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An instability of the standard model of cosmology creates the anomalous acceleration without dark energy

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We identify the condition for smoothness at the centre of spherically symmetric solutions of Einstein's original equations without the cosmological constant or dark energy. We use this to derive a universal phase portrait which describes general, smooth, spherically symmetric solutions near the centre of symmetry when the pressure $p=0$. In this phase portrait, the critical $k=0$ Friedmann space–time appears as a saddle rest point which is unstable to spherical perturbations. This raises the question as to whether the Friedmann space–time is observable by redshift versus luminosity measurements looking outwards from any point. The unstable manifold of the saddle rest point corresponding to Friedmann describes the evolution of local uniformly expanding space–times whose accelerations closely mimic the effects of dark energy. A unique simple wave perturbation from the radiation epoch is shown to trigger the instability, match the accelerations of dark energy up to second order and distinguish the theory from dark energy at third order. In this sense, anomalous accelerations are not only consistent with Einstein's original theory of general relativity, but are a prediction of it without the cosmological constant or dark energy.

1. Introduction

We identify the condition for smoothness at the centre of spherically symmetric solutions of Einstein's original equations of general relativity (without the cosmological

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constant), and use this to derive a universal phase portrait which describes the evolution of smooth solutions near the centre of symmetry when the pressure $p=0$.[1] In this phase portrait, the $k=0$, $p=0$ Friedmann space–time appears as an unstable saddle rest point. Earlier attempts to identify an instability in the standard model of cosmology[2] (SM) were inconclusive;[3] cf. [3,4]. The condition for smoothness is that all odd-order derivatives with respect to r of metric components and scalar functions, and all even derivatives of the velocity, should vanish at the centre $r=0$ when the space–time metric is expressed in standard Schwarzschild coordinates (SSCs) (cf. (1.1) below). We prove that this condition is preserved by the evolution of the Einstein equations, and remark that this smoothness condition appears not to have been identified in previous studies based on Lemaitre–Tolman–Bondi (LTB) coordinates. Here, we propose that the correct invariant condition for smoothness of a spherically symmetric space–time metric given in radial coordinates (r, ϕ, θ) is the condition that all odd-order r-derivatives of metric components vanish at $r=0$ in SSC coordinates.

The constraint of smoothness at the centre provides a new ansatz for Taylor expanding smooth spherically symmetric solutions about the centre of symmetry in SSC, and we show the anstatz *closes* in SSC at even orders when $p=0$. The effect of imposing smoothness reduces the solution space and implies that the local phase portrait is valid with errors one order of magnitude larger than one would obtain if (as in prior LTB studies) non-zero SSC derivatives of odd order were allowed at the centre. From this we prove that smooth perturbations of the Friedmann space–time trigger an instability when the pressure drops to zero, and the effect of spherical perturbations, as described by the unstable manifold, is to create local uniformly expanding space–times with accelerated expansion rates. These space–times introduce a new global space–time geometry given in closed form when the higher-order corrections affecting the space–time far from the centre are neglected. We show that, in the under-dense case, these local space–times mimic almost exactly the effects of dark energy, producing *precisely* the same range of quadratic corrections to redshift versus luminosity during the evolution from the end of radiation to the present time, as are produced by the cosmological constant in the theory of dark energy. Based on this we conclude the following. (1) The Friedmann space–time is unstable in Einstein's original theory of general relativity (GR) without the cosmological constant, and given this, we should not expect to observe it by redshift versus luminosity measurements looking outward from any point taken as the centre, when $p=0$. (2) Because under-dense perturbations create space–times that locally mimic the effects of dark energy, the anomalous acceleration[4] observed in the supernova data is not only consistent with Einstein's original theory, but one could interpret this as a prediction of it. Statements (1) and (2) remain valid independently of whether or not the instability of Friedmann actually *is* the source of the anomalous acceleration observed in the supernova data.

It is natural, then, to test the consistency of the accelerations which are created by the instability with the accelerations observed in the supernova data. In fact, all these ideas arose out of the authors' earlier attempts to explain the anomalous acceleration of the galaxies within Einstein's original theory without dark energy. These ideas followed from a self-contained line of reasoning stemming from questions that naturally arose from earlier investigations on incorporating a shock wave into SM, [5–7]. There have been a number of other attempts to model cosmic acceleration by assuming that we live in an under-dense region of the universe (cf. [8], and references [37–65] of [9], including [3,10–15] listed below). Such a class of models is called the void models.

[1] By *smooth* we mean arbitrary orders of derivatives exist on the scale for which the Friedmann approximation is valid. Making sure appropriate smoothness conditions are imposed on solutions is of fundamental importance to mathematics and physics.

[2] Assuming the so-called cosmological principle, that the universe is uniform on the largest scale, the evolution of the universe on that scale is described by a Friedman space–time, which is determined by the equation of state in each epoch [1]. In this paper, we let SM denote the approximation to the standard model of cosmology without dark energy given by the critical $k=0$ Friedman universe with equation of state $p=(c^2/3)\rho$ during the radiation epoch, and $p=0$ thereafter (cf. the ΛCDM model with $\Lambda=0$ [2]).

[3] Cf. [3] for inconclusive attempts to identify the instability in SM by taking a *long wavelength* limit in LTB coordinates.

[4] In this paper, we use the term *anomalous acceleration* to refer to the corrections to redshift versus luminosity from the predictions of the $k=0$, $p=0$ Friedmann space–time, as observed in the supernova data. We take these to be given exactly by the corrections obtained by assuming $\Omega_\Lambda=0.7$

Prior void models have been based on spherically symmetric $p = 0$ solutions represented in LTB coordinates (coordinates which typically take the radial coordinate to be co-moving with the fluid; cf. [1]) and, until now, a smoothness condition at the centre was never identified for the purpose of characterizing smooth solutions in LTB. Although the void models are still discussed and taken seriously, it is generally believed that unless we live in an extreme vicinity of the centre of a spherically symmetric space, it would be in contradiction with the observation of cosmic microwave background radiation. Moreover, *central weak singularities* have been shown to exist in LTB at the centre in models that appear to account for the anomalous acceleration [3,4,15]. The fine-tuning problem of being near the centre, and the existence of mild singularities at the centre, have both been put forth as possible reasons to rule out the void model explanation for the cosmic acceleration. While we do not address these problems here, we point out there are in fact large-scale angular anomalies in the microwave background radiation, [16], and the fine-tuning problem persists whether we fine-tune the model to be near a centre, or fine-tune it to make the cosmological constant on the order of the energy density of the universe (required to correct the redshift versus luminosity relations by the cosmological constant [7]). The void models in LTB are essentially based on choosing initial data to match the observations at the present time, and then proposing the LTB time reversal of such solutions as the cosmological model. Here, we take a different approach by exploring the consequences of assuming that the *instability* in SSC created the under-density. This is fundamentally different because we identify a *mechanism*, the instability, by which the redshift versus luminosity data is altered in a specific way from the SM values as a direct consequence of the Einstein equations.

Based on this, we explore the connection between the local accelerations created by the instability and the anomalous acceleration observed in the supernova data, *making no assumptions* about the space–time far from the centre.[5] The universal phase portrait applies up to fourth-order errors (in distance from the centre) in the density variable and third-order errors in the velocity, implying that neglecting these errors, the phase portrait only affects the linear and quadratic terms in the observed redshift versus luminosity relations. We prove that the accelerations created by the instability are consistent with the supernova observations out to second order in the redshift factor z. However, to obtain a third-order correction which provides a prediction different from dark energy, some assumption must be made about the third-order velocity term. For this prediction, we propose that the under-density is created (at that order) by a distinguished one-parameter family of smooth perturbations of the Friedmann space–time that exist during the radiation epoch, when $p = (c^2/3)\rho$ [18,19]. In [7], the authors identified these self-similar perturbations, and proposed them as a possible source of the anomalous acceleration observed in the supernova data without dark energy. Now it is commonly stated that the radiation epoch ends, and the pressure drops approximately to zero, about *one order of magnitude* (1 power of 10) before the uncoupling of radiation and matter, the latter occurring approximately 300 000–400 000 years after the Big Bang. To make precise the connection between these self-similar solutions from the radiation epoch and the instability they trigger when the pressure drops to $p = 0$, we make the simplifying assumption that the pressure drops discontinuously to zero at some temperature between 3000 and 9000 K. That is, we model the continuous drop in pressure from the radiation epoch to the matter-dominated epoch as a discontinuous process, but allow the temperature at which the drop takes place to be essentially arbitrary. The approximation of a discontinuous drop in the pressure is commonly made in Cosmology. Indeed, to quote Longair [2, p. 276], '…the transition to the *radiation-dominated era* would take place at redshift $z \approx 6000$. At redshifts less than this value, the Universe was *matter-dominated* and the dynamics were described by the standard Friedman models [with scale factor] $t^{2/3}$ [the case $p = 0$]…'. Thus our assumption that the pressure drops precipitously to zero at a temperature $3000\,\mathrm{K} \le T_* \le 9000\,\mathrm{K}$ is reasonable. As our numerics show that the results are independent of that temperature, we are confident that the conclusion would not change significantly if a continuous process were

[5]Author's work in [17] shows how solutions with positive velocity can be extended beyond a given radius with arbitrary initial density and velocity profiles.

modelled. Thus the conclusions derived from the assumption of a precipitous drop in pressure to $p=0$ are justified. By numerical simulation we identify a unique wave in the family that accounts for the same values of the Hubble constant and quadratic correction to redshift versus luminosity as are implied by the theory of dark energy with $\Omega_\Lambda \approx 0.7$, and the numerical simulation of the third-order correction associated with that unique wave establishes the testable prediction that distinguishes this theory from the theory of dark energy. Here, we characterize the sought-after instability, show it is triggered by a family of simple wave perturbations from the radiation epoch and, as a bonus, obtain a testable alternative mathematical explanation for the anomalous acceleration of the galaxies that does not invoke dark energy.

