \quad \textbf{[4.3]}$$

and the SSC velocity $v = v(\xi)$ is given by

$$v(\xi) = v_1(\xi) + O(|a-1|\xi^3). \quad \textbf{[4.4]}$$

Here ψ_0 is the time-scaling parameter, $a = 1$ corresponds to FRW, $v_1(\xi)$ now denotes the SSC velocity of the Standard Model given in Eqs. **2.6–2.8**, *and $a \neq 1$ introduces a new acceleration parameter, which gives the leading order perturbation of FRW. In particular Eq.* **2.7** *gives*

$$v_1(\xi) = \frac{\psi_0\xi}{2} + O(1)\xi^3, \quad \textbf{[4.5]}$$

so Eq. **4.4** *implies that the velocity v is independent of the parameter a up to second order in ξ.*

In light of Eq. **3.14**, when $a = 1$, Eq. **4.1** reduces exactly to the FRW metric

$$A(\xi) = 1 - v_1(\xi)^2, \quad \textbf{[4.6]}$$

$$B(\xi) = \frac{1}{\psi_0^2(1-v_1(\xi)^2)}. \quad \textbf{[4.7]}$$

Note that the SSC representation of FRW depends only on H and $\bar{r}$, both of which are invariant under the scaling $r \to \alpha r$, $R \to R/\alpha$ of the FRW metric (Eq. **2.1**), so the SSC representation of FRW is independent of α and therefore independent of our choice of scale for $R(t)$. Thus without loss of generality, we can take $\psi_0 = 1$, and we can also assume throughout that the FRW metric is scaled exactly so that

$$R(t) = \sqrt{t}; \quad \textbf{[4.8]}$$

compare Eq. **2.1** and ref. 1.

Comoving Coordinates and Comparison with the Standard Model

To get insight into the geometry of the spacetime metric (Eq. 5.4) when $a \neq 1$, consider the extension of the FRW (t, r) coordinate transformation (Eqs. **2.4** and **2.5**) to $a \neq 1$ defined by

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$$\bar{t} = \left\{1 + \frac{a^2\zeta^2}{4}\right\} t, \qquad [5.1]$$

$$\bar{r} = t^{a/2} r. \qquad [5.2]$$

A straightforward caculation shows that the metric of Eq. **4.1** transforms to (t, r)-coordinates as

$$ds^2 = -dt^2 + t^a dr^2 + \bar{r}^2 d\Omega^2 + a(1-a)\zeta dt d\bar{r}. \qquad [5.3]$$

The metric of Eq. **5.3** takes the form of a $k = 0$ Friedmann–Robertson–Walker metric with a small correction to the scale factor, ($R_a(t) = t^{a/2}$ instead of $R(t) = t^{1/2}$), and a small corrective mixed term. In particular, the time slices $t = const.$ in Eq. **5.3** are all flat space $\mathcal{R}^3$, as in FRW, and the $\bar{r} = const.$ slices agree with the FRW metric modified by scale factor $R_a(t)$. It follows that the $t = const.$ surfaces given by Eqs. **5.1** and **5.2** define a foliation of spacetime into flat three dimensional spacelike slices. Thus, Eq. **5.3** exhibits many of the flat space properties characteristic of FRW.

The metric of Eq. **5.3** is not comoving, even at the leading order, when $a \neq 1$. To obtain (an approximate) comoving frame, note that Eq. **4.5** is independent of a up to order ξ^3, so it follows that even when $a \neq 1$, the inverse of the transformation (Eqs. **2.4** and **2.5**) gives, to leading order in ξ, a comoving coordinate system for Eq. **4.1** in which we can compare the Hubble constant and redshift vs. luminosity relations for Eq. **4.1** when $a \neq 1$ to the Hubble constant and redshift vs. luminosity relations for FRW as measured by Eqs. **2.1** and **2.2**. Thus, from here on, we take the Standard Model coordinate map to FRW coordinates Eqs. **2.4** and **2.5**, which corresponds to taking $a = 1$ in Eqs. **5.1** and **5.2**.