We now discuss the perturbations from the radiation epoch in more detail. Most of the expansion of the universe before the pressure drops to $p \approx 0$ is governed by the radiation epoch, a period in which the large-scale evolution is approximated by the equations of pure radiation. These equations take the form of the relativistic p-system [19,20] of shock-wave theory, and for such highly nonlinear equations one expects complicated solutions to become simpler. Solutions of the p-system typically decay to a concatenation of self-similar *simple waves*, solutions along which the equations reduce to ODEs [7,21,22]. Based on this, together with the fact that large fluctuations from the radiation epoch (like the baryonic acoustic oscillations) are typically spherical [2], the authors began the program in [23] by looking for a family of spherically symmetric solutions that perturb the SM during the radiation epoch when the equation of state $p=(c^2/3)\rho$ holds, and on which the Einstein equations reduce to ODEs. In [6,7], we identified a unique family of such solutions which we refer to as a-waves, parametrized by the so-called *acceleration parameter* $a>0$, normalized so that $a=1$ is the SM.[6] The a-waves are the only known family of solutions of the Einstein equations which both perturb Friedman space–times, and reduce the Einstein equations to ODEs, [7,24,25]. As when $p=0$, *under-densities* relative to the SM are a natural mechanism for creating anomalous *accelerations* (less matter present to slow the expansion implies a larger expansion rate [2]), we restrict to the perturbations $a<1$ which induce under-densities relative to the SM [6,7]. Thus our starting hypothesis in [6,7] was that the anomalous acceleration of the galaxies is due to a local under-density relative to the SM, on the scale of the supernova data, created by a perturbation that has decayed (locally near the centre) to an a-wave, $a<1$, by the end of the radiation epoch.[7] Here, we use the a-waves to obtain a third-order correction to redshift versus luminosity to be compared with dark energy. We can now state the results precisely.

In this paper, we prove the following. (i) The $k=0$, $p=0$ Friedman space–time is *unstable*, and smooth spherical perturbations evolve, locally to leading order near the centre, according to a *universal phase portrait* in which the SM appears as an unstable saddle rest point *SM* (cf. figure 1; italic *SM* denotes the rest point). (ii) Under-dense perturbations of *SM* at the end of radiation trigger evolution along the unstable manifold from *SM* to *M*, and this describes the formation of a local region of accelerated expansion (one order of magnitude larger in extent than would be expected if the smoothness condition were not imposed), which extends further and further outwards from the centre, becoming more flat and more uniform, as time evolves. Comparing these local uniformly expanding solutions generated by the phase portrait to the critical uniformly expanding Friedmann space–time accelerated by the cosmological constant, we find that evolution along the unstable manifold produces *precisely* the same range of quadratic corrections Q to redshift versus luminosity as dark energy—for apparently a completely different reason. (iii) A unique a-wave perturbation at the end of radiation which creates the same H_0 and Q at the present time as dark energy provides a predictive third-order correction C that has the same order, but a different sign, from dark energy.

[6]This family of waves was first discovered from a different point of view in the fundamental paper [24]; cf. also the *self-similarity hypothesis* in [25]. As far as we know, ours is the first attempt to connect this family of waves with the anomalous acceleration.

[7]As time asymptotic wave patterns typically involve multiple simple waves, we make no hypothesis regarding the space–time far from the centre of the a-wave.

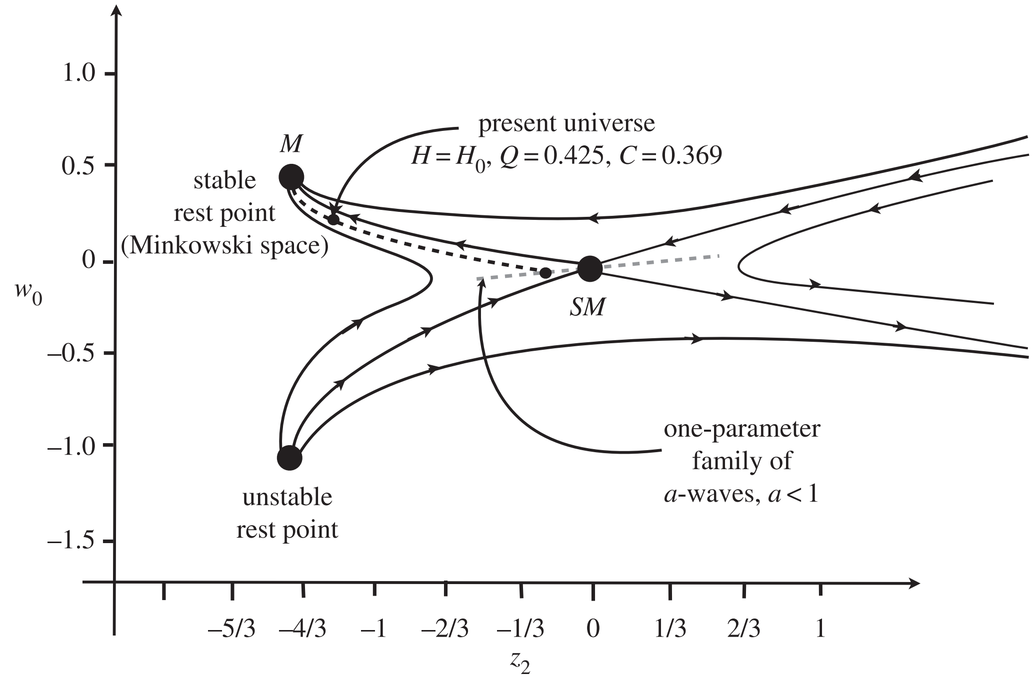

Figure 1. Phase portrait for central region.

Spherically symmetric space–times can generically be transformed near the centre to SSC where the metric takes the canonical form

$$ds^2 = -B(t,r)\,dt^2 + \frac{1}{A(t,r)}\,dr^2 + r^2\,d\Omega^2, \quad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2, \tag{1.1}$$

$d\Omega$ giving the standard line element on the unit 2-sphere [18]. Letting

$$H\,d_\ell = z + Qz^2 + Cz^3 + O(z^4) \tag{1.2}$$

denote the relation between redshift factor z and luminosity distance d_ℓ at a given value of the Hubble constant H as measured at the centre,[8] the value of the quadratic correction Q increases from the value $Q = 0.25$ at rest point SM at the end of radiation, to the value $Q = 0.5$ for orbits evolving along the unstable manifold to M as $t \to \infty$. This is precisely the same range of values Q takes on in dark energy theory as the fraction Ω_Λ of dark energy to classical energy increases from its value of $\Omega_\Lambda \approx 0$ at the end of radiation, to $\Omega_\Lambda = 1$ as $t \to \infty$. In particular, this holds for any $a < 1$ near $a = 1$, and for any value of the cosmological constant $\Lambda > 0$, assuming only that a and Λ both induce a negligibly small under-dense correction to the SM value $Q = 0.25$ at the end of radiation.[9] Indeed, this holds for *any* under-dense perturbation that follows the unstable trajectory of rest point SM into the rest point M (cf. figure 1).

These results are recorded in the following theorem. Here, we let the present time in a given model denote the time at which the Hubble constant H (as defined in (1.2)) reaches its present measured value $H = H_0$, this time being different in different models. We refer to the model in which the anomalous acceleration is created by an a-wave from radiation, the *wave model* [26].

[8]For FRW, Q is determined by the value of the so-called *deceleration parameter* q_0, and C is determined by the *jerk* j; cf. [2]. The deceleration parameter gives Q through $H_0 d_\ell = z - ((3 + q_0)/2)z^2 + O(z^3)$, with $q_0 = -\frac{10}{3} < 0$ in SM.

[9]We qualify with this latter assumption only because, in dark energy theory, the value of Ω_Λ is small but not exactly equal to zero at the end of radiation; and in the wave model, the value of Q jumps down slightly below $Q = 0.25$ at the end of radiation before it increases to $Q = 0.5$ from that value as $t \to \infty$.

Theorem 1.1. *Let $t = t_0$ denote the present time in the wave model and $t = t_{\mathrm{DE}}$ the present time in the dark energy*[10] *model. Then there exists a unique value of the acceleration parameter $\underline{a} = 0.999999426 \approx 1 - 5.74 \times 10^{-7}$ corresponding to an under-density relative to the SM at the end of radiation, such that the subsequent $p = 0$ evolution starting from this initial data evolves to time $t = t_0$ with $H = H_0$ and $Q = 0.425$, in agreement with the values of H and Q at $t = t_{\mathrm{DE}}$ in the dark energy model. The cubic correction at $t = t_0$ in the wave model is then $C = 0.359$, while dark energy theory gives $C = -0.180$ at $t = t_{\mathrm{DE}}$. The times are related by $t_0 \approx 0.95 t_{\mathrm{DE}}$.*

In principle, adding acceleration to a model increases the expansion rate H and consequently the age of the universe because it then takes longer for the Hubble constant H to decrease to its present small value H_0. The numerics confirm that the age of the universe well approximates the age obtained by adding in dark energy.

We emphasize that t_0, Q and C in the wave model are determined by a alone. Indeed, the initial data at the end of radiation, which determine the $p = 0$ evolution, depend, at the start, on two parameters: the acceleration parameter a of the self-similar waves and the initial temperature T_* at which the pressure is assumed to drop to zero. But our numerics show that the dependence on the starting temperature is negligible for T_* in the range $3000\,\mathrm{K} \leq T_* \leq 9000\,\mathrm{K}$ (covering the range assumed in cosmology [2]). Thus for the temperatures appropriate for cosmology, t_0, Q and C are determined by a alone.

A measure of the *severity* of the instability created by the $a = \underline{a}$ perturbation of the SM, is quantified by the numerical simulation. For example, comparing the initial density ρ_{wave} for $a = \underline{a}$ at the centre of the wave to the corresponding initial density ρ_{sm} in the SM at the end of radiation $t = t_*$ gives $\rho_{\mathrm{wave}}/\rho_{\mathrm{sm}} \approx 1 - (1.88) \times 10^{-6} \approx 1$. During the $p = 0$ evolution, this ratio evolves to a *sevenfold* under-density in the wave model relative to the SM by the present time, i.e. $\rho_{\mathrm{wave}}/\rho_{\mathrm{sm}} = 0.144$ at $t = t_0$.

Our wave model is based on the self-similarity variable $\xi = r/ct < 1$, which we introduce as a natural measure of the outward distance from the centre of symmetry $r = 0$ in the inhomogeneous space–times we describe in SSC. We call ξ the *fractional distance to the Hubble radius* because $1/ct$ is the Hubble radius in the Friedman space–time, and t is chosen to be proper time at $r = 0$ in our SSC gauge. Thus it is convenient also to define $1/ct$ to be the Hubble radius in our inhomogeneous space–times. Moreover, the SSC radial variable approximately measures arclength distance at fixed time in our SSC space–times when $\xi \ll 1$, and exactly measures arclength at fixed time in the Friedman space–time in co-moving coordinates. Thus when $\xi \ll 1$, ξ tells approximately how far out relative to the Hubble radius an observer at the centre of our inhomogeneous space–times would conclude an object observed at ξ were positioned, if he mistakenly thought he were in a Friedman space–time.[11] We show below (cf. §3c) that if we neglect errors $O(\xi^4)$, and then further neglect small errors between the wave metric and the Minkowski metric which tend to zero, at that order, with approach to the stable rest point M, and also neglect errors due to relativistic corrections in the velocities of the fluid relative to the centre (where the velocity is zero), the resulting space–time is, like a Friedman space–time, *independent of the choice of centre*. Thus the central region of approximate uniform density at present time $t = t_0$ in the wave model extends out from the centre $r = 0$ at $t = 0$ in SSC to radial values r small enough so that the fractional distance to the Hubble radius $\xi = r/ct_0$ satisfies $\xi^4 \ll 1$.

The cubic correction C to redshift versus luminosity is a *verifiable* prediction of the wave model which distinguishes it from dark energy theory. In particular, $C > 0$ in the wave model and $C < 0$ in the dark energy model implies that the cubic correction *increases* the right-hand side of (1.2) (i.e. increases the discrepancy between the observed redshifts and the predictions of the SM) far from the centre in the wave model, while it *decreases* the right-hand side of (1.2) far from the centre in the dark energy theory. Now the anomalous acceleration was originally derived from a collection of data points, and the $\Omega_\Lambda \approx 0.7$ critical FRW space–time is obtained

[10]By the dark energy model we refer to the critical $k = 0$ Friedman universe with cosmological constant, taking the present value $\Omega_\Lambda = 0.7$ as the best fit to the supernova data among the two parameters (k, Λ) [27].

[11]Here, ξ is just a measure of distance in SSC, and need not have a precise physical interpretation for $\xi \gg 1$ [1,7,18].

as the best fit to Friedman space–times among the parameters (k, Λ). Presently it is not clear to the authors whether or not there are indications in the data that could distinguish $C < 0$ from $C > 0$.

In §2, we give a physical motivation for our smoothness condition imposed at the centre $r = 0$ of a spherically symmetric space–time in SSC. Our results are presented in §3. In §3a, we derive an alternative formulation of the $p = 0$ Einstein equations in spherical symmetry, and in §3b we prove that the evolution preserves smoothness. In §3c, we introduce our new asymptotic ansatz for corrections to the SM which are consistent with the condition at $r = 0$ for smooth solutions derived in §2. In §3d, we use the exact equations together with our ansatz to derive asymptotic equations in (t, ξ) for the corrections, and use these to derive the universal phase portrait. In §3e, we derive the correct redshift versus luminosity relation for the SM including the corrections. In §3f, we introduce a gauge transformation that converts the a-waves at the end of radiation into initial data that are consistent with our ansatz. In §3g, we present our numerics that identifies the unique a-wave $a = \underline{a}$ in the family that meets the conditions $H = H_0$ and $Q = 0.425$ at $t = t_0$, and explain our predicted cubic correction $C = 0.359$. In §3h, we discuss the uniform space–time created at the centre of the perturbation. Concluding remarks are given in §4. Details are omitted in this announcement. We use the convention $c = 1$ when convenient.

2. Smoothness at the centre of spherically symmetric space–times

The results of this paper rely on the validity of approximating solutions by finite Taylor expansions about the centre of symmetry, so the main issue is to guarantee that solutions are indeed smooth in a neighbourhood of the centre. Of course the universe is not smooth on small scales, so our assumption is simply that the centre is not special regarding the level of smoothness assumed in the large-scale approximation of the universe. Smoothness at a point P in a space–time manifold is determined by the atlas of coordinate charts defined in a neighbourhood of P, the smoothness of tensors being identified with the smoothness of the tensor components as expressed in the coordinate systems of the atlas. Now spherically symmetric solutions given in LTB and SSC in GR employ spherical coordinates (r, ϕ, θ) for the spacelike surfaces at constant time, and the subtlety here is that $r = 0$ is a coordinate singularity in spherical coordinates, and functions are defined only for radial coordinate $r \geq 0$, but a coordinate system must be specified in a *neighbourhood* of $r = 0$ to impose the conditions for smoothness at the centre. Of course, once we have the metric represented as smooth in coordinate system x on an initial data surface in a neighbourhood of $r = 0$, the local existence theorem giving the smooth evolution of solutions from smooth initial data for the Einstein equations would not alone suffice to obtain our smoothness condition, as one would still have to prove that this evolution preserved the metric ansatz.

We begin by showing that this issue can be resolved relatively easily in SSC because the SSC coordinates are precisely the spherical coordinates associated with Euclidean *coordinate charts* defined in a neighbourhood of $r = 0$. Based on this, we show below that the condition for smoothness of metric components and functions in SSC is simply that all odd-order derivatives should vanish at $r = 0$.

Consider now in more detail the problem of representing a smooth, spherically symmetric perturbation of the Friedman space–time in GR. To start, assume the existence of a solution of Einstein's equations representing a large, smooth under-dense region of space–time that expands from the end of radiation out to the present time. For smooth perturbations, there should exist a coordinate system in a neighbourhood of the centre of symmetry, in which the solution is represented as smooth. Assume we have such a coordinate system $(t, \mathbf{x}) \in \mathcal{R} \times \mathcal{R}^3$ with $\mathbf{x} = 0$ at the centre, and use the notation $x = (x^0, x^1, x^2, x^3) \equiv (t, \mathbf{x})$, $\mathbf{x} \equiv (x, y, z)$ (there should be no confusion with the ambiguity in x). Spherical symmetry makes it convenient to represent the spatial Euclidean coordinates $\mathbf{x} \in \mathcal{R}^3$ in spherical coordinates (r, θ, ϕ), with $r = |\mathbf{x}|$. As generically, any spherically symmetric metric can be transformed locally to SSC form [18], we assume the space–time represented in the coordinate system (t, r, θ, ϕ) takes the SSC form (1.1). This is equivalent to the metric in Euclidean coordinates $\mathbf{x}$ taking the form

$$\mathrm{d}s^2 = -B(|\mathbf{x}|, t)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A(|\mathbf{x}|, t)} + |\mathbf{x}|^2\,\mathrm{d}\Omega^2, \tag{2.1}$$

with

$$\left.\begin{aligned} r^2 = x^2 + y^2 + z^2, \quad \mathrm{d}r = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z}{r}, \\ \mathrm{d}r^2 = \frac{x^2\,\mathrm{d}x^2 + y^2\,\mathrm{d}y^2 + z^2\,\mathrm{d}z^2 + 2xy\,\mathrm{d}x\,\mathrm{d}y + 2xz\,\mathrm{d}x\,\mathrm{d}z + 2yz\,\mathrm{d}y\,\mathrm{d}z}{r^2} \end{aligned}\right\} \tag{2.2}$$

and

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{2.3}$$

To guarantee the smoothness of our perturbations of Friedman at the centre, we assume a gauge in which

$$B(t, r) = 1 + O(r^2) \tag{2.4}$$

and

$$A(t, r) = 1 + O(r^2), \tag{2.5}$$

so also

$$\frac{1}{A(t, r)} = 1 + O(r^2) \equiv 1 + \hat{A}(t, r)r^2, \tag{2.6}$$

where the smoothness of A is equivalent to the smoothness of $\hat{A}$ for $r > 0$. This sets the SSC time gauge to proper geodesic time at $r = 0$ and makes the SSC coordinates locally inertial at $r = 0$ for each time $t > 0$, a first step in guaranteeing that the spherical perturbations of Friedman which we study are smooth at the centre. Keep in mind that the SSC form is invariant under arbitrary transformation of time, so we are free to choose geodesic time at $r = 0$; and the locally inertial condition at $r = 0$ simply imposes that the corrections to Minkowski at $r = 0$ are second order in r. (These assumptions make physical sense, and their consistency is guaranteed by reversing the steps in the argument to follow). In particular, the SSC metric (1.1) tends to Minkowski at $r = 0$. We now ask what conditions on the metric functions A, B are imposed by assuming the SSC metric be *smooth* when expressed in our *original* Euclidean coordinate chart $(t, \mathbf{x})$ defined in a neighbourhood of a point at $r = 0$, $t > 0$.