Theorem 5. *Take $\psi_0 = 1$, and set $\zeta = \bar{r}/t$. Then the inverse of the coordinate transformation, Eqs.* **2.4** *and* **2.5**, *maps Eq.* **4.1** *over to (t, r)-coordinates as*

$$ds^2 = F_a(\zeta)^2\{-dt^2 + t dr^2\} + \bar{r}^2 d\Omega^2, \qquad [5.4]$$

where

$$F_a(\zeta)^2 = 1 + (a^2 - 1)\frac{\zeta^2}{4} + O(|a-1|\zeta^4), \qquad [5.5]$$

and the SSC velocity v in Eq. **4.5** *maps to the (t, r)-velocity*

$$\tilde{v} = O(|a-1|\zeta^3). \qquad [5.6]$$

The errors in Eqs. **5.5** *and* **5.6** *are written in terms of the comoving coordinate variable ζ, which by Eqs.* **2.4** *and* **2.5** *satisfies $\zeta = O(\xi)$ as $\xi \to 0$, and $\bar{r} = R(t)r$ exactly measures arclength at $t = const.$ in FRW when $a = 1$.*

Note that $\zeta = \bar{r}/t$ is a natural dimensionless perturbation parameter that has a physical interpretation in (t, r)-coordinates because (assuming $c = 1$ or $t \equiv ct$), ζ ranges from 0 to 1 as $\bar{r}$ ranges from zero to the horizon distance in FRW (approximately the Hubble distance c/H), a measure of the furthest one can see from the center at time t units after the Big Bang (2); that is,

$$\zeta \approx \frac{Dist}{Hubble\ Length}. \qquad [5.7]$$

Thus expanding in ζ gives an expansion in the fractional distance to the Hubble length (cf. ref. 1). Note also that when $a = 1$ we obtain the FRW metric (Eq. **2.1**), where we have used $R(t) = \sqrt{t}$ (compare with Eq. **4.8**).

Now for a first comparison of the relative expansion at $a \neq 1$ to the expansion of FRW, define the Hubble constant at parameter value a, by

$$H_a(t, \zeta) = \frac{1}{R_a}\frac{\partial}{\partial t}R_a,$$

where

$$R_a(t, \zeta) = F_a(\zeta)\sqrt{t},$$

equals the square root of the coefficient of dr^2 in Eq. **5.4**. Then one can show

$$H_a(t, \zeta) = \frac{1}{2t}\left\{1 - \frac{3}{8}(a^2 - 1)\zeta^2 + O(|a^2 - 1|\zeta^4)\right\}.$$

We conclude that the fractional change in the Hubble constant due to the perturbation induced by expanding waves $a \neq 1$ relative to the FRW of the Standard Model $a = 1$, is given by

$$\frac{H_a - H}{H} = \frac{3}{8}(1 - a^2)\zeta^2 + O(|a^2 - 1|\zeta^4).$$

Redshift vs. Luminosity Relations

In this section we obtain the first-order corrections to the redshift vs. luminosity relation of FRW, as measured by an observer positioned at the center $\zeta = 0$ of the expanding wave spacetimes described by the metric of Eq. **4.1** when $a \neq 1$.[‖] Recall that $\zeta \equiv \bar{r}/t$ measures the fractional distance to the horizon (compare with Eq. **5.7**). The physically correct coordinate system in which to do the comparison with FRW ($a = 1$) should be comoving with respect to the sources. Thus we restrict to the coordinates (t, r) defined by Eqs. **2.4** and **2.5**, in which our one-parameter family of expanding wave spacetimes are described, to leading order in ζ, by the metric of Eq. **5.4**. Note that Eq. **5.4** reduces exactly to the FRW metric when $a = 1$ (compare Eq. **2.1** with Eq. **4.8**). For our derivation of the redshift vs. luminosity relation for Eq. **5.4** we follow the development in ref. 12.