To transform the SSC metric (1.1) to $(t, \mathbf{x})$ coordinates, use (2.3) to eliminate the $r^2 d\Omega^2$ term and (2.2) to eliminate the dr^2 term to obtain

$$\begin{aligned} \mathrm{d}s^2 = -B(|\mathbf{x}|, t)\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \hat{A}(|\mathbf{x}|, t))\{x^2\,\mathrm{d}x^2 + y^2\,\mathrm{d}y^2 + z^2\,\mathrm{d}z^2 \\ + 2xy\,\mathrm{d}x\,\mathrm{d}y + 2xz\,\mathrm{d}x\,\mathrm{d}z + 2yz\,\mathrm{d}y\,\mathrm{d}z\}. \end{aligned} \tag{2.7}$$

The smoothness of $\hat{A}$ is equivalent to the smoothness of A, and the smoothness of A and B for $r > 0$ guarantees the smoothness of the Euclidean space–time metric (2.7) in $(t, \mathbf{x})$ coordinates everywhere except at $\mathbf{x} = 0$. For smoothness at $\mathbf{x} = 0$, we impose the condition that the metric components in (2.7) should be smooth functions of $(t, \mathbf{x})$ at $x = 0$ as well. (Again, imposing smoothness in $(t, \mathbf{x}) = 0$ coordinates at $\mathbf{x} = 0$ is correct in the sense that it is preserved by the Einstein evolution equations; cf. §3b below.) We now show that smoothness at $\mathbf{x} = 0$ *in this sense* is equivalent to requiring that the metric functions A and B satisfy the condition that all *odd* r-derivatives vanish at $r = 0$. To see this, observe that a function $f(r)$ represents a smooth spherically symmetric function of the Euclidean coordinates $\mathbf{x}$ at $r = |\mathbf{x}| = 0$ if and only if the function

$$g(x) = f(|\mathbf{x}|)$$

is smooth at $\mathbf{x} = 0$. Assuming f is smooth for $r \geq 0$ (by which we mean f is smooth for $r > 0$, and one-sided derivatives exist at $r = 0$) and taking the n'th derivative of g from the left and right and setting them equal gives the smoothness condition $f^n(0) = (-1)^n f^n(0)$. We state this formally as follows.

Lemma 2.1. *A function $f(r)$ of the radial coordinate $r = |\boldsymbol{x}|$ represents a smooth function of the underlying Euclidean coordinates $\boldsymbol{x}$ if and only if f is smooth for $r \geq 0$, and all odd derivatives vanish at $r = 0$. Moreover, if any odd derivative $f^{(n+1)}(0) \neq 0$, then $f(|\boldsymbol{x}|)$ has a jump discontinuity in its $n+1$ derivative, and hence a kink singularity in its n'th derivative at $r = 0$.*

As an immediate consequence we obtain the condition for smoothness of SSC metrics at $r = 0$.

Corollary 2.2. *The SSC metric (1.1) is smooth at $r = 0$ in the sense that the metric components in (2.7) are smooth functions of the Euclidean coordinates $(t, \boldsymbol{x})$ if and only if the component functions $A(r, t)$, $B(r, t)$ are smooth in time and smooth for $r > 0$, all odd one-sided r-derivatives vanish at $r = 0$, and all even r-derivatives are bounded at $r = 0$.*

To conclude, solutions of the Einstein equations in SSC have four unknowns, the metric components A, B, the density ρ and the scalar velocity v. It is easy to show that if the SSC metric components satisfy the condition that all odd-order r-derivatives vanish at $r = 0$, then the components of the unit 4-velocity vector u associated with smooth curves that pass through $r = 0$ will have the same property,[12] and the scalar velocity $v = (1/\sqrt{AB})(\mathrm{d}r/\mathrm{d}t)$ will have the property that all even derivatives vanish at $r = 0$ (because v is an outward velocity which picks up a change of sign when represented in x). Thus smoothness of SSC solutions at $r = 0$ at fixed time is equivalent to requiring that the metric components satisfy the condition that all odd r-derivatives vanish at $r = 0$. These then give conditions on SSC solutions equivalent to the condition that the solutions are smooth in the ambient Euclidean coordinate systems x. Theorem 3.1 of §3b below proves that smoothness in the coordinate system x at $r = 0$ at each time in this sense is preserved by the Einstein evolution equations for SSC metrics when $p = 0$. In particular, this demonstrates that our condition for smoothness of SSC metrics at $r = 0$ is equivalent to the well-posedness of solutions in the ambient Euclidean coordinates defined in a neighbourhood of $r = 0$. Thus we obtain the condition for smoothness of SSC metrics at $r = 0$ based on the Euclidean coordinate systems associated with SSC, and show this is preserved by the evolution of the Einstein equations. As smoothness of the SSC metric components in this sense is equivalent to smoothness of the x-coordinates with respect to arclength along curves passing through $r = 0$, in this sense, our condition for smoothness is *geometric*.

3. Presentation of results

In §3a–e, we derive equations and formulae for smooth spherically symmetric solutions in SSC in the case $p = 0$ sufficient to determine the quadratic correction Q in (1.2) and the phase portrait in figure 1. Our analysis employs the SSC forms of the SM in which metric components as well as density and velocity variables depend only on the SSC self-similar variable $\xi = r/t$. In §3f,g, we incorporate the inhomogeneous self-similar a-waves that exist for $p = (c^2/3)\rho$ and reduce to the critical Friedman space–time for pure radiation when $a = 1$, to obtain the third-order prediction C in (1.2). Recall that when $p = 0$, no such self-similar perturbations of Friedmann exist [6,7,25,26,28]. The asymptotics employed is based on Taylor expanding the solutions in even powers of ξ about the centre in SSC.

(a) The $p = 0$ Einstein equations in coordinates aligned with the physics

In this section, we introduce a new formulation of the $p = 0$ Einstein equations that describe outwardly expanding spherically symmetric solutions employing the SSC metric form (1.1). We start with the SSC equations in [17], introduce new dimensionless density and velocity variables (z, w) and transform equations over to (t, ξ) coordinates, where $\xi = r/t$. Recall that the SSC metric form is invariant under transformations of t, and there exists a time coordinate in which SM is self-similar in the sense that the metric components A, B, the velocity v and ρr^2 are functions of ξ alone. This self-similar form exists, but is different for $p = (c^2/3)\rho$ and $p = 0$ [7,25,28]. Taking

[12]This implies that the coordinates are smooth functions of arclength along curves passing through $r = 0$.

$p=0$, letting v denote the SSC velocity and ρ the co-moving energy density, and eliminating all unknowns in terms of v and the Minkowski energy density $T_{\mathrm{M}}^{00}=\rho/(1-(v/c)^2)$ (cf. [17]), the locally inertial formulation of the Einstein equations $G=\kappa T$ introduced in [17] reduce to

$$
\begin{aligned}
&(\kappa T_{\mathrm{M}}^{00} r^2)_t + \left\{\sqrt{AB}\frac{v}{r}(\kappa T_{\mathrm{M}}^{00} r^2)\right\}_r = -2\sqrt{AB}\frac{v}{r}(\kappa T_{\mathrm{M}}^{00} r^2),\\
&\left(\frac{v}{r}\right)_t + r\sqrt{AB}\left(\frac{v}{r}\right)\left(\frac{v}{r}\right)_r = -\sqrt{AB}\left\{\left(\frac{v}{r}\right)^2 + \frac{1-A}{2Ar^2}\left(1-r^2\left(\frac{v}{r}\right)^2\right)\right\},\\
&r\frac{A'}{A} = \left(\frac{1}{A}-1\right) - \frac{1}{A}\kappa T_{\mathrm{M}}^{00} r^2,\\
&r\frac{B'}{B} = \left(\frac{1}{A}-1\right) + \frac{1}{A}\left(\frac{v}{c}\right)^2 \kappa T_{\mathrm{M}}^{00} r^2,
\end{aligned}
$$

where prime denotes $\mathrm{d}/\mathrm{d}r$. Note that the $1/r$ singularity is present in the equations because incoming waves can amplify without bound. We resolve this for outgoing expansions by assuming that $w=v/\xi$ is positive and finite at $r=\xi=0$. Making the substitution $D=\sqrt{AB}$, taking $z=\kappa T_{\mathrm{M}}^{00} r^2$ as the dimensionless density, $w=v/\xi$ as the dimensionless velocity with $\xi=r/t$ and rewriting the equations in terms of (t,ξ), we obtain

$$tz_t + \xi\{(-1+Dw)z\}_\xi = -Dwz, \tag{3.1}$$

$$tw_t + \xi(-1+Dw)w_\xi = w - D\left\{w^2 + \frac{1-\xi^2 w^2}{2A}\left[\frac{1-A}{\xi^2}\right]\right\}, \tag{3.2}$$

$$\xi A_\xi = (1-A) - z, \tag{3.3}$$

and

$$\frac{\xi D_\xi}{D} = \frac{1}{A}\left\{(1-A) - \frac{(1-\xi^2 w^2)}{2} z\right\}. \tag{3.4}$$

That is, as the sound speed is zero when $p=0$, $w(t,0)>0$ restricts us to expanding solutions in which all information from the fluid propagates outwards from the centre.

(b) Smoothness of solutions in the ambient Euclidean coordinate system in a neighbourhood of $r=0$

In this section, we prove that smoothness in the ambient Euclidean coordinate system $x=(x^0,x^1,x^2,x^3)=(t,x,y,z)$ associated with spherical SSC coordinates is preserved by the evolution of the Einstein equations. By lemma 2.1, smoothness of SSC solutions at $r=0$ is imposed by the condition that odd-order r-derivatives of the metric components and the density vanish at $r=0$, and even derivatives of the velocity v vanish at $r=0$. As $\xi=r/t$, imposing this condition on r-derivatives at $t>0$ is equivalent to imposing it on ξ-derivatives, and as $w=\xi v$, $D=\sqrt{AB}$, $z=\rho r^2$, smoothness at $r=0$ is equivalent to the condition that all odd derivatives of (z,w,A,D) vanish at $\xi=0$, $t>0$. The following theorem establishes that smoothness in the ambient coordinate system x is preserved by the evolution of the Einstein equations in SSC.

Theorem 3.1. *Assume $z(t,\xi), w(t,\xi), A(t,\xi), D(t,\xi)$ are a given smooth solution of our $p=0$ equations (3.1)–(3.4) satisfying*

$$z=O(\xi^2),\quad w=w_0(t)+O(\xi^2) \tag{3.5}$$

and

$$A=1+O(\xi^2)\quad \text{and}\quad D=1+O(\xi^2), \tag{3.6}$$

for $0<t_0\le t<t_1$, and assume that at $t=t_0$ the solution agrees with initial data

$$z(t_0,\xi)=\bar{z}(\xi)\quad \text{and}\quad w(t_0,\xi)=\bar{w}(\xi) \tag{3.7}$$

and

$$A(t_0,\xi)=\bar{A}(\xi)\quad \text{and}\quad D(t_0,\xi)=\bar{D}(\xi), \tag{3.8}$$

such that each initial data function $\bar{z}(\xi)$, $\bar{w}(\xi)$, $\bar{A}(\xi)$, $\bar{D}(\xi)$ satisfies the condition that all odd ξ-derivatives vanish at $\xi = 0$. Then, all odd ξ-derivatives of the solution $z(t,\xi), w(t,\xi), A(t,\xi), D(t,\xi)$ vanish at $\xi = 0$ for all $t_0 < t < t_1$.

Proof. Start with equations (3.1)–(3.4) in the form

$$tz_t = -\xi\{(-1 + Dw)z\}_\xi - Dwz, \tag{3.9}$$

$$tw_t = -\xi(-1 + Dw)w_\xi + w - D\left\{w^2 + \frac{1-A}{2A\xi^2}(1 - \xi^2 w^2)\right\}, \tag{3.10}$$

$$\xi A_\xi = (1 - A) - z \tag{3.11}$$

and

$$\xi D_\xi = \frac{D}{2A}\{2(1-A) - z + \xi^2 w^2 z\}. \tag{3.12}$$

First note that products and quotients of smooth functions that satisfy the condition that all odd derivatives vanish at $\xi = 0$ also have this property. Now for a function $F(t,\xi)$, let $F_\xi^{(n)}(t)$ denote the n'th partial derivative of F with respect to ξ at $\xi = 0$. We prove the theorem by induction on n. For this, assume $n \geq 1$ is odd, and make the induction hypothesis that, for all odd $k < n$, $F_\xi^{(k)}(t) = 0$ for all $t \geq t_0$ and all functions $F = z, w, A, D$ (functions of (t,ξ)). We prove that $F_\xi^{(n)}(t) = 0$ for $t > t_0$. For this we employ the following simple observation: if n is odd, and the n'th derivative of the product of m functions,

$$\frac{\partial^n}{\partial \xi^n}(F_1 \cdots F_m)$$

is expanded into a sum by the product rule, the only terms that will not have a factor containing an odd derivative of order less than n are the terms in which all the derivatives fall on the same factor. This follows from the simple fact that if the sum of k integers is odd, then at least one of them must be odd. Taking the n'th derivative of (3.9) and setting $\xi = 0$ gives the ODE at $\xi = 0$:

$$t\frac{\mathrm{d}}{\mathrm{d}t}z_\xi^{(n)} = -n\frac{\partial^n}{\partial \xi^n}((-1 + Dw)z) - \frac{\partial^n}{\partial \xi^n}(DWz). \tag{3.13}$$

As all odd derivatives of order less than n are assumed to vanish at $\xi = 0$, we can apply the observation and the assumptions (3.5), (3.6) that $D = 1$, $w = w_0(t)$ and $z = 0$ at $\xi = 0$, to see that only the n'th order derivative $z_\xi^{(n)}$ survives on the RHS of (3.13). That is, by the induction hypothesis, (3.13) reduces to

$$t\frac{\mathrm{d}}{\mathrm{d}t}z_\xi^{(n)} = [n - (n+1)w_0(t))]z_\xi^{(n)}. \tag{3.14}$$

As under the change of variable $t \to \ln(t)$, (3.14) is a linear first-order homogeneous ODE in $z_\xi^{(n)}(t)$ with $z_\xi^{(n)}(t_0) = 0$, it follows by uniqueness of solutions that $z_\xi^{(n)}(t) = 0$ for all $t \geq t_0$. This proves the theorem for the solution component $z(t,\xi)$.

Consider next equation (3.11). Differentiating both sides n times with respect to ξ and setting $\xi = 0$ gives

$$(n+1)A_\xi^{(n)}(t) = -z_\xi^{(n)}(t) = 0, \tag{3.15}$$

thus

$$A_\xi^{(n)}(t) = 0 \tag{3.16}$$

for $t \geq t_0$, which verifies the theorem for component A.

Consider equation (3.12). Differentiating both sides n times with respect to ξ, setting $\xi=0$ and applying the observation and the induction hypothesis gives

$$\begin{aligned} nD_{\xi}^{(n)} &= \frac{\partial^n}{\partial \xi^n}\left(D\frac{1-A}{A}\right) \\ &= D_{\xi}^{(n)}\left(\frac{1-A}{A}\right) + \sum_{k<n \text{ odd}} c_k D_{\xi}^{(k)} + D\left(\frac{1-A}{A}\right)_{\xi}^{(n)} \\ &= 0 \end{aligned} \tag{3.17}$$

for $t \geq t_0$ because $A=1$ at $\xi=0$, all lower-order odd derivatives are assumed to vanish at $\xi=0$ and we have already verified the theorem for the component A. This proves

$$D_{\xi}^{(n)}(t) = 0 \tag{3.18}$$

for $t \geq t_0$, verifying the theorem for component D.

Consider lastly the equation (3.10). Differentiating both sides n times with respect to ξ, setting $\xi=0$ and applying our observation gives

$$\begin{aligned} t\frac{\mathrm{d}}{\mathrm{d}t}w_{\xi}^{(n)} &= -n(-1+w_0(t))w_{\xi}^{(n)} + w_{\xi}^{(n)} - \frac{\partial^n}{\partial \xi^n}(w^2) \\ &= -n(-1+w_0(t))w_{\xi}^{(n)} + w_{\xi}^{(n)} - 2ww_{\xi}^{(n)} \\ &= [-n(-1+w_0(t)) + 1 - 2w]w_{\xi}^{(n)} \end{aligned} \tag{3.19}$$

for $t \geq t_0$ because $A=1$ and $\xi=0$, all lower-order odd derivatives are assumed to vanish at $\xi=0$, and we have established the theorem for the component A. Thus $w_{\xi}^{(n)}(t)$ solves the first-order homogeneous ODE

$$t\frac{\mathrm{d}}{\mathrm{d}t}w_{\xi}^{(n)} = [-n(-1+w_0(t)) + 1 - 2w]w_{\xi}^{(n)}, \tag{3.20}$$

starting from zero initial data at $t=t_0$, and hence again we conclude

$$w_{\xi}^{(n)}(t) = 0 \tag{3.21}$$

for $t \geq t_0$. This verifies the theorem for the final component w, thereby completing the proof of theorem 3.1. ■

(c) A new Ansatz for corrections to standard model of cosmology

In this section, we derive the phase portrait which describes any spherical perturbation of the $k=0$, $p=0$ Friedman space–time which is smooth in SSC coordinates. Our condition for smooth solutions is that (z, w, A, B) are smooth functions away from $\xi=0$, all time derivatives are smooth and all odd ξ-derivatives vanish at $\xi=0$. As solutions are assumed smooth at $\xi=0$, $t>0$, Taylor's theorem is valid at $\xi=0$, so the following ansatz for corrections to SM near $\xi=0$ is valid in a neighbourhood of $\xi=0$, $t>0$, with errors bounded by derivatives of the corresponding functions at the corresponding orders.

$$z(t,\xi) = z_{\mathrm{sm}}(\xi) + \Delta z(t,\xi) \qquad \Delta z = z_2(t)\xi^2 + z_4(t)\xi^4, \tag{3.22}$$

$$w(t,\xi) = w_{\mathrm{sm}}(\xi) + \Delta w(t,\xi) \qquad \Delta w = w_0(t) + w_2(t)\xi^2, \tag{3.23}$$

$$A(t,\xi) = A_{\mathrm{sm}}(\xi) + \Delta A(t,\xi) \qquad \Delta A = A_2(t)\xi^2 + A_4(t)\xi^4 \tag{3.24}$$

and

$$D(t,\xi) = D_{\mathrm{sm}}(\xi) + \Delta D(t,\xi) \qquad \Delta D = D_2(t)\xi^2, \tag{3.25}$$

where $z_{\mathrm{sm}}, w_{\mathrm{sm}}, A_{\mathrm{sm}}, D_{\mathrm{sm}}$ are the expressions for the unique self-similar representation of the SM when $p=0$, given by [28],

$$z_{\mathrm{sm}}(\xi) = \tfrac{4}{3}\xi^2 + \tfrac{40}{27}\xi^4 + O(\xi^6) \quad \text{and} \quad w_{\mathrm{sm}}(\xi) = \tfrac{2}{3} + \tfrac{2}{9}\xi^2 + O(\xi^4) \tag{3.26}$$

and

$$A_{\text{sm}}(\xi) = 1 - \tfrac{4}{9}\xi^2 - \tfrac{8}{27}\xi^4 + O(\xi^6) \quad \text{and} \quad D_{\text{sm}}(\xi) = 1 - \tfrac{1}{9}\xi^2 + O(\xi^4). \tag{3.27}$$