To start, assuming radiation is emitted by a source at time t_e at wave length λ_e and received at $\zeta = 0$ at later time t_0 at wave length λ_0, define

$$L \equiv \textit{Absolute Luminosity} = \frac{\text{Energy Emitted by Source}}{\text{Time}} \qquad [6.1]$$

$$\ell \equiv \textit{Apparent Luminosity} = \frac{\text{Power Recieved}}{\text{Area}} \qquad [6.2]$$

and let

$$d_\ell \equiv \textit{Luminosity Distance} = \left(\frac{L}{4\pi\ell}\right)^{1/2} \qquad [6.3]$$

$$z \equiv \textit{Redshift Factor} = \frac{\lambda_0}{\lambda_e} - 1. \qquad [6.4]$$

Then using two serendipitous properties of the metric of Eq. **5.4**, namely, the metric is diagonal in comoving coordinates, and there is no a-dependence on the spheres of symmetry, it follows that the arguments in ref. 12, section 11.8, can be modified to give the following result.

Theorem 6. *The redshift vs. luminosity relation, as measured by an observer positioned at the center $\zeta = 0$ of the spacetime described by the metric of Eq.* **4.1**, *is given to leading order in redshift factor z by*

$$d_\ell = 2\sqrt{t_0}z\left\{1 + \frac{1}{2}(a^2 - 1)z + O(1)|a - 1|z^2\right\} \qquad [6.5]$$

where we used the fact that z and ζ are of the same order as $\zeta \to 0$.

Note that when $a = 1$, Eq. **6.5** reduces to the well-known FRW linear relation,

$$d_\ell = 2\sqrt{t_0}z,$$

‖This is of course a theoretical relation, as the pure radiation FRW spacetime is not transparent.

correct for the radiation phase of the Standard Model (12). Thus the bracket in Eq. **6.5** gives the leading order quadratic correction to the redshift vs. luminosity relation implied by the change in the Hubble expansion law corresponding to expanding wave perturbations of the FRW spacetime when $a \neq 1$. Since $(a^2 - 1)$ appears in front of the leading order correction in Eq. **6.5**, it follows (by continuous dependence of solutions on parameters) that the leading order part of any anomalous correction to the redshift vs. luminosity relation of the Standard Model, observed at a time after the radiation phase, can be accounted for by suitable adjustment of parameter a. In particular, note that the leading order corrections in Eq. **6.5** imply a blue-shifting of radiation relative to the Standard Model, as observed in the supernova data, when $a > 1$, (12).

Concluding Remarks

We have constructed a one-parameter family of general relativistic expansion waves, which, at a single parameter value, reduces to what we call the FRW spacetime, the Standard Model of Cosmology during the radiation epoch. The discovery of this family is made possible by a remarkable coordinate transformation that maps the FRW metric in standard comoving coordinates, over to SSC in such a way that all quantities depend only on the single self-similar variable $\xi = \bar{r}/\bar{t}$. Note that it is not evident from the FRW metric in standard comoving coordinates that self-similar variables even exist, and if they do exist, by what ansatz one should extend the metric in those variables to obtain nearby self-similar solutions that solve the Einstein equations exactly. The main point is that our coordinate mapping to SSC form explicitly identifies the self-similar variables as well as the metric ansatz that together accomplish such an extension of the metric.

The self-similarity of the FRW metric in SSC suggested the existence of a reduction of the SSC Einstein equations to a new set of ODEs in ξ. Deriving this system from first principles then establishes that the FRW spacetime does indeed extend to a three-parameter family of expanding wave solutions of the Einstein equations. This three-parameter family reduces to an (implicitly defined) one-parameter family by removing a scaling invariance and imposing regularity at the center. The remaining parameter a changes the expansion rate of the spacetimes in the family, and thus we call it the acceleration parameter. Transforming back to comoving coordinates, the resulting one-parameter family of metrics is amenable to the calculation of a redshift vs. luminosity relation, to second order in the redshift factor z, leading to the relation of Eq. **6.5**. It follows by continuity that the leading order part of an anomalous correction to the redshift vs. luminosity relation of the Standard Model observed after the radiation phase, can be accounted for by suitable adjustment of parameter a.

These results suggest an interpretation that we might call a Conservation Law Scenario of the Big Bang. That is, it is well-known that highly oscillatory interactive solutions of genuinely nonlinear conservation laws decay in time to noninteracting waves (shock waves and expanding waves), by the mechanism of shock wave dissipation. The subtle point is that even though dissipation terms are neglected in the formulation of the equations, there is a canonical dissipation and consequent loss of information due to the nonlinearities, and this can be modeled by shock wave interactions that drive solutions to noninteracting wave patterns. [This viewpoint is well-expressed in celebrated works (3, 5, 6)]. Since the one fact most certain about the Standard Model is that our universe arose from an earlier hot dense epoch in which all sources of energy were in the form of radiation, and since it is approximately uniform on the largest scale but highly oscillatory on smaller scales,** one might reasonably conjecture that decay to a noninteracting expanding wave occurred during the radiation phase of the Standard Model, via the highly nonlinear evolution driven by the large sound speed, and correspondingly large modulus of GN. Our analysis has shown that FRW is just one point in a family of noninteracting expanding waves, and as a result we conclude that some further explanation is required as to why, on some length scale, decay during the radiation phase of the Standard Model would not proceed to a member of the family satisfying $a \neq 1$. If decay to $a \neq 1$ did occur, then the galaxies that formed from matter at the end of the radiation phase (some 379,000 years after the Big Bang), would be displaced from their anticipated positions in the Standard Model at present time, and this displacement would lead to a modification of the observed redshift vs. luminosity relation. In principle such a mechanism could account for the anomalous acceleration of the galaxies as observed in the supernova data. Of course, if $a \neq 1$, then the spacetime has a center, and this would violate the so-called Copernican Principle, a simplifying assumption generally accepted in cosmology (compare with the discussions in ‡ and refs. 8 and 13). As a consequence, if the earth did not lie within some threshold of the center of expansion, the expanding wave theory would imply large angular variations in the observed expansion rate.†† In any case, the expanding wave theory presented here can in principle be tested. For such a test of Eq. **6.5**, one must first evolve the relation, valid at the end of the radiation phase, up through the $p \approx 0$ stage to present time in the Standard Model, thereby obtaining (an approximation to) the special value of a that gives the leading order correction to the redshift vs. luminosity relation observed in the supernova data. Then a derivation of the next order correction to (6.5) during the radiation phase, at that special a-value, evolved up through the $p \approx 0$ stage to present time, would make a prediction of the next order correction to redshift vs. luminosity at present time, and this could be compared with an accurate plot of the supernove data.

To summarize, the expanding wave theory could in principle give an explanation for the observed anomalous acceleration of the galaxies within classical general relativity, with classical sources. In the expanding wave theory, the so-called anomalous acceleration is not an acceleration at all, but is a correction to the Standard Model due to the fact that we are looking outward into an expansion wave. The one parameter family of noninteracting general relativistic expansion waves derived here, are all equally possible end-states that could result after dissipation by nonlinear wave interaction during the radiation phase of the Standard Model, is done; and when $a \neq 1$ they introduce an anomalous acceleration into the Standard Model of cosmology. Unlike the theory of dark energy, this provides a possible explanation for the anomalous acceleration of the galaxies that is not ad hoc in the sense that it is derivable exactly from physical principles and a mathematically rigorous theory of expansion waves. That is, this explanation does not require the ad hoc assumption of a universe filled with an as yet unobserved form of energy (dark energy) with antigravitational properties in order to fit the data. The idea that the anomalous acceleration might be accounted for by a local under-density in a neighborhood of our galaxy was expounded in a recent article (8). Our results here might then give an accounting for the source of such an under-density.

In conclusion, these expanding wave solutions of the Einstein equations provide a paradigm to test against the Standard Model. Moreover, even if these general relativistic expansion waves do not in the end explain the anomalous acceleration of the galaxies, their presence represents an instability in the Standard Model in the sense that an explanation is required as to why small-scale oscillations have to settle down to large-scale $a = 1$ expansions instead of $a \neq 1$ expansions (either locally or globally), during the radiation phase of the Big Bang.

** In the Standard Model, the universe is approximated by uniform density on a scale of a billion light years or so, about a tenth of the radius of the visible universe (2). The stars, galaxies, and clusters of galaxies are then evidence of large oscillations on smaller scales.

†† The size of the center, consistent with the angular dependence that has been observed in the actual supernova and microwave data, has been estimated to be ≈15 megaparsecs, approximately the distance between clusters of galaxies, ≈1/200 the distance across the visible universe (cf. refs. 2, 8 and 14).

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