This gives

$$z(t,\xi) = \left(\tfrac{4}{3} + z_2(t)\right)\xi^2 + \left\{\tfrac{40}{27} + z_4(t)\right\}\xi^4 + O(\xi^6) \tag{3.28}$$

and

$$w(t,\xi) = \left(\tfrac{2}{3} + w_0(t)\right) + \left\{\tfrac{2}{9} + w_2(t)\right\}\xi^2 + O(\xi^4). \tag{3.29}$$

Consistent with theorem 3.1, we verify the equations close within this ansatz, at order ξ^4 in z and order ξ^2 in w with errors $O(\xi^6)$ in z and $O(\xi^4)$ in w. Corrections expressed in this ansatz create a uniform space–time of density $\rho(t)$, constant at each fixed t, out to errors of order $O(\xi^4)$. That is, as the ansatz,

$$z(\xi,t) = \kappa\rho(t,\xi)r^2 + O(\xi^4) = \left(\tfrac{4}{3} + z_2(t)\right)\xi^2 + O(\xi^4), \tag{3.30}$$

neglecting the $O(\xi^4)$ error gives $\kappa\rho = \left(\tfrac{4}{3} + z_2(t)\right)/t^2$, a function of time alone. For the SM , $z_2 \equiv 0$ and this gives $\kappa\rho(t) = (\tfrac{4}{3})t^{-2}$, which is the exact evolution of the density for the SM Friedman space–time with $p = 0$ in co-moving coordinates [18]. For the evolution of our specific under-densities in the wave model, we show $z_2(t) \to -\tfrac{4}{3}$ as the solution tends to the stable rest point, implying that the instability creates an accelerated drop in the density in a large uniform space–time expanding outwards from the centre. (Cf. §3h below.)

(d) Asymptotic equations for corrections to standard model of cosmology

Substituting the ansatz (3.22)–(3.25) for the corrections into the Einstein equations $G = \kappa T$, and neglecting terms $O(\xi^4)$ in w and $O(\xi^6)$ in z, we obtain the following closed system of ODEs for the corrections $z_2(\tau)$, $z_4(\tau)$, $w_0(\tau)$, $w_2(\tau)$, where $\tau = \ln t$, $0 < \tau \le 11$. (Introducing τ renders the equations autonomous, and solves the long time simulation problem.) Letting prime denote $\mathrm{d}/\mathrm{d}\tau$, the equations for the corrections reduce to the autonomous system

$$z_2' = -3w_0\left(\tfrac{4}{3} + z_2\right), \tag{3.31}$$

$$w_0' = -\tfrac{1}{6}z_2 - \tfrac{1}{3}w_0 - w_0^2, \tag{3.32}$$

$$z_4' = 5\left\{\tfrac{2}{27}z_2 + \tfrac{4}{3}w_2 - \tfrac{1}{18}z_2^2 + z_2w_2\right\} + 5w_0\left\{\tfrac{4}{3} - \tfrac{2}{9}z_2 + z_4 - \tfrac{1}{12}z_2^2\right\} \tag{3.33}$$

and

$$w_2' = -\tfrac{1}{10}z_4 - \tfrac{4}{9}w_0 + \tfrac{1}{3}w_2 - \tfrac{1}{24}z_2^2 + \tfrac{1}{3}z_2w_0 + \tfrac{1}{3}w_0^2 - 4w_0w_2 + \tfrac{1}{4}w_0^2z_2. \tag{3.34}$$

We prove that for the equations to close within the ansatz (3.22)–(3.25), it is necessary and sufficient to assume that the initial data satisfies the gauge conditions

$$A_2 = -\tfrac{1}{3}z_2, \quad A_4 = -\tfrac{1}{5}z_4 \quad \text{and} \quad D_2 = -\tfrac{1}{12}z_2. \tag{3.35}$$

We prove that if these constraints hold initially, then they are maintained by the equations for all time. Conditions (3.35) are not invariant under time transformations, even though the SSC metric form is invariant under arbitrary time transformations, so we can interpret (3.35), and hence the ansatz (3.22)–(3.25), as fixing the time coordinate gauge of our SSC metric.

The autonomous 4×4 system (3.31)–(3.34) contains within it the closed, autonomous 2×2 subsystem (3.31), (3.32). This subsystem describes the evolution of the corrections (z_2, w_0), which we show in §3e determines the quadratic correction Qz^2 in (1.2). Thus the subsystem (3.31), (3.32) gives the corrections to SM at the order of the observed anomalous acceleration, accurate within the central region where errors $O(\xi^4)$ in z and orders $O(\xi^3)$ in $v = w/\xi$ can be neglected. The phase portrait for subsystem (3.31), (3.32) exhibits an unstable saddle rest point at $SM = (z_2, w_0) = (0, 0)$ corresponding to the SM, and a stable rest point at $(z_2, w_0) = (-\tfrac{4}{3}, \tfrac{1}{3})$. These are the rest points referred to in the introduction. From the phase portrait (figure 1), we see that perturbations of SM corresponding to small under-densities will evolve away from the SM near the unstable manifold of $(0,0)$, and towards the stable rest point M. By (3.35), $A_2 = \frac{4}{9}, D_2 = \frac{1}{9}$ at $(z_2, w_0) = (-\frac{4}{3}, \frac{1}{3})$, hence by (3.24) and (3.25) the metric components A and B are equal to $1 + O(\xi^4)$, implying that the metric at the stable rest point M is Minkowski up to $O(\xi^4)$. Thus during evolution towards the stable rest point, the metric tends to flat Minkowski space–time with $O(\xi^4)$ errors.

Note that we have only assumed a smooth SSC solution and expanded in finite Taylor series about the centre, so our only asymptotic assumption has been that ξ is small, not that the perturbation from the $k=0, p=0$ Friedman space–time is small. Thus the phase portrait in figure 1 is universal in that it describes the evolution of every SSC smooth solution in a neighbourhood of $\xi = 0, t > 0$. We state this as a theorem.

Theorem 3.2. *Let (z, w, A, B) be an SSC solution which is smooth in the ambient Euclidean coordinate system x associated with the spherical SSC coordinates, and meeting condition (2.6). Then there exists an SSC time gauge in which the solution satisfies equations (3.31)–(3.34) and (3.35) up to the appropriate orders. Thus the phase portrait of figure 1 is valid in a neighbourhood of $\xi = 0$ with errors $O(1)\xi^6$ in z and $O(1)\xi^4$ in w, where by Taylor's theorem, the $O(1)$ errors are bounded by the maximum of the sixth and fourth derivatives of the solution components z and w, respectively.*

(e) Redshift versus luminosity relations for the ansatz

In this section, we obtain formulae for Q and C in (1.2) as a function of the corrections z_2, w_0, z_4, w_2 to the *SM*; we compare this to the values of Q and C as a function of Ω_Λ in DE theory, and we show that, remarkably, Q passes through the same range of values in both theories.

Recall that Q and C are the quadratic and cubic corrections to redshift versus luminosity as measured by an observer at the centre of the spherically symmetric perturbation of the SM determined by these corrections.[13] The calculation requires taking account of all of the terms that affect the redshift versus luminosity relation when the space–time is not uniform, and the coordinates are not co-moving.

The redshift versus luminosity relation for the $k=0$, $p=\sigma\rho$, FRW space–time, at any time during the evolution, is given by

$$Hd_\ell = \frac{2}{1+3\sigma}\{(1+z) - (1+z)^{(1-3\sigma)/2}\}, \tag{3.36}$$

where only H evolves in time [29]. For pure radiation $\sigma = \frac{1}{3}$, which gives $Hd_\ell = z$, and when $p = \sigma = 0$, we get (cf. [7])

$$Hd_\ell = z + \tfrac{1}{4}z^2 - \tfrac{1}{8}z^3 + O(z^4). \tag{3.37}$$

The redshift versus luminosity relation in the case of dark energy theory, assuming a critical Friedman space–time with the fraction of dark energy Ω_Λ, is

$$Hd_\ell = (1+z)\int_0^z \frac{\mathrm{d}y}{\sqrt{\mathcal{E}(y)}}, \tag{3.38}$$

where

$$\mathcal{E}(z) = \Omega_\Lambda(1+z)^2 + \Omega_M(1+z)^3 \tag{3.39}$$

and $\Omega_M = 1 - \Omega_\Lambda$, the fraction of the energy density due to matter (cf. (11.129), (11.124) of [29]). Taylor expanding gives

$$Hd_\ell = z + \frac{1}{2}\left(-\frac{\Omega_M}{2} + 1\right)z^2 + \frac{1}{6}\left(-1 - \frac{\Omega_M}{2} + \frac{3\Omega_M^2}{4}\right)z^3 + O(z^4), \tag{3.40}$$

where Ω_M evolves in time, ranging from $\Omega_M = 1$ (valid with small errors at the end of radiation) to $\Omega_M = 0$ (the limit as $t \to \infty$). From (3.40) we see that, in dark energy theory, the quadratic term

[13] The uniformity of the centre out to errors $O(\xi^4)$ implies that these should be good approximations for observers somewhat off-centre with the coordinate system of symmetry for the waves.

Q increases exactly through the range

$$0.25 \leq Q \leq 0.5 \tag{3.41}$$

and the cubic term decreases from $-\frac{1}{8}$ to $-\frac{1}{6}$, during the evolution from the end of radiation to $t \to \infty$, thereby verifying the claim in theorem 1.1. In the case $\Omega_M = 0.3$, $\Omega_\Lambda = 0.7$, representing present time $t = t_{DE}$ in dark energy theory, this gives the exact expression

$$H_0 d_\ell = z + \tfrac{17}{40} z^2 - \tfrac{433}{2400} z^3 + O(z^4), \tag{3.42}$$

verifying that $Q = 0.425$ and $C = -0.180$, as recorded in theorem 1.1.

In the case of a general non-uniform space–time in SSC, the formula for redshift versus luminosity as measured by an observer at the centre is given by (see [29])

$$d_\ell = (1+z)^2 r_e = t_0 (1+z)^2 \xi_e \left(\frac{t_e}{t_0}\right), \tag{3.43}$$

where (t_e, r_e) are the SSC coordinates of the emitter and $(0, t_0)$ are the coordinates of the observer. A calculation based on using the metric corrections to obtain ξ_e and t_e/t_0 as functions of z, and substituting this into (3.43), gives the following formula for the quadratic correction $Q = Q(z_2, w_0)$ and cubic correction $C = C(z_2, w_0, z_4, w_2)$ to redshift versus luminosity in terms of arbitrary corrections w_0, w_2, z_2, z_4 to *SM*. We record the formulae in the following theorem.

Theorem 3.3. *Assume a GR space–time in the form of our ansatz (3.22)–(3.25), with arbitrary given corrections $w_0(t), w_2(t), z_2(t), z_4(t)$ to SM. Then the quadratic and cubic corrections Q and C to redshift versus luminosity in (1.2), as measured by an observer at the centre $\xi = r = 0$ at time t, is given explicitly by*

$$H d_\ell = z \left\{ 1 + \left[\tfrac{1}{4} + E_2\right] z + \left[-\tfrac{1}{8} + E_3\right] z^2 \right\} + O(z^4), \tag{3.44}$$

where

$$H = \left(\frac{2}{3} + w_0(t)\right) \frac{1}{t},$$

so that

$$Q(z_2, w_0) = \tfrac{1}{4} + E_2 \quad \text{and} \quad C(w_0, w_2, z_2, z_4) = -\tfrac{1}{8} + E_3, \tag{3.45}$$

where $E_2 = E_2(z_2, w_0)$, $E_3 = E_3(z_2, w_0, z_4, w_2)$ are the corrections to the $p = 0$ standard model values in (3.37). The function E_2 is given explicitly by

$$E_2 = \frac{24 w_0 + 45 w_0^2 + 3 z_2}{4(2 + 3w_0)^2}. \tag{3.46}$$

The function E_3 is defined by the following chain of variables:

$$
\begin{aligned}
E_3 &= 2I_2 + I_3, \\
I_{2,3} &= J_2 + \frac{9w_0}{2(2+3w_0)}, \quad J_3 + 3\left[-1 + \left(\frac{8 - 8J_2 + 3w_0 - 12 J_2 w_0}{2(2+3w_0)^2}\right)\right], \\
J_2 &= \frac{1}{4}\left\{1 - \frac{1 + 9K_2}{(1 + (3/2)w_0)^2}\right\}, \\
J_3 &= \frac{5}{8}\left\{1 - \frac{1 - (18/5)K_2 - (81/5)K_2^2 + (9/5)w_0 + (27/5)K_3 + (81/10)Q_3 w_0}{(1+(3/2)w_0)^4}\right\}, \\
K_{2,3} &= \frac{2}{3}w_0 + \frac{1}{2}w_0^2 - \frac{1}{12}z_2, \ \frac{2}{9}w_0 + w_0^2 + \frac{1}{2}w_0^3 + w_2 - \frac{1}{18}z_2 - \frac{1}{3}z_2 w_0.
\end{aligned} \tag{3.47}
$$

From (3.46) one sees that Q depends only on (z_2, w_0), $Q(0,0) = 0.25$ (the exact value for the SM), $Q(-\frac{4}{3}, \frac{1}{3}) = 0.5$ (the exact value for the stable rest point), and from this it follows that Q increases through precisely the same range (3.41) of DE, from $Q \approx 0.25$ to $Q = 0.5$, along the orbit of (3.31), (3.32) that takes the unstable rest point $SM = (z_2, w_0) = (0, 0)$ to the stable rest point $M = (z_2, w_0) = (-\frac{4}{3}, \frac{1}{3})$ (cf. figure 1).

(f) Initial data from the radiation epoch

In this section, we compute the initial data for the $p = 0$ evolution from the restriction of the one-parameter family of self-similar a-waves to a constant temperature surface $T = T_*$ at the end of radiation, and convert this to initial data on a constant time surface $t = t_*$, these two surfaces being different when $a \neq 1$. We then define a gauge transformation that converts the resulting initial data to equivalent initial data that meets the gauge conditions (3.35). (Recall that condition (3.35) fixes a time coordinate, or gauge, for the underlying SSC metric associated with our ansatz, and the initial data for the a-waves is given in a different gauge because time since the big bang depends on the parameter a, as well as on the pressure, so it changes when p drops to zero.) The equation of state of pure radiation is derived from the Stefan–Boltzmann Law, which relates the initial density ρ_* to the initial temperature T_* in degrees Kelvin by

$$\rho_* = \frac{a_s c}{4} T_*^4, \tag{3.48}$$

where a_s is the Stefan–Boltzmann constant [30]. According to current theories in cosmology (see [30]), the pressure drops precipitously to zero at a temperature $T = T_*$ somewhere between $3000\,\mathrm{K} \leq T_* \leq 9000\,\mathrm{K}$, corresponding to starting times t_* roughly in the range $10\,000\,\mathrm{yr} \leq t_* \leq 30\,000\,\mathrm{yr}$ after the Big Bang. We make the assumption that the pressure drops discontinuously to zero at some temperature T_* within this range. That our resulting simulations are numerically independent of the starting temperature (cf. §3g), justifies the validity of this assumption. Using this assumption, we can take the values of the a-waves on the surface $T = T_*$ as the initial data for the subsequent $p = 0$ evolution. Using the equations we convert this to initial data on a constant time surface $\bar{t} = \bar{t}_*$, where $\bar{t}$ is the time coordinate used in the self-similar expression of the a-waves which assumes $p = (c^2/3)\rho$. Our first theorem proves that there is a gauge transformation $\bar{t} \to t$ which converts the initial data for a-waves at the end of radiation at $\bar{t} = \bar{t}_*$, to initial data that meet both the assumptions of our ansatz (3.22)–(3.25), and the gauge conditions (3.35).

Theorem 3.4. *Let $\bar{t}$ be the time coordinate for the self-similar waves during the radiation epoch, and define the transformation $\bar{t} \to t$ by*

$$t = \bar{t} + \tfrac{1}{2}\mu(\bar{t} - \bar{t}_*)^2 - t_B, \tag{3.49}$$

where μ and t_B are given by

$$\mu = \frac{a^2}{2(2 - a^2)} \tag{3.50}$$

and

$$t_B = \bar{t}_*(1 - \alpha), \tag{3.51}$$

where

$$\alpha = 4\frac{2 - a^2}{7 - 4a^2}. \tag{3.52}$$

Then, upon performing the gauge transformation (3.49), the initial data from the a-waves at the end of radiation $\bar{t} = \bar{t}_*$ meets the conditions for the ansatz (3.22)–(3.25), as well as the gauge conditions (3.35).

Our conclusions are summarized in the following theorem.

Theorem 3.5. *The initial data for the $p=0$ evolution determined by the self-similar a-wave on a constant time surface $t=t_*$ with temperature $T=T_*$ at $r=0$, is given as a function of the acceleration parameter a and the temperature T_*, by*

$$z_2(t_*)=\hat{z}_2,\quad z_4(t_*)=\hat{z}_4+3\hat{w}_0\left(\tfrac{4}{3}+\hat{z}_2\right)\gamma,$$

$$w_0(t_*)=\hat{w}_0,\quad w_2(t_*)=\hat{w}_2+\left(\tfrac{1}{6}\hat{z}_2+\tfrac{1}{3}\hat{w}_0+\hat{w}_0^2\right)\gamma,$$

where $\hat{z}_2, \hat{z}_4, \hat{w}_0, \hat{w}_2$ and γ are functions of acceleration parameter a given by

$$\hat{z}_2=\frac{3a^2\alpha^2}{4}-\frac{4}{3},\quad \hat{z}_4=2\alpha^3(1-\alpha)\bar{\gamma}Z_2+\alpha^4 Z_4-\frac{40}{27},$$

$$Z_2=\frac{3a^2}{4},\quad Z_4=\left[\frac{9a^2}{16}+\frac{15a^2(1-a^2)}{40}\right],$$

$$\hat{w}_0=\frac{\alpha}{2}-\frac{2}{3},\quad \hat{w}_2=\alpha^2(1-\alpha)\bar{\gamma}W_0+\alpha^3W_2-\frac{2}{9},$$

$$W_0=\frac{1}{2},\quad W_2=\left[\frac{1}{8}+\frac{(1-a^2)}{20}\right],$$

where

$$\gamma=\alpha\bar{\gamma}=\alpha\left(\frac{2-a^2}{4}\right), \tag{3.53}$$

and α is given in (3.52).

The time t_* is then given in terms of the initial temperature T_* by

$$t_*=\frac{a\alpha}{2}\sqrt{\frac{3}{\kappa\rho_*}},\quad \rho_*=\frac{a_s}{4c}T_*^4. \tag{3.54}$$

Taking the leading-order part of the initial data gives a curve parametrized by a in the (z_2, w_0)-plane that cuts through the saddle point SM in system (3.31), (3.32), between the stable and unstable manifold (the lighter dashed line in figure 1). This implies that a small under-density corresponding to $a<1$ will evolve to the stable rest point $M=(z_2,w_0)=\left(-\frac{4}{3},\frac{1}{3}\right)$ (cf. figure 1).

(g) The numerics

In this section, we present the results of our numerical simulations. We simulate solutions of (3.31)–(3.34) for each value of the acceleration parameter $a<1$ in a small neighbourhood of $a=1$ (corresponding to small under-densities relative to the SM), and for each temperature T_* in the range $3000\,\mathrm{K}\le T_*\le 9000\,\mathrm{K}$. We simulate up to the time t_a, the time depending on the acceleration parameter a at which the Hubble constant is equal to its present measured value $H=H_0=100h_0(\mathrm{km/s\,mpc})$, with $h_0=0.68$. From this we conclude that the dependence on T_* is negligible. We then asked for the value of a that gives $Q(z_2(t_a), w_0(t_a))=0.425$, the value of Q in dark energy theory with $\Omega_\Lambda=0.7$. This determines the unique value $a=\underline{a}=0.999999426$ and the unique time $t_0=t_{\underline{a}}$. These results are recorded in the following theorem.

Theorem 3.6. *At present time t_0 along the solution trajectory of (3.31)–(3.34) corresponding to $a=\underline{a}$, our numerical simulations give $H=H_0$, $Q=0.425$, together with the following:*

$$z(t_0,\xi)=(-1.142)\xi^2+(1.385)\xi^4+O(\xi^6),$$

$$w(t_0,\xi)=0.247-(0.348)\xi^2+O(\xi^4)$$

$$A(t_0,\xi)=1+(0.381)\xi^2-(0.277)\xi^4, \tag{3.55}$$

and

$$D(t_0,\xi)=1+(0.095)\xi^2+O(\xi^4). \tag{3.56}$$

The cubic correction to redshift versus luminosity as predicted by the wave model at $a = \underline{a}$ *is*

$$C = 0.359. \tag{3.57}$$

Note that (3.55) and (3.56) imply that the space–time is very close to Minkowski at the present time up to errors $O(\xi^4)$, so the trajectory in the (z_2, w_0)-plane is much closer to the stable rest point M than to the SM at the present time (cf. figure 1). The cubic correction associated with dark energy theory with $k = 0$ and $\Omega_\Lambda = 0.7$ is $C = -0.180$, so (3.57) is a theoretically verifiable prediction which distinguishes the wave model from dark energy theory. A precise value for the actual cubic correction corresponding to C in the relation between redshift versus luminosity for the galaxies appears to be beyond current observational data.

(h) The uniform space–time at the centre

In this section, we describe more precisely the central region of accelerated uniform expansion triggered by the instability due to perturbations that meet the ansatz (3.22)–(3.25). By (3.30) we have seen that neglecting terms of order ξ^4 in z, the density $\rho(t)$ depends only on the time. Further neglecting the small errors between (z_2, w_0) and the stable rest point $\left(-\frac{4}{3}, \frac{1}{3}\right)$ at present time t_0 when $a = \underline{a}$, we prove that the space–time is Minkowski with a density $\rho(t)$ that drops like $O(t^{-3})$, so the instability creates a central region that appears to be a flat version of a uniform Friedman universe with a larger Hubble constant, in which the density drops at a faster rate than the $O(t^{-2})$ rate of the SM.

Specifically, as $t \to \infty$, our orbit converges to $\left(-\frac{4}{3}, \frac{1}{3}\right)$, the stable rest point for the (z_2, w_0) system

$$\begin{pmatrix} z_2 \\ w_0 \end{pmatrix}' = \begin{pmatrix} -3w_0\left(\frac{4}{3} + z_2\right) \\ -\frac{1}{6}z_2 - \frac{1}{3}w_0 - w_0^2 \end{pmatrix}. \tag{3.58}$$

Setting $z_2 = -\frac{4}{3} + \bar{z}(t)$, $w_0 = \frac{1}{3} + \bar{w}(t)$ and discarding higher-order terms, we obtain the linearized system at rest point $\left(-\frac{4}{3}, \frac{1}{3}\right)$:

$$\begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ -\frac{1}{6} & -1 \end{pmatrix} \begin{pmatrix} \bar{z} \\ \bar{w} \end{pmatrix}. \tag{3.59}$$

The matrix in (3.59) has the single eigenvalue $\lambda = -1$ with single eigenvector $R = (0, 1)$. From this we conclude that all orbits come into the rest point $\left(-\frac{4}{3}, \frac{1}{3}\right)$ from below along the vertical line $z_2 = -\frac{4}{3}$. This means that $z_2(t)$ and $\rho(t) = z_2(t)/t^2$ can tend to zero at algebraic rates as the orbit enters the rest point, but $w_0(t)$ must come into the rest point exponentially slowly, at rate $O(\mathrm{e}^{-t})$. Thus our argument that $\bar{w} = w_0 - \frac{1}{3}$ is constant on the scale where $\rho(t) = k_0/t^\alpha$ gives the precise decay rate,

$$\rho(t) = \frac{k_0}{t^{3(1+\bar{w})}}. \tag{3.60}$$

That is, $\bar{w} \equiv \bar{w}(t) \to 0$ and $k_0 \equiv k_0(t)$ are changing exponentially slowly, but the density is dropping at an inverse cube rate, $O(1/t^{3(1+\bar{w})})$, which is *faster* than the $O(1/t^2)$ rate of the standard model.

Therefore, neglecting terms of order ξ^4 together with the small errors between the metric at present time t_0 and the stable rest point, the space–time is Minkowski with a density $\rho(t)$ that drops like $O(t^{-3})$, a faster rate than the $O(t^{-2})$ of the SM. Furthermore, we show that neglecting relativistic corrections to the velocity of the fluid near the centre where the velocity is zero, evolution towards the stable rest point creates a flat, centre-independent space–time which evolves outward from the origin, and whose size is proportional to the Hubble radius.

We conclude that the effect of the instability triggered by a perturbation of the SM consistent with ansatz (3.22)–(3.25) near the stable rest point $\left(-\frac{4}{3}, \frac{1}{3}\right)$ is to create an acceleration consistent with the anomalous acceleration of the galaxies in a large, flat, uniform, centre-independent space–time, expanding outwards from the centre of the perturbation.

4. Conclusion

This is a culmination in the authors' ongoing research programme to identify a possible mechanism that might account for the anomalous acceleration of the galaxies within Einstein's original theory, without the cosmological constant or dark energy. We have found such a mechanism, namely, our discovery of an instability in the Friedmann space–time characterized by a universal phase portrait (figure 1) which describes smooth spherical perturbations about any point. It is universal in the sense that it describes the evolution near the centre of *any* $p=0$ spherically symmetric space–time that solves the Einstein equations in SSC and is smooth at $r=0$ in the ambient Euclidean coordinate system that corresponds to SSC. The phase portrait places SM at an unstable saddle rest point *SM*, and the unstable manifold of *SM* provides a specific mechanism which induces anomalous accelerations into the SM without the cosmological constant. This mechanism induces *precisely* the same range of quadratic corrections to redshift versus luminosity as does the cosmological constant, without assuming it. The phase portrait of the instability shows that only under-dense and over-dense perturbations of *SM* are observable (not *SM* itself), and the under-dense case would imply that we live within a large (order $|\xi|^4 \ll 1$) region of approximate uniform density that is expanding outwards from us at an accelerated rate relative to the SM. The central region created by the instability is different from, but looks a lot like, a speeded-up Friedman universe tending more rapidly to flat Minkowski space than the SM. Finally, we prove that a one-parameter family of exact perturbations from the radiation epoch trigger the instability, and provide a third-order correction to redshift versus luminosity that makes a prediction which can be compared to the predictions of dark energy.

Given that SM is unstable, the paper raises the fundamental question as to whether it is reasonable to expect to observe an unperturbed Friedman space–time, with or without dark energy, on the scale of the supernova data. But the paper does not purport to solve all the problems of Cosmology. We have made no assumptions regarding the space–time far from the centre of the perturbations that trigger the instabilities in the SM. The consistency of this model with other observations in astrophysics would require additional assumptions that apply far from the centre. Naively, one might wonder whether a local perturbation, neglecting higher-order terms, perhaps only lies at a scale below the large scale on which the Friedmann metric is assumed to apply (like voids or galaxies). But of course, our theory then implies that the Friedmann space–time is also unstable on that larger scale where it is also assumed to apply. The instability raises the question as to the observability of the Friedmann space–time, with or without dark energy, on any scale. Regarding the higher-order terms, we note that when $p=0$, the fluid velocity is the only sound speed, so solutions far from the centre should not constrain solutions near the centre so long as the velocity remains positive. In the light of [17], solutions near the centre should be extendable on an initial data surface by arbitrary density and velocity profiles, and this reflects the freedom to impose coefficients of higher-order powers of ξ on an initial data surface. So there is a great deal of freedom to extend beyond these local space–times, and the extensions would ultimately determine the size of the central region. But to explore further assumptions concerning the space–time far from the centre in this paper would obscure the clarity of the theory presented. Applications of this theory are topics of the authors' future research.

Data accessibility. This work does not have any experimental data.
Authors' contributions. This is all joint work.
